NONCOMMUTATIVITY AND SINGULARITIES

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1. Singularities

2. Noncommutativity: how it helps

3. The noncommutative pillow

RIEMANNIAN GEOMETRY

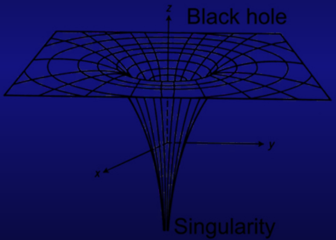
The basis of todays physics is geometry: mathematical description of a space, equipped with additional structure: *topology* (continuity), *differentiability* (smoothness) and *metric* (distances).

One of the consequences of General Relativity and Einstein equations is the appearance of *singularities* - something, which is usually defined as a sort of *pathological behaviour* of a physical theory.

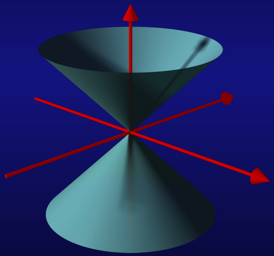
Usually one distinguishes two types of singularities: *physical singularities*, where at some ”point” (or at the limit of some points) a physical quantity becomes degenrate (whatever it might be), and *mathematical singularities*, which appear in the mathematical constructions.

SINGULARITIES

Typical examples of singularities are:



curvature singularities



conical singularities.

WHERE DO CONICAL SINGULARITIES COME FROM ?

In principle conical singularity is best described by an action of a finite group:

$$z \mapsto e^{\frac{2\pi i}{n}} z,$$

is an action of $\mathbb{Z}_n$ on the complex plane such that the quotient space is the cone.

Remark

As a space (even as a topological space !) the cone *is homeomorphic* – (topologically indistinguishable) to the plane !

WHAT ARE ORBIFOLDS

An orbifold is a generalization of the notion of manifold, allowing presence of points, such that their neighbourhood is diffeomorphic to a quotient of neighourhood of $0 \in \mathbb{R}^n$ by a finite group.

Usually, orbifolds are not *global*, that is they are not global quotients of manifolds by a global action of a finite group. If they are – they are *good orbifolds*.

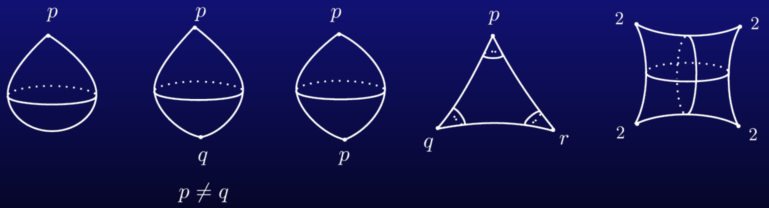
Orbifolds might have any number of sigular points.

Orbifolds are usually *good* topological spaces (Hausdorff).

THE SIMPLEST ORBIFOLDS.

The simplest orbifolds come from the torus and the sphere, they all have the *topology of a sphere* but might have many singular points:

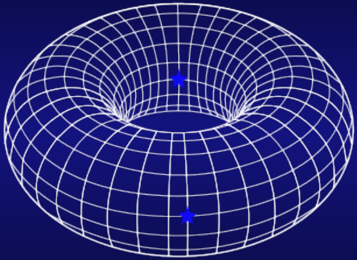
- 1-pillow (**teardrop**)
- 2-pillow (**football**)
- k-pillow (**is always a good orbifold for** $k > 2$)



$2 - d$ spherical orbifolds

THE PILLOW

4-pillow



action on torus

THE NONCOMMUTATIVE PILLOW

Looking on the algebra of functions we have:

$$f(s,t) \mapsto f(-s,-t),$$

so for the generiting functions:

$$U = e^{2\pi i s}, \quad V = e^{2\pi i t},$$

we have:

$$U \mapsto U^*, \quad V \mapsto V^*.$$

For the noncommutative torus:

The above action remains an *automorphism* of the algebra of the *noncommutative torus*:

$$UV = e^{2\pi i\theta} VU,$$

for ant $0 \leq \theta < 1$.

THE NONCOMMUTATIVE PILLOW

DEFINITION

We define the fixed point algebra of the noncommutative torus under the above action of the group $\mathbb{Z}_2$ as the *noncommutative pillow*.

The generators of the algebra of NC pillow are:

$$x = U + U^*, \quad y = V + V^*, \quad z = UV^* + VU^*,$$

with following relations ($\lambda = e^{2\pi i\theta}$):

$$\begin{aligned} xz - \bar{\lambda}zx &= (1-\lambda^2)y \\ zy - \bar{\lambda}yz &= (1-\bar{\lambda}^2)x \\ xy - \lambda yx &= (1-\lambda^2)z \\ x^2 + y^2 + \bar{\lambda}^2 z^2 - xzy &= 2(1-\bar{\lambda}^2) \end{aligned}$$

THE DIFFERENTIAL CALCULUS OVER NC PILLOW

Since the pillow algebra could be embedded in the algebra of the noncommutative torus we could check what happens with the differential calculus.

Theorem

If $\lambda^4 \neq 1$ then the module of two-forms is generated by a single two-form, which generates the module of two-forms over the noncommutative torus.

$$dx = (V - V^*)\omega_V, \quad dy = (W - W^*)\omega_W, \quad dz = (VW^* - V^*W) * (\omega_V - \omega_W),$$

The generating two-form is:

$$\begin{aligned} \omega = & \lambda^3 \, x \, dz \, dy + (1 - \lambda^2)\lambda(1 + \lambda^4) \, z \, dy \, dx \\ & - (1 + \lambda^2)\lambda(1 + \lambda^4) \, y \, dx \, dz - (3\lambda^4 + 1)\lambda \, y \, dz \, dx, \\ = & (3\lambda^4 + 1)(1 - \lambda^4) \, \omega_V \, \omega_W. \end{aligned}$$

THE SPECTRAL TRIPLE PICTURE

DEFINITION

Spectral triple A spectral triple is $(\mathcal{A}, \mathcal{H}, D)$ - an algebra $\mathcal{A}$, faithfully represented on a Hilbert space $\mathcal{H}$, and a densely defined, unbounded, selfadjoint operator with a compact resolvant such that

$$[D, a] \in B(\mathcal{H}), \quad \forall a \in \mathcal{A}.$$

+ several conditions, which we choose to omit...

Over the noncommutative torus there exists a well-known and well-studied spectral triple made of the Dirac operator acting on a dense subspace of $L^2(\mathcal{A}) \otimes \mathbb{C}^2$:

$$D = \sigma^1 \delta_1 + \sigma^2 \delta_2,$$

where δ_i, $i = 1, 2$ are the derivations on the noncommutative torus:

$$\delta_1(U) = U, \;\; \delta_2(U) = 0, \;\; \delta_1(V) = 0, \;\; \delta_2(V) = V.$$

THE SPECTRAL TRIPLE:

One of the *extra* conditions is the existence of the Hoschchild cycle $\sum a_0 \otimes a_1 \otimes a_2$ such that:

$$\gamma = \sum a_0[D, a_1][D, a_2],$$

For the Noncommutative Torus we have:

$$U^* V^* \otimes V \otimes U - V^* U^* \otimes U \otimes V,$$

gives the desired cycle.

Remark

It has been shown by Rennie & Varilly that this condition (for any chain not necessarily a Hochschild chain) cannot be satisfied on orbifolds.

SPECTRAL TRIPLE OVER NC PILLOW

Imagine we restrict the spectral triple that we have over the NC Torus to the NC Pillow - this could be easily done (just take a smaller algebra and the smaller Hilbert space, while keeping the Dirac operator restricted to the smaller subspace.

Is the cycle (chain) conditions satisfied or not ?

Result:

$$(\lambda + \bar{\lambda})z\,[D,x]\,[D,y] - \lambda[D,yz]\,[D,x] + \lambda y\,[D,z]\,[D,x] + \\ + (1+\lambda^2)z\,[D,y]\,[D,x] - [D,xz]\,[D,y] + x\,[D,z][D,y] = 2(\lambda^2 - \bar{\lambda}^2)\gamma.$$

BUT: we demonstrate the existence of chain and *not a Hochschild cycle* !

CONCLUSIONS

REMARK 1

This is only for some specific deformation - *but* more examples exist !

REMARK 2

There are further arguments (homological), which suggest that singularities disappear.

REMARK 3

The deformation and the „orbifolds” are global (good).

REMARK 4

There is no general theory (yet).

CONCLUSIONS

REMARK 5

There are many interesting questions:

- can one see desingularisation on the space of states ?
- can one see desingularisation locally ?
- if these are noncommutative manifolds: what are their properties ?
- what are the most general description of such objects ?

REMARK 6

How can we use that to describe (potentially) geometrical singularities in physics ?
Can we trade singularities for noncommutativity ?

THANK YOU !