Causality and Noncommutativity

Andrzej Sitarz*
INSTITUTE OF PHYSICS, JAGIELLONIAN UNIVERSITY,
REYMONTA 4, 30-059 KRAKÓW, POLAND
AND
COPERNICUS CENTER FOR INTERDISCIPLINARY STUDIES,
UL. SŁAWKOWSKA 17, 31-016 KRAKÓW, POLAND

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Abstract

Noncommutative Geometry offers a modern mathematical approach to the formulation of physical models, which comprise gravity and gauge theories. We review its basic ideas, applications to models with Lorentzian geometry and challenges it poses to our understanding of causality.

1 Introduction

Within the last century our understanding of the world improved enormously both at the micro scale (particle physics) as well as at the macro scale (cosmology). However, still there is no theory, which could offer a comprehensive view of all fundamental physics. On one hand we have gauge theories, with roots in the mathematical theory of principal fibre bundles and connections, which nicely allow the formulation of classical and perturbative quantum field theory. On the other hand we have again a purely geometry theory of gravity, mathematically set in the Riemannian (or pseudo-Riemannian) geometry, which gives no hints whatsoever as to its quantum version, the *mythical* quantum gravity. Both theories are well-tested and define the basics of our current knowledge. Yet, there is no unique and satisfactory theory (nor even an idea) how to combine both of them into one single mathematical framework.

So far, the attempts were directed towards the modifications of General Relativity, so that it is presented like a gauge theory, which could lead to some quantum formalism. However, an attempt to look to opposite direction, which originated some time ago from works of Alain Connes, appears to be at least worth looking at. How does it work ? Instead of looking at gravity as some sort of gauge theory we should rather look at gauge theories, as some modifications of pure gravity. However, in order to do that we need

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to enlarge our understanding of geometry using a set of new tools and methods, which goes under the name of *Noncommutative Geometry* (see [5, 11] for details).

2 The Language of Noncommutative Geometry

One of the main difficulties of noncommutative geometry is its language. In all modern physical theories, the fundamental concepts are always connected with the notion of *points* and *spaces*. This allows easily not only to draw pictures and visualize many concepts, but also to include a very intuitive concept of locality. Even in the case of Quantum Mechanics, or Quantum Field Theory, when working with operators on Hilbert spaces, one very often can visualize them as local, or differential operators on $L^2(X)$, for a suitable space X.

Contrary to that, Noncommutative Geometry starts from a completely different point, namely an algebra and its representation(s) on a Hilbert space. Of course, just like a space alone is not sufficient to do geometry, we need also structure to be imposed on the algebra to work with some really interesting objects. The one, relevant for topology is that of C^* algebras:

Definition 2.1. Let $\mathcal{A}$ be an involutive Banach algebra (that is a complex[1] normed algebra, which is complete as a topological space in the norm). If:

$$||aa^*|| = ||a||^2,$$

then $\mathcal{A}$ is a C^*-algebra.

Example 1. If X is a (locally) compact Hausdorff space and $C(X)$ is the algebra of continuous functions on X, then $C(X)$ is a commutative (non) unital C^*-algebra. Such algebra is always abelian.

What is, however, more important, is the converse of the above statement, which forms the renowned Gelfand-Naimark theorem:

Theorem 2.1 (Gelfand-Naimark). *If a C^* algebra is commutative then the Gelfand transform, $\mu : \mathcal{A} \to C_0(X_{\mathcal{A}})$, where $X_{\mathcal{A}}$ is the Gelfand spectrum of $\mathcal{A}$ (space of homomorphisms from $\mathcal{A}$ to $\mathbb{C}$):*

$$\mu(a)(\chi) = \chi(a), \quad a \in \mathcal{A}, \chi \in X_{\mathcal{A}},$$

is an isometric $*$-isomorphism.

Any such map $\xi \in X_{\mathcal{A}}$ is called a character of the algebra, and, roughly speaking, characters of abelian algebras could be identified with points of the underlying space. Here begins the problem with the intuition, since this cannot be translated to the case of nonabelian algebras. These object may have very few or even no characters at all, so we cannot think of them as functions on some space.

A bit of help (though certainly more abstract) comes from the second important fact:

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[1] All algebras we consider are over $\mathbb{C}$.

Theorem 2.2 (Gelfand-Naimark-Segal). *Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^* algebra of bounded operators on a Hilbert space $\mathcal{H}$. If the algebra $\mathcal{A}$ is separable then we can take $\mathcal{H}$ to be separable.*

So, instead of thinking of some abstract algebra we are able to make the notion very precise (though not in a unique way), identifying the algebra with an algebra (norm closed) of some bounded operators on a Hilbert space. A good example of C^* algebras is that of finite-dimensional ones:

Example 2. Any finite dimensional C^* algebras is of the form

$$M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}),$$

for some $0 < n_1 \leq n_2 \leq \cdots \leq n_k$, $n_i \in \mathbb{Z}$.

Extending the Hilbert space to an infinite dimensional one (but with a countable basis) we arrive at the next-largest C^*-algebra (metaphorically speaking), which are *compact operators*. The algebra $\mathcal{K}(\mathcal{H})$ is a closure of the algebra of operators of finite rank (a closure of all finite-dimensional C^*-algebras) and it is a C^* algebra. Taking again a separable Hilbert space $\mathcal{H}$ it is not difficult to see that all bounded operators, $B(\mathcal{H})$, form a C^* algebra, which is the largest possible (by GNS construction) C^*-algebra.

To summarize that brief review we can present a dictionary showing corresponding terms:

SPACES	ALGEBRAS
(locally compact) topological space	C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact space	unital C^*-algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone-Čech compactification	multiplier algebra
cartesian product	tensor product

Noncommutative Geometry aims to extend this *dictionary* and equivalence to geometry, or more precisely to differential geometry and Riemannian geometry.

3 Connes' way to Riemannian Geometry

Having a space, or even a topological space, or a manifold, is not enough for the purposes of physics. What one needs is a concept of a *Riemannian manifold*, that is a space equipped with a metric, which has suitable properties. Since we have no points there is no way to introduce the metric in the manner of prescribing the distances. Furthermore, since there is no generic construction of a tangent bundle or even the differential calculus of the first order, we cannot introduce (in a satisfactory way, at least) the metric tensor.

The way, which bypasses both problems has been to choose a construction, which appears to be natural in the language we adapt and which contains in the classical situation all the necessary ingredients of Riemannian geometry. The construction is that of the Dirac operator, which might worry a bit, since that would mean that we need to restrict the notion only to the counterparts of spin or spin^c - geometries. Setting this apart (and keeping in mind that here are ways to circumvent the problem) we shall briefly recall here the main ingredients of the definition.

Definition 3.1. We say that a triple $(A, \mathcal{H}, D)$ is a (unital) spectral triple if:

- $\mathcal{H}$ is a separable Hilbert space,
- A is a unital pre-C^*-algebra, faithfully represented as bounded operators on H, $\pi(a) \in B(\mathcal{H})$ for any $a \in A$,
- D is an unbounded and selfadjoint operator on H, so that its resolvent is compact on $\mathcal{H}$,
- For all $a \in A$ the operators $[D, \pi(a)]$ are bounded.

These are the most important conditions, which give us a *bare* spectral triple. Then we need to add some more conditions:

- Spectral triples are $\mathbb{Z}_2$-graded. We say that it is *even* if there exists an operator $\gamma = \gamma^\dagger$, $\gamma^2 = 1$, $[\gamma, \pi(a)] = 0$ for all $a \in A$ and $\gamma D + D\gamma = 0$. Otherwise we say that the spectral triple is *odd*.
- We say that the spectral triple satisfies *smoothness condition* if for all $a \in A$ both $\pi(a)$ and $[D, \pi(a)]$ belong to the domain of δ, $\delta(x) = [|D|, x]$.
- We say that the spectral triple satisfies a *finiteness condition* if the space of smooth vectors (intersection of domains of δ^m for all $m > 0$ is a finitely generated projective module over A.
- If there exists a nonnegative integer n such that the sequence of first k the eigenvalues of $|D|^{-1}$ (we ignore the kernel of D, which is finite-dimensional) arranged in a decreasing order and counting the multiplicities behaves like $k^{\frac{1}{n}}$.
- We say that a spectral triple is **real** if there exists an antilinear unitary J such that:

  $$[J\pi(a)J^{-1}, \pi(b)] = 0, \quad \forall a, b \in A,$$

  and

  $$\left[J\pi(a)J^{-1}, [D, \pi(b)]\right] = 0, \quad \forall a, b \in A.$$

  The last requirement is called the *order-one condition*. We say that the real spectral triple is of KR-dimension n if: the following conditions are satisfied:

  $$DJ = \epsilon JD, \quad J^2 = \epsilon', \quad J\gamma = \epsilon'' \gamma J,$$

  where $\epsilon, \epsilon', \epsilon''$ are ± 1 depending on n modulo 8 according to the following rules:

n mod 8	0	1	2	3	4	5	6	7
ϵ	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$
ϵ'	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
ϵ''	$+$		$-$		$+$		$-$	

- Finally, we say that the spectral triple is of *homological* dimension n, if there exists an n-Hochschild cycle with coefficients in the bimodule $A \otimes A^{op}$,

  $$a_0 \otimes b_0 \otimes a_1 \otimes \cdots \otimes a_n = c \in Z_n(A, A \otimes A^{op}),$$

  for which

  $$\pi(c) = \pi(a_0)\left(J\pi(b_0)J^{-1}\right)[D, \pi(a_1)] \cdots [D, \pi(a_n)] = \begin{cases} 1 & \text{odd} \\ \gamma & \text{even} \end{cases}$$

There are numerous examples of spectral triples, starting with the ones presented originally with the definition [6], the examples coming from the so-called isospectral deformations [7]. A general theory of equivariant spectral triples [22] led to a family of real spectral triples on the noncommutative torus, which correspond to the different spin structures [20] and the construction of spectral triples on the Podles sphere [9] and the $SU_q(2)$ algebra [8]. The examples of locally compact geometries, like Moyal deformation has also been incorporated as those admitting a spectral triple geometry. It should be mentioned, however, that not all of the above axioms can be met in all the examples and it is still uncertain which of them are essentially important.

4 Can we have Lorentzian Noncommutative Geometry ?

In principle, it is not very difficult to rewrite some part of the axioms for spectral triple for the case of an a arbitrary signature. The crucial problem, however, lies in the properties of the Dirac operator, which, unless like in an Euclidean case, fails to be an operator with a compact resolvent. Therefore one needs an additional ingredient of the theory, which corresponds to the choice of time orientation (in the easiest case of one time direction only).

The axioms, proposed in [21] are, in fact, analogous to those for the Euclidean case and are based on the idea that to each Lorentzian spectral geometry, it should be possible to associate a corresponding Euclidean one, as motivated by the classical situation [1].

Definition 4.1. *A geometric real (odd or even) Lorentzian spectral triple* of signature $(1, q)$ is given by the data $(A, \pi, \mathcal{H}, D, J, \gamma, \beta)$, where:

- A is an involutive algebra, π its faithful, bounded, star representation on a Hilbert space $\mathcal{H}$,
- in the even case, $1 + q \in 2\mathbb{Z}$, $\gamma = \gamma^\dagger$, $\gamma^2 = 1$ is a $\mathbb{Z}_2$ grading, commuting with the representation of A,

- J is an antilinear isometry such that:

  $$[J\pi(a)J^{-1}, \pi(b)] = 0, \quad \forall a, b \in A,$$

- $\beta = -\beta^\dagger$, $\beta^2 = -1$ is the $\mathbb{Z}_2$-grading (associated with the Krein-space structure), commuting with the representation of the algebra A,

- D an unbounded, densely defined operator, which is β-selfadjoint, that is:

  $$D^\dagger = \beta D \beta,$$

  and such that $[D, \pi(a)]$ is bounded for every $a \in A$, and $D\gamma = -\gamma D$.

- The operator

  $$\langle D \rangle = \sqrt{\frac{1}{2}(DD^\dagger + D^\dagger D)} \tag{4.1}$$

  has compact resolvent. In addition it is required that $[\langle D \rangle, [D, \pi(a)]]$ is bounded for all $a \in \mathcal{A}$.

- The grading, reality structure and the Dirac operator satisfy:

  $$DJ = \epsilon JD, \quad J^2 = \epsilon', \quad J\gamma = \epsilon'' \gamma J. \tag{4.2}$$

  where $\epsilon, \epsilon', \epsilon''$ are ± 1 depending on $1-q$ modulo 8 according to the following rules:

$1-q$ mod 8	0	1	2	3	4	5	6	7
ϵ	$+$	$+$	$+$	$-$	$+$	$+$	$+$	$-$
ϵ'	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$+$
ϵ''	$+$		$-$		$+$		$-$	

- the Krein-space structure satisfies:

  $$\beta\gamma = -\gamma\beta, \quad \beta J = -\epsilon^q J\beta,$$

- The Dirac operator satisfies the order-one condition:

  $$\left[J\pi(a)J^{-1}, [D, \pi(b)]\right] = 0, \quad \forall a, b \in A.$$

- There exists a Hochschild cycle of dimension $n = 1 + q$, valued in $A^{op} \otimes A$,

  $$c = b_o^{op} \otimes a_{i_0} \otimes a_{i_1} \otimes \cdots \otimes a_{i_n},$$

  such that:

  $$J\pi(b_o)J^{-1}\pi(a_{i_0})[D, \pi(a_{i_1})] \ldots [D, \pi(a_{i_n})] = \begin{cases} \gamma & n \text{ even} \\ 1 & n \text{ odd} \end{cases},$$

- We say that the Lorentzian spectral triple has time-orientation if there exist $a_i^o, a_i, b_i \in A$ such that

$$\beta = \sum_i J\pi(b_o)J^{-1}\pi(a_i)[D, \pi(b_i)].$$

If we do not assume existence of J, we have a spectral triple without real structure but in such a case it is difficult to say whether we are in a typically Lorentzian case, with one *time-like* noncommutative direction, as we might as well be facing a geometry with a signature (p, q) where $p - q = 1 \mod 2$.

The above definition does not touch analytic properties (like summability, finiteness conditions etc), as these could be worked out using the operator $< D >$ and will be similar to the ones of the Euclidean case (see [11], for instance).

Remark 1. The definition can be in a straightforward way extended to the case of arbitrary signature (p, q). Our notation is that the Euclidean spectral geometry of dimension 0 is identified with the $(0, n)$ signature.

The sign relations for arbitrary signature have been studied before in the context of *spectral geometry axioms* by various people [12, 16, 14].

5 Quantum Field Theory and the Concept of Causality

The classical concept of causality, that is the relationship between *cause* and *effect*, is based again on the notion of *points* which label the events taking place in space-time. However, using the language of algebras it might be relevant to start with the notion of *Einstein causality*, which has been developed for the purposes of Relativistic Quantum Field Theory. First, if we use the nets of algebras, that is algebras of quantum field theory operators, which are assigned to a region of space-time, we come across the following definitions of causality.

Let us also mention two related notions of macro- and micro-causality:

- **Einstein causality:** causal locality in QFT in terms of space-time indexed operator algebras $\mathcal{A}(O)$:

  $$[A, B] = 0, A \in \mathcal{A}(O), B \in \mathcal{A}(O'),$$

  for spacelike sparated regions

  $$\mathcal{A}(O) = \mathcal{A}(O'')$$

  causal shadow property; O'' is causal completion of O.

- **Macrocausality:** relates causality in QFT to some analytical properties of the S-matrix (see [15]), which classically reflects some properties of the retarded Green functions.

- **Microcausality:** is a simple condition for quantum fields evaluated at distinct points x and x', it is obeyed if:

  $$[\Phi(x), \Phi(x')] = 0, \quad (x - x')^2 < 0.$$

  for a quantum field Φ (understood as operators on a Hilbert space).

As we can see - even in the quantum field theory approach the basic underlying structure is that of a space-time, points and their space-like or time-like separation. For this reason, any field theory based on noncommutative geometry cannot be tested using any of the proposed properties of causality.

6 Challenges for Causality from Noncommutativity

Since there is no good notion of **points** and **locality** in noncommutative geometry and, moreover, the field theories on noncommutative spaces could be thought of as *nonlocal* ones, it appears that there is a long way before one could formulate a consistent notion of locality and - in the case of Lorentzian geometries - to formulate some principle of causality.

In most cases, the approach is similar as for the **nonclocal** theories, with the deformation parameter considered **small** so that there exists a **classical** limit. In the end, possible limitations of the theories are considered only in the perturbative expansions. This leads often to some contradicting results. Some consequences of the Lorentzian models have been discussed for the Moyal deformation, which is one of the best known noncommutative spaces, with the algebra generated by selfadjoint elements subject to the commutation relations of the quantum-mechanics type:

$$[x^\mu, x^\nu] = \theta^{\mu\nu}.$$

A.P.Balachandran et al. [2] observe that spacetime noncommutativity deforms statistics and therefore: ,,generically violates causality in noncommutative quantum theories". On the other hand, such violations are potentially a rich source of modifications of existing physics at the subatomic scale. Some of potential features such as modification of the Pauli principle causing forbidden atomic transitions or correlations of observables in spacelike regions are mentioned (though without any specific analysis). Potential cosmological consequences also include anisotropies of the cosmic microwave background radiation.

A completely different conclusion has been reached by D.Bahns, in her work [3] she finds some remnants of causality at large distances: the cluster decomposition property, which remains intact on the noncommutative Minkowski space. Both approaches, however, use the interpretation of the underlying Euclidean space on which the noncommutative product is defined, as the space of *points* we observe, thus, in fact, treating the theory as a type of nonlocal quantum field theory. A treatment of the problem, which is based on the macro causality approach, come from the computations of perturbative quantum field theory for a scalar field on a Moyal space [4]. The results suggest, that, depending on the procedure assumed (as to the choice of which there are no independent indications) one has to give up give up causality or unitarity in noncommutative field theories.

As a sort of remedy, a different approach has been proposed by M.A.Soloviev [23], who proposed a concept of θ-locality, which could replace the micro locality condition listed above. Though this appears to be a relevant concept for the isospectral and Moyal-type deformations, it is limited to those type of non commutativity only.

Finally, let us mention that an alternative to *locality*, which is a so-called *wedge-locality* has been proposed and studied for some of noncommutative models in [13]. The model contains a family of fields, labelled by different noncommutativity parameters, so that fields are related to each other by Lorentz transformations, which act also on the space of non commutativity parameters. Then one can create models, which obey the principles of covariance and locality, that is the fields localized in space-like separated wedges commute with each other.

7 A proposal: causal structure on the states

The discussions of physical models concentrate around the Moyal deformation for an easy reason: this deformation allows presentation as a star product and it in its properties very similar to the usual, commutative geometry. In particular, the derivations, differential calculi, integrals are just the standard ones, moreover, the dual space (momenta) are completely commutative. Therefore there is no need even to depart from the usual scheme of thoughts - and there is no need for the approach using spectral triples.

However, if we consider some more general deformations, we need to start with the general setup, and so far, the one based on spectral triples appears to be the best suited for the purposes. Let us begin with the discussion of the metric, understood as the metric on the space of states $\mathcal{X}$: linear functionals from the algebra to $\mathbb{C}$. In the Euclidean spectral triple geometry one sets the distance between two states ϕ, ψ as:

$$d(\phi, \psi) = sup_{||[D,a]|| \leq 1} |\phi(a) - \psi(a)|.$$

This leads, in the classical situation, where one consider pure states, which correspond to points, to the usual metric on the space, which by the Galfand-Naimark theorem corresponds to the commutative algebra. By definition such metric is positive definite and as such cannot be used in the Lorentzian case. So far the attempts to construct a Lorentzian version of the metric formula as presented above were limited to the classical situation. The first attempt was in [19], later extended to a more elaborate one in [17], however, they were not using the Dirac but rather the classical Laplace-Beltrami-D'Alembert operator.

We would like to indicate here, that using the notion of Lorentzian spectral triples, as presented, we can go much further, using all building blocks of Lorentzian geometry, in particular, the operator β, which determines the time orientation.

Using β and the noncommutative integral (which is still an accessible tool even in the Lorentzian case, provided we use the Euclidean volume form) we can define the *spatial* elements of the algebra $A_s \subset A$ in the following way (n denotes here the dimension of space-time and the metric dimension of $\langle D \rangle$):

$$a \in A_s \Leftrightarrow \,⨍\, \beta[D,a]\langle D\rangle^{-n} = 0.$$

Since in the classical situation β is a form dual to a global time-oriented vector field, the above orthogonality condition means that the gradient of a (which is computed through the commutation with the Dirac operator) is *spatial*. Similarly, one might define elements of the algebra, which are *time-like*, $A_t \subset A$:

$$a \in A_t \Leftrightarrow \,⨍\, [D,b][D,a]\langle D\rangle^{-n} = 0, \;\; \forall b \in A_s.$$

Then one can define the *spatial* distance between the states ϕ, ψ,

$$d_s(\phi,\psi) = sup_{||[D,a]|| \leq 1, a \in A_s} |\phi(a) - \psi(a)|,$$

as well as the *time-like* distance:

$$d_t(\phi,\psi) = sup_{||[D,a]|| \leq 1, a \in A_t} |\phi(a) - \psi(a)|.$$

Having both, it is rather easy to precise that we can consider two states to be casually linked, if their time-like distance is bigger or equal their spatial distance. So, in fact, using the structure of Lorentzian spectral triples one is able to determine the causal structure on the space of states.

Of course, one should be aware of the possible shortcomings of the suggested definition. First of all, the definition might be ill-posed in some degenerate cases, where the metric dimension is 0. Furthermore, the spaces of spatial or time-like elements might be too small to really distinguish between the states. Finally, we should be aware that all this is relative to the choice of β, which plays the fundamental role in the definition of spectral triple and in defining the spatial elements of the algebra. Having a slightly different β (which is by no means unique) might result in a different subset of spatial and time-like elements and therefore in a completely different results for spatial and time-like distances.

We shall not discuss here the details of the definitions as well its limitations and applications, leaving it for some future work. Certainly, one would want to require, for instance, that the causal structure on the space of states remains independent of β. Whether it is indeed the case, we could only conjecture at the moment. We hope that the above presented or a similarly constructed definition of causal structure for the states might be the basics of a better understanding of the relations between non commutativity and causality in physical models.

8 Conclusions

The models of noncommutative geometry pose a big challenge in the considerations of locality and causality. First of all, the notion of Lorentz (and Poincaré) symmetry might be deformed and we possibly cannot use the classical symmetries but rather to replace it with the Hopf algebras. Can we formulate a version of causality, which would correspond to the deformed symmetries ?

Furthermore, most of our intuitions and approaches, also in the quantum field theory, is based on the notion of time as a continuous variable. Therefore, we need some *time ordering* methods, for example. Now, what can we do if the time is supposed to be a noncommutative variable ? Can one allow for a noncommutative time at all ?

Of course, all these questions have their classical counterparts. We indeed have a little grasp what really locality and causality mean at a very small distances or, for instance, in a situation of energy scales comparable to the Planck mass. A solution to these problems would be a major step towards reaching the more abstract ones, which are related with the noncommutative geometry.

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