GRAVITY AND COSMOLOGICAL MODELS ON (ALMOST) NONCOMMUTATIVE SPACES

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GEOMETRY THROUGH SPECTRAL TRIPLES: WHY ?

DEFINITION: THE SPECTRAL TRIPLE

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions:

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5. ...+ conditions of „analysis” type

THEOREM [CONNES]

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

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HOW TO CONSTRUCT THEM?

No general method: there are only very few examples.

GEOMETRY AND NUMERS: SPECTRAL DATA

THE SPECTRAL PROPERTIES OF THE DIRAC OPERATOR

Classically it is known that the spectrum of the Dirac is discrete (separate eigenvalues), with finite multiplicities and no point of convergence apart form ∞. Roughly speaking - spectrum of Dirac squared is like the spectrum of Laplace operator. The properties do not change if we modify the operator (change the metric, add connections, torsion etc etc.)

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SO ONE CAN COMPUTE SOME SPECTRAL FUNCTIONALS

$$S(\mathcal{D}, f) = \mathrm{tr}\left(f(\mathcal{D}^2) \right),$$

where f is a suitable function such that the expression makes sense and tr is the usual trace on the Hilbert space.

GEOMETRY AND NUMERS: SPECTRAL DATA

THE EXOTIC TRACE

Although the usual trace does not extend to the operators like Dirac (or its powers) there are some *exotic traces* which might be interpreted as regularized traces (something of the form of ζ-function regularization).

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THE SPECTRAL FUNCTIONAL (2)

Using this exotic traces we can postulate the spectral functional to be (for example)

$$S(\mathcal{D}, \Lambda) = \sum_n \Lambda^n \fint \mathcal{D}^{-n},$$

where $\fint$ is that exotic trace. For most n that would be 0 but some terms would be nonzero.

THE UBIQUITOUS HEAT KERNEL!

THEOREM (GILKEY)

For a Riemannian n-dimensional manifold M, with a metric compatible connection, ∇, on a vector bundle E over M and a Laplace-type operator is of the form:

$$L = -g^{ab}\nabla_a\nabla_b - Z,$$

we have:

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we have:

$$tr\, e^{-tL} = \sum_{k=0}^{n} \left(\frac{1}{4\pi t}\right)^{\frac{k-n}{2}} \int_M a_{[k]}(x) + o(t).$$

where $a_{[k]}(x)$ are functions on M (De Witt–Seeley–Gilkey).

HEAT-KERNEL ASYMPTOTICS

$$a_{[0]} = \mathrm{rank}(E), \quad a_{[1]} = 0, \quad a_{[2]} = \mathrm{tr}(-\frac{1}{6}R + Z).$$

where R is the scalar curvature.

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REMARK

First term gives nothing else but the so-called Weyl's theorem about the growth of eigenvalues of the Laplace operator.

THE ALGEBRA AND ITS GEOMETRY

Let M be a two dimensional Riemannian manifold which we assume to be closed, connected and oriented with a fixed spin structure. Let $D : L^2(S) \to L^2(S)$ denote the Dirac operator of M acting on the space of spinors.

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Consider the algebra $\mathcal{A} = C^\infty(M) \otimes M_n(\mathbb{C})$ of matrix valued smooth functions on M and let $h \in \mathcal{A}$ be a positive element. The operator D naturally acts on the Hilbert space $H = L^2(S) \otimes \mathbb{C}^n$.

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We would like to address the question: does the Gauss-Bonnet theorem hold for D_h? Can we compute the *scalar curvature* ?

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Then:

$$hDh = UHU^* DUHU^* = U\left(H\left(D + U^*[D, U]\right)H\right)U^*.$$

Therefore the spectrum of D_h is the same as the spectrum of $D_{A,H} = H(D + A)H$, where A is a matrix valued one-form.

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The aim is to show that the Gauss-Bonnet holds for the family of conformally rescaled Dirac operators with possible fluctuations $D_{A,H} = H(D + A)H$ where the rescaling is a diagonal matrix.

THE ALGEBRA AND ITS GEOMETRY

THEOREM

Let X be a compact connected Hausdorff space such that $H^2(X,\mathbb{Z}) = 0$. Then any continuous map $H : X \to M_n(\mathbb{C})$ with values in positive definite matrices with simple spectrum is continuously diagonalizable.

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It should be noted that if X is a smooth manifold and H is smooth as well, then the roots $\lambda_i(x)$ will be smooth functions on X. It follows that the line bundles L_i will be smooth and in that case H will be smoothly diagonalizable, provided of course $H^2(X, \mathbb{Z}) = 0$.

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THEOREM

If X is a compact Riemann surface then there exists a non-diagonalizable smooth positive 2×2 matrix.

COMPUTING THE CURVATURE

We take the symbols of D^2 and compute the symbols of D^{-2}:

$$\begin{aligned} b_0 &= (a_2+1)^{-1}, \\ b_1 &= -\left(b_0 a_1 + \partial_k(b_0)\delta_k(a_2)\right) b_0, \\ b_2 &= -\Big(b_1 a_1 + b_0 a_0 + \partial_k(b_0)\delta_k(a_1) + \partial_k(b_1)\delta_k(a_2) \\ &\qquad + \frac{1}{2}\partial_k\partial_j(b_0)\delta_k\delta_j(a_2)\Big) b_0, \end{aligned}$$

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The computations of b_2 yield three parts, which we treat separately: independent of A, linear in A, quadratic in A and terms depending on the derivate of A.

$$\zeta_{f,D}(0) = \int_{\mathbb{T}^2} \mathrm{Tr}\, f\, R = \int_{\mathbb{T}^2} \int_{\xi} \mathrm{Tr}\, f\, b_2.$$

TERMS LINEAR IN A

$$R^{(1)}(H, A) = \sum_{i=1,2} 2\pi H G(\Delta)(A_i)\delta_i(H),$$

where G is the following function:

$$G(s) = \frac{(1+\sqrt{s})\sqrt{s}}{(s-1)^3}\left((s+1)\ln(s) - 2(s-1)\right),$$

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and a second term, surprisingly enough:

$$R^{(2)}(H,A)=\sum_{i=1,2}-2\pi H^{-2}\delta_i(H)G(\Delta)(A_i)H,$$

with the same function $G(s)$. Note that after taking the trace both terms shall cancel each other independently of its value at $s=1$, which is $G(1)=\frac{2}{3}$.

$$\Delta(x)=H^{-4}xH^4.$$

TERMS LINEAR IN $\delta_i(A_i)$

In this case we have:

$$b_2(H, \delta_i(A_j)) = -b_0 H^3 \delta_i(A_i) b_0 H + b_0^2 H^5 \delta_i(A_i) b_0 H^3 \xi^2 \\ + b_0^2 H^7 \delta_i(A_i) b_0 H \xi^2,$$

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and again explicit integration over ξ gives:

$$\pi H^{-1} F(\Delta)(\delta_i(A_i)) H,$$

where

$$F = -\frac{(1+\sqrt{s})\sqrt{s}}{(s-1)^2} \ln(s) + \frac{\sqrt{s}+1}{s-1}.$$

Again, it is not difficult to check that $F(1) = 0$ and the expression vanishes after we take the trace of it, so:

$$\mathrm{Tr}\ \left(R(H, \delta_i(A_j))\right) = 0.$$

QUADRATIC TERMS IN A_i

$$R(H, A^2) = -\pi H^{-1} Q(\Delta^{(1)}, \Delta^{(2)})(A_i \cdot A_i) H$$

where

$$Q(s,t) = \frac{\sqrt{s}(\sqrt{t}+s)}{(s-1))(s-t)} \ln s - \frac{\sqrt{s}\sqrt{s}}{(s-t)\sqrt{t}} \ln t$$

To compute the trace we first take $t = 1$:

$$F(s) = Q(s,1) = \frac{(s+1)\ln s + 2(1-s)}{(s-1)^2},$$

and observe that due to the trace property:

$$\mathrm{Tr}\left(H^{-1} F(\Delta)(A_i) A_i H\right) = \mathrm{Tr}\left(A_i F(\Delta)(A_i)\right) = \mathrm{Tr}\left(F(\Delta^{-1})(A_i) A_i\right).$$

But:

$$F(\frac{1}{s}) = \frac{-(\frac{1}{s}+1)\ln s + 2(1-\frac{1}{s})}{(\frac{1}{s}-1)^2} = \frac{-(s+1)\ln s - 2(1-s)}{(s-1)^2} = -F(s),$$

and therefore $\mathrm{Tr}\, R(H, A^2) = 0$, so the quadratic term vanishes as well.

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Conclusion: still, differential geometry methods do work for almost commutative geometries *but* the computations are becoming really difficult.

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INTO 4 DIMENSIONS: MINIMALITY

THEOREM (AS, J. PSEUDO-DIFFER. OPER. APPL. (2014) 5: 305)

In 4 dimensions the conformally rescaled Laplace operator (as proposed by Khalkhali and Fatzizadeh) is not minimal:

$$\Delta_h = \sum_{a=1}^{4} h^{-2}\delta_a(h^2\delta_a)h^{-2},$$

does not minimize the functional: $\Phi(\Delta_h) = Wres(\Delta_{h^{-1},Y})$.

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PROOF.

Using the calculus of pseudodifferential operators on the noncommutative torus and the formula for Wodzicki residue expressed in terms of symbol or order -4 of Δ_h^{-1}:

$$\Delta_h = h^{-2}\left(\sum_a \delta_a^2\right) + \sum_a Y^a \delta_a,$$

Using:

$$Y_a = \delta_a(h^{-2}) + T_a,$$

we have

THEOREM

The Wodzicki residue of Δ^{-2} *depends only on* h*:*

$$Wres(\Delta^{-2}) = 2\pi^2\, \mathfrak{t}(h^4).$$

whereas for Δ^{-1}*:*

$$Wres(\Delta^{-1}) = \frac{\pi^2}{2}\Big(\mathfrak{t}(h^2 T_a h^2 T_a h^2) + \mathfrak{t}(h^2[T_a, \delta_a(h^2)]) \\ - \mathfrak{t}(\delta_a(h^2) h^{-2} \delta_a(h^2))\Big)\,.$$

A SIMPLE TOY MODEL

Consider a $4D$ toroidal geometry with the full algebra of matrix-valued functions and the Dirac operator:

$$D = \gamma^1 \partial_1 + A(x^1) \left(\gamma^i \partial_i \right) + \gamma^5 \gamma^1 T(x^1).$$

where A, T are matrix-valued differentiable functions over the circle.

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Question: how does the Einstein-Hilbert action looks like ?

In fact, we are only interested (for now) in the T-dependent terms or in the case $T = 0$..

THE $T = 0$ CASE

The Einstein-Hilbert action for the matrix Robertson-Walker geometry (alone - no torsion, no extra fields)
The action reads:

$$\int \mathrm{Tr}\left(A^{-7} F(\Delta)(\dot{A}^2)(\dot{A}^2) + A^{-5} G(\Delta)(\dot{A})\dot{A} - \frac{1}{2}\frac{\pi^2}{A^5}\ddot{A}^2 \right),$$

where

$$F(s) = -\frac{\pi^2}{4}\,\frac{5s^7 + 20s^6 + 30s^5 + 20s^4 - 3s^3 - 32s^2 - 64s - 16}{(s+1)^4 s^3},$$

$$G(s) = \frac{2(s+2)\pi^2}{s(s+1)^2}.$$

$s = 1$:

$$\frac{5\pi^2}{8}(\dot{A}^2)(\dot{A}^2) + \frac{3\pi^2}{2} A^{-5}\dot{A}\dot{A} - \frac{1}{2}\frac{\pi^2}{A^5}\ddot{A}^2 = \pi^2 A^{-5}\left(3(\dot{A})^2 - A\dot{A}\right)$$

THE T-TERMS

:

$$b_2 \sim [T, A] \left(b_0 A T b_0^2 + b_0^2 A T b_0 \right) \xi_i^2 - b_0 T^2 b_0,$$

and then after integrating out ξ and we get the *torsion terms* as (using the modification of the so-called Lesch rearrangement Lemma that we used with LD to study the asymmetric noncommutative torus):

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$$2\pi^2 \mathrm{Tr}\,([T, A] A F(\Delta)(T)) - \frac{1}{4}\pi^2 \mathrm{Tr}\, T^2,$$

where

$$F(s) = \frac{1}{(s+1)s} \left(1 + \frac{1}{s} + \frac{1}{s^2} \right).$$

and

$$\Delta(x) = A^{-2} x A^2.$$

ALMOST NONCOMMUTATIVE TOROIDAL FLRW GEOMETRY.

First, let us write the Dirac operator for the toroidal FLRW geometry with the metric:

$$ds^2 = (dx^1)^2 + a(t)^2 \left((dx_2)^2 + (dx_3)^2 + (dx_4)^2\right).$$

when acting on the Hilbert space with the **flat** torus metric:

$$D_f = a(t)^{-\frac{3}{2}} D a(t)^{\frac{3}{2}} = \gamma^1 \partial_1 + \frac{1}{a(t)} \left(\gamma^2 \partial_2 + \gamma^3 \partial_3 + \gamma^4 \partial_4\right),$$

ALMOST NONCOMMUTATIVE TOROIDAL FLRW GEOMETRY.

First, let us write the Dirac operator for the toroidal FLRW geometry with the metric:

$$ds^2 = (dx^1)^2 + a(t)^2 \left((dx_2)^2 + (dx_3)^2 + (dx_4)^2 \right).$$

when acting on the Hilbert space with the **flat** torus metric:

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Now, make it slightly noncommutative ("two sheets" geometry):

$$D_f = \begin{pmatrix} D_t + A_1(t) D_3 & \gamma\phi \\ \gamma\phi^* & D_t + A_2(t) D_3 \end{pmatrix},$$

THE CURVATURE

Again we can compute (using the Wodzicki residue technique) the Einstein-Hilbert term for this model. First, there will be kinetic part for the a_1 and a_2 parameters:

$$\sum_i \pi^2 \frac{3\dot{A_i}^2 - A_i \ddot{A_i}}{A_i^5}.$$

and the term that mixes both Φ and a_1, a_2:

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$$\sum_i \pi^2 \frac{3\dot{A_i}^2 - A_i \ddot{A_i}}{A_i^5}.$$

and the term that mixes both Φ and a_1, a_2:

$$2\pi^2 |\Phi|^2 \frac{(A_1 - A_2)^2}{(A_1 + A_2) A_1 A_2} \left(1 + \frac{1}{A_1^2} + \frac{1}{A_2^2} + \frac{1}{A_1 A_2} \right)$$

ARE THERE ANY CONSEQUENCES ?

We can look for small perturbations around the classical solution of the Einstein equations (we are in the Euclidean case but this still makes sense). Assume that:

$$A_1(t) = A(t)(1 + \varepsilon a(t)), \qquad A_2(t) = A(t)(1 - \varepsilon a(t)),$$

and that $A(t)$ is the solution of the Einstein equations (for example with no cosmological constant):

$$A(t) \sim (H_0 t)^{-\frac{2}{3}},$$

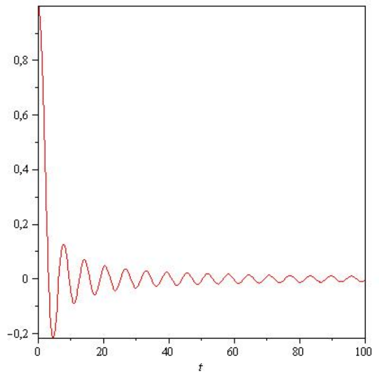
and see what we get for the perturbations $a(t)$.

SMALL PERTURBATIONS

There are two regimes: first for small t and then for large t (say: near the Big Bang and now):

For $t >> 0$:

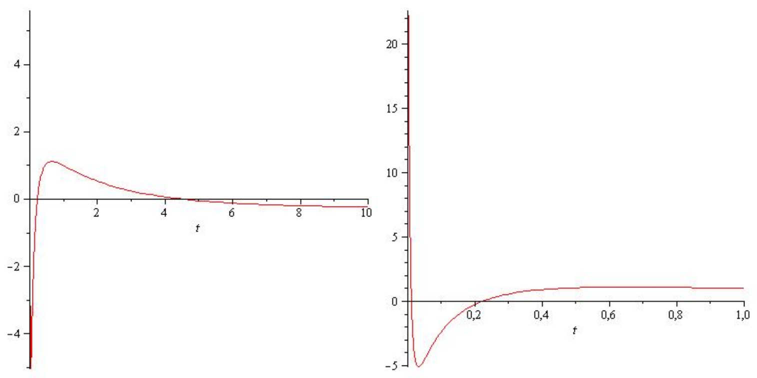
$$a(t) \sim \frac{\sin(ct)}{t},$$



SMALL PERTURBATIONS TOWARDS $t = 0$

The (qualitative) solution at small t is of the form:

$$a(t) \sim \frac{\cos(c\ln(t))}{\sqrt{t}},$$



CONCLUSIONS

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What are *minimal operators* in NCG ? *Good* operators ?

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FIFTH PROBLEM:

Are the models *computable* ?