QUOTIENTS OF NONCOMMUTATIVE TORI AND DIFFERENTIAL SMOOTHNESS.

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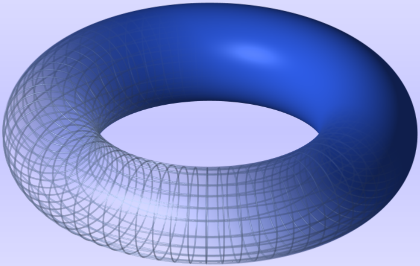
12.09.2017, Boulder

The Pillow

4-pillow



torus

The Noncommutative Pillow

Looking on the algebra of functions we have:

$$f(s,t) \mapsto f(-s,-t),$$

so for the generiting functions:

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For the noncommutative torus:

The above action remains an *automorphism* of the algebra of the *noncommutative torus*:

$$UV = e^{2\pi i\theta} VU,$$

for any $0 \leq \theta < 1$.

The Noncommutative Pillow

Definition

We define the fixed point algebra of the noncommutative torus under the above action of the group $\mathbb{Z}_2$ as the *noncommutative pillow*.

The generators of the algebra of NC pillow are:

$$x = U + U^*, \quad y = V + V^*, \quad z = UV^* + VU^*,$$

with following relations ($\lambda = e^{2\pi i\theta}$):

$$\begin{aligned} xz - \bar{\lambda}zx &= (1-\lambda^2)y \\ zy - \bar{\lambda}yz &= (1-\bar{\lambda}^2)x \\ xy - \lambda yx &= (1-\lambda^2)z \\ x^2 + y^2 + \bar{\lambda}^2 z^2 - xzy &= 2(1-\bar{\lambda}^2) \end{aligned}$$

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Is this a manifold or an orbifold ?

The Differential calculus over NC Pillow

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The generating two-form is:

$$\begin{aligned}\omega &= \lambda^3 x \, dz \, dy + (1-\lambda^2)\lambda(1+\lambda^4) \, z \, dy \, dx \\ &\quad - (1+\lambda^2)\lambda(1+\lambda^4) \, y \, dx \, dz - (3\lambda^4+1)\lambda \, y \, dz \, dx, \\ &= (3\lambda^4+1)(1-\lambda^4) \, \omega_V \, \omega_W.\end{aligned}$$

Differential forms and integral forms

Dually to a differential calculus on A one considers its *integral calculus*. Let ΩA be a differential calculus on A. The space of n-forms $\Omega^n A$ is an A-bimodule. Let $\mathcal{I}_n A$ denote the right dual of $\Omega^n A$, i.e. the space of all right A-linear maps $\Omega^n A \to A$. Each of the $\mathcal{I}_n A$ is an A-bimodule with the actions

$$(a \cdot \phi \cdot b)(\omega) = a\phi(b\omega), \qquad \text{for all} \quad \phi \in \mathcal{I}_n A, \ \omega \in \Omega^n A, \ a, b \in A.$$

The direct sum of all the $\mathcal{I}_n A$, denoted $\mathcal{I}A = \oplus_n \mathcal{I}_n A$, is a right ΩA-module with action

$$(\phi \cdot \omega)(\omega') = \phi(\omega \wedge \omega'), \qquad \text{for all} \quad \phi \in \mathcal{I}_{n+m} A, \ \omega \in \Omega^n A, \ \omega' \in \Omega^m A.$$

A *divergence* on A is a linear map $\nabla : \mathcal{I}_1 A \to A$, such that

$$\nabla(\phi \cdot a) = \nabla(\phi) a + \phi(da), \qquad \text{for all} \quad \phi \in \mathcal{I}_1 A, \ a \in A.$$

Integrable differential calculi

Definition

An n-dimensional differential calculus ΩA is said to be *integrable* if ΩA admits a complex of integral forms $(\mathcal{I}A, \nabla)$ for which there exist an algebra automorphism ν of A and A-bimodule isomorphisms $\Theta_k : \Omega^k A \to {}^\nu\mathcal{I}_{n-k}A$, $k = 0, \ldots, n$, rendering commutative the following diagram:

$$\begin{array}{ccccccccccc}
A & \xrightarrow{d} & \Omega^1 A & \xrightarrow{d} & \Omega^2 A & \xrightarrow{d} & \ldots & \xrightarrow{d} & \Omega^{n-1} A & \xrightarrow{d} & \Omega^n A \\
\downarrow{\scriptstyle \Theta_0} & & \downarrow{\scriptstyle \Theta_1} & & \downarrow{\scriptstyle \Theta_2} & & & & \downarrow{\scriptstyle \Theta_{n-1}} & & \downarrow{\scriptstyle \Theta_n} \\
{}^\nu\mathcal{I}_n A & \xrightarrow{\nabla_{n-1}} & {}^\nu\mathcal{I}_{n-1} A & \xrightarrow{\nabla_{n-2}} & {}^\nu\mathcal{I}_{n-2} A & \xrightarrow{\nabla_{n-3}} & \ldots & \xrightarrow{\nabla_1} & {}^\nu\mathcal{I}_1 A & \xrightarrow{\nabla} & {}^\nu A .
\end{array}$$

The n-form $\omega := \Theta_n^{-1}(1) \in \Omega^n A$ is called an *integrating volume form*.

Integrable calculi and volume form

Theorem (Equivalent conditions)

- *ΩA is an integrable differential calculus.*
- *ΩA has a volume form ω such that all left multiplication maps*

$$\ell^k_{\pi_\omega} : \Omega^k A \to \mathcal{I}_{n-k} A, \qquad \omega' \mapsto \pi_\omega \cdot \omega', \qquad k = 1, \dots, n-1,$$

are bijective, where:

$$\pi_\omega(\omega a) = a, \qquad \text{for all } a \in A,$$

and the right multiplication is:

$$(\phi \cdot \omega)(\omega') = \phi(\omega \wedge \omega').$$

Smoothness

Definition (Smoothness)

An algebra with integer Gelfand-Kirillov dimension n is said to be *differentially smooth* if it admits an n-dimensional connected integrable differential calculus.

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Gelfand-Kirillov dimension

Let us write $\mathcal{V}(n)$ for the subspace of A spanned by 1 and all words in generators of A of length at most n. The algebra A is said to have *polynomial growth* if there exist $c \in \mathbb{R}$ and $\nu \in \mathbb{N}$ such that $\dim \mathcal{V}(n) \leq cn^{\nu}$ for all sufficiently large n. The *Gelfand-Kirillov dimension* of A is a real number defined as

$$\mathrm{GKdim}(A) := \inf\{\nu \mid \dim \mathcal{V}(n) \leq n^{\nu},\ n \gg 0\},$$

The Spectral Triple Picture

Definition

Spectral triple A spectral triple is $(\mathcal{A}, \mathcal{H}, D)$ - an algebra $\mathcal{A}$, faithfully represented on a Hilbert space $\mathcal{H}$, and a densely defined, unbounded, selfadjoint operator with a compact resolvant such that

$$[D, a] \in B(\mathcal{H}), \quad \forall a \in \mathcal{A}.$$

+ several conditions, which we choose to omit...

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Over the noncommutative torus there exists a well-known and well-studied spectral triple made of the Dirac operator acting on a dense subspace of $L^2(\mathcal{A}) \otimes \mathbb{C}^2$:

$$D = \sigma^1 \delta_1 + \sigma^2 \delta_2,$$

where δ_i, $i = 1, 2$ are the derivations on the noncommutative torus:

$$\delta_1(U) = U, \ \ \delta_2(U) = 0, \ \ \delta_1(V) = 0, \ \ \delta_2(V) = V.$$

THE SPECTRAL TRIPLE:

One of the *extra* conditions is the existence of the Hoschchild cycle $\sum a_0 \otimes a_1 \otimes a_2$ such that:

$$\gamma = \sum a_0[D, a_1][D, a_2],$$

The spectral triple:

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$$U^* V^* \otimes V \otimes U - V^* U^* \otimes U \otimes V,$$

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Remark

It has been shown by Rennie & Varilly that this condition (for any chain not necessarily a Hochschild chain) cannot be satisfied on orbifolds.

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Imagine we restrict the spectral triple that we have over the NC Torus to the NC Pillow - this could be easily done (just take a smaller algebra and the smaller Hilbert space, while keeping the Dirac operator restricted to the smaller subspace.

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Result:

$$(\lambda+\bar{\lambda})z\,[D,x]\,[D,y]-\lambda[D,yz]\,[D,x]+\lambda y\,[D,z]\,[D,x]+$$
$$+(1+\lambda^2)z\,[D,y]\,[D,x]-[D,xz]\,[D,y]+x\,[D,z][D,y]$$
$$=2(\lambda^2-\bar{\lambda}^2)\gamma.$$

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WARNING: we demonstrate the existence of chain *not a Hochschild cycle !*

Bieberbach groups and manifolds

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Bieberbach manifolds are compact manifolds, which are quotients of the Euclidean space by a free, properly discontinuous and isometric action of a discrete group. The first nontrivial low-dimensional examples appear in dimension three and have been already described in the seminal works of Biebierbach.

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- Could be made noncommutative (slightly)

Free actions of finite groups on 3-torus

name	group G	gen$G$	action of G on U, V, W
B2	$\mathbb{Z}_2$	e	$e \triangleright U = -U,\ e \triangleright V = V^*,\ e \triangleright W = W^*$
B3	$\mathbb{Z}_3$	e	$e \triangleright U = e^{\frac{2}{3}\pi i} U,\ e \triangleright V = W^*,\ e \triangleright W = W^* V$
B4	$\mathbb{Z}_4$	e	$e \triangleright U = iU,\ e \triangleright V = W,\ e \triangleright W = V^*$
B5	$\mathbb{Z}_2 \times \mathbb{Z}_2$	e_1, e_2	$e_1 \triangleright U = -U,\ e_1 \triangleright V = V^*,\ e_1 \triangleright W = W^*$ $e_2 \triangleright U = U^*,\ e_2 \triangleright V = -V,\ e_2 \triangleright W = -W^*$
B6	$\mathbb{Z}_6$	e	$e \triangleright U = e^{\frac{1}{3}\pi i} U,\ e \triangleright V = W,\ e \triangleright W = WV^*$

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N1	$\mathbb{Z}_2$	e	$e \triangleright U = -U,\ e \triangleright V = V,\ e \triangleright W = W^*$
N2	$\mathbb{Z}_2$	e	$e \triangleright U = -U,\ e \triangleright V = VW,\ e_1 \triangleright W = W^*$
N3	$\mathbb{Z}_2 \times \mathbb{Z}_2$	e_1, e_2	$e_1 \triangleright U = -U,\ e_1 \triangleright V = V^*,\ e_1 \triangleright W = W^*$ $e_2 \triangleright U = U,\ e_2 \triangleright V = -V,\ e_2 \triangleright W = W^*$
N4	$\mathbb{Z}_2 \times \mathbb{Z}_2$	e_1, e_2	$e_1 \triangleright U = -U,\ e_1 \triangleright V = V^*,\ e_1 \triangleright W = W^*$ $e_2 \triangleright U = U,\ e_2 \triangleright V = -V,\ e_2 \triangleright W = -W^*$

Noncommutative Bieberbach Manifolds

Let's take a noncommutative 3-torus realized as a twisted group algebra $C^*(\mathbb{Z}^3, \omega_\theta)$ with the cocycle over $\mathbb{Z}^3$:

$$\omega_\theta(\vec{m}, \vec{n}) = e^{\pi i \sum_{j,k=1}^{3} \theta_{jk} m_j n_k}, \quad \vec{m}, \vec{n} \in \mathbb{Z}^3,$$

where θ_{jk} is a real antisymmetric matrix ($0 \leq \theta_{jk} < 1$).

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Find all possible values of the matrix θ_{jk} such that the actions of the finite group G (as discussed earlier) are compatible with the cocycle.

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where θ_{jk} is a real antisymmetric matrix ($0 \leq \theta_{jk} < 1$).
Find all possible values of the matrix θ_{jk} such that the actions of the finite group G (as discussed earlier) are compatible with the cocycle.
We say that the action of the finite group G is compatible with the cocycle ω_θ if:

$$g \triangleright (a *_{\omega_\theta} b) = (g \triangleright a) *_{\omega_\theta} (g \triangleright b), \quad \forall a, b \in C^*(\mathbb{Z}^3), g \in G.$$

Noncommutative Bieberbach Manifolds

Definition

Let $C(\mathbb{T}^3_\theta)$, be a twisted group algebra over $\mathbb{Z}^3$ corresponding to a cocycle obtained from $\theta_{12} = \theta_{21} = 0$ and $\theta_{23} = -\theta$ for an irrational $0 < \theta < 1$. Then the generating unitaries U, V, W satisfy relations:

$$UV = VU, \quad UW = WU, \quad WV = e^{2\pi i\theta} VW.$$

We define the algebras of noncommutative Bieberbach manifolds as the fixed point algebras of the following actions of G on $C(\mathbb{T}^3_\theta)$:

name	group	action of $\mathbb{Z}_N$ on U, V, W
$\mathrm{B2}_\theta$	$\mathbb{Z}_2$	$e \triangleright U = -U,\ e \triangleright V = V^*,\ e \triangleright W = W^*,$
$\mathrm{B3}_\theta$	$\mathbb{Z}_3$	$e \triangleright U = e^{\frac{2}{3}\pi i}U,\ e \triangleright V = e^{-\pi i\theta} V^* W,\ e \triangleright W = V^*,$
$\mathrm{B4}_\theta$	$\mathbb{Z}_4$	$e \triangleright U = iU,\ e \triangleright V = W,\ e \triangleright W = V^*,$
$\mathrm{B6}_\theta$	$\mathbb{Z}_6$	$e \triangleright U = e^{\frac{1}{3}\pi i}U,\ e \triangleright V = W\ ,\ e \triangleright W = e^{-\pi i\theta} V^* W,$
$\mathrm{N1}_\theta$	$\mathbb{Z}_2$	$e \triangleright U = U^*,\ e \triangleright V = -V,\ e \triangleright W = W,$
$\mathrm{N2}_\theta$	$\mathbb{Z}_2$	$e \triangleright U = U^*,\ e \triangleright V = -V,\ e \triangleright W = WU^*,$

Properties of the action

Freeness of a coaction of a Hopf algebra H on a C^*-algebra A means (for a right coaction) that the spans of $(a \otimes \mathrm{id})\Delta(b)$ and $\Delta(b)(a \otimes \mathrm{id})$, $a, b \in A$ are dense in $A \otimes H$ for a minimal tensor product.

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Theorem (P.Olczykowski, AS)

The actions of the cyclic groups $\mathbb{Z}_N$, $N = 2, 3, 4, 6$ on the noncommutative three-torus, as given in the previous table, are free.

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The actions of the cyclic groups $\mathbb{Z}_N$, $N = 2, 3, 4, 6$ on the noncommutative three-torus, as given in the previous table, are free.

Proof

Easy: in the B2,B3,B4,B6 case the coaction of the dual Hopf algebra to the group algebra $\mathbb{Z}_N$ is simply: $\Delta U = U \otimes \tilde{e}$, where $\tilde{e}$ is the generator of $\mathbb{C}(\mathbb{Z}_N)$. Since U (and its powers) are invertible, it is evident that $(a \otimes \mathrm{id})\Delta(U^n)$ and $\Delta(U^n)(a \otimes \mathrm{id})$ are dense in $A \otimes \mathbb{C}(\mathbb{Z}_N)$. In the $\mathrm{N1}_\theta, \mathrm{N2}_\theta$ case the same argument applies when we take V instead.

The K-theory groups of Bieberbachs

Theorem (P.Olczykowski, AS)

$$K_0(\mathrm{B}2_\theta) = \mathbb{Z}^2 \oplus (\mathbb{Z}_2)^2, \qquad K_1(\mathrm{B}2_\theta) = \mathbb{Z}^2,$$
$$K_0(\mathrm{B}3_\theta) = \mathbb{Z}^2 \oplus \mathbb{Z}_3, \qquad K_1(\mathrm{B}3_\theta) = \mathbb{Z}^2,$$
$$K_0(\mathrm{B}4_\theta) = \mathbb{Z}^2 \oplus \mathbb{Z}_2, \qquad K_1(\mathrm{B}4_\theta) = \mathbb{Z}^2,$$
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Remarks

The result is valid also for rational θ (requires some arguments).

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Remarks

The result is valid also for rational θ (requires some arguments).
The method is straightforward (though technically complicated).

The K-theory groups of Bieberbachs

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Almost all Bieberbachs have torsion in K-theory.

Classification of flat real spectral triples

- Step 1: Prove that any spectral triple comes from the NC 3-torus.

Lemma

Let $\{\mathrm{BN}_\theta, \mathcal{H}, J, D\}$ be a real spectral triple over a noncommutative Bieberbach manifold BN_θ. Then, there exists a spectral triple over three-torus, such that this triple is its reduction.

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- Compute spectral action, spectral invariants, some perturbations...

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EXAMPLE $N = 3$ AND RESULT

LEMMA

The only D-equivariant and real-equivariant action of $\mathbb{Z}_N$ on $\mathcal{H}$, which implements the action of $\mathbb{Z}_N$ on the algebra is possible if:

N	3	4	6
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THEOREM (P.OLCZYKOWSKI, AS)

Number of possible noncommutative spin structures for Bieberbach BN_θ:

N	2	3	4	6
#	8	2	4	2

CONCLUSIONS

- Quotients of NC tori to study some ideas.

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THANK YOU!