Riemannian Geometry of Quantum Phase Space

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WHY MOYAL AND NONCOMMUTATIVITY ?

- Classical gravity $\mapsto$ Riemannian geometry with commutative coordinates
- string theory motivation: candidate of quantum gravity
- new kind of spacetime geometry
- extended objects
- effective nonlocal interactions

STRING MOTIVATION

Open string theory in the background B field

$$= \frac{1}{4\pi\alpha'} \int_\Sigma \left(g_{ij} \partial_a X^i \partial^a X^j - 2\pi\alpha' B_{ij} \epsilon^{ab} \partial_a X^i \partial^b X^j \right)$$

Taking mode expansion:

$$X^i = x^i + 2\alpha' \left(g^{ij} p_j t - 2\pi\alpha' B^{ij} p_j \sigma \right)$$
$$+ \sqrt{2\alpha'} \sum_n \frac{e^{-int}}{n} \left(i g^{ij} \alpha_{nj} \cos n\sigma - 2\pi\alpha' B^{ij} \alpha_{jn} \sin n\sigma \right)$$

one obtains from the usual canonical commutation relations of field theory:

$$[x^i, x^j] = i 2\pi\alpha' \left((g + 2\pi\alpha' B)^{-1}_A \right)_{ij} .$$

SPHERE AND PLANE

What is a difference between a sphere and a plane ?

A sphere is a one-point compactification of the plane.
Looking at the geometry of the sphere could be translated to the geometry of the plane.

Metrics on the plane

In a similar manner we can consider matrics on the plane and ask, which metrics would be *regular* in infinity.
One can pose the following question: what are the metrics on $\mathbb{R}^2$ such that their curvature is constant ?
Looking for the family of conformally rescaled metrics:

$$ds^2 = k^2(r)(dx^2 + dy^2)$$

We have the curvature

$$R(k) = \frac{k''(r)}{k^3(r)} + \frac{k'(r)}{rk^3(r)} - \frac{k'(r)k'(r)}{k^4(r)}$$

Solution

The problem (despite its nonlinearity) has a solution:

$$k(r) = A\frac{r^{a-1}}{b + r^{2a}}$$

which gives constant curvature

$$R(k) = \frac{8a^2 b}{A^2}$$

and finite volume:

$$V(k) = \frac{\pi A^2}{ba}$$

So the Gauss-Bonnet term is:

$$\int_{\mathbb{R}^2} \sqrt{g} R = 8\pi a.$$

Fubini-Study metric

Of course, out of the family of metrics with constant scalar curvature only these with $a = 1$ are regular (strictly positive at $r = 0$ and $r = \infty$).
The rest are actually positive metrics but rather on a cylinder ($\mathbb{R}^2 \setminus 0$)
But one can show that

Lemma

Assuming that $k(r)$ is regular (nonzero) at $r = 0$ and that $k(r)$ behaves like $r^{-\alpha}$ at $r \to \infty$ we obtain:

$$\int_{\mathbb{R}^2} \sqrt{g} R(g) = 4\pi\alpha.$$

CONSTANT CURVATURE PHASE-SPACE?

So far all the examples of *noncommutative manifolds* were flat - or very *symmetric* - that is it - the metric (in any of its disguises) was not really allowed to vary.
One of the interesting new ideas is that there is a way to *vary* the metric using the notion of *conformal rescaling* or *partial conformal rescaling* of the Dirac operator (Laplace operator).
The idea is to replace D by D_h:

$$D_h = hDh,$$

where h is from the commutant of the algebra A.
In a similar way we can change Δ to $h\Delta h$ (at least in the two-dimensional case).
Then using spectral ζ-function we might look for the corresponding Gauss-Bonnet term.

MOYAL FACTS

What is the Moyal plane (Moyal deformation of the plane) ?

$$[x_1, x_2] = \theta$$

But this is not the best description (as both coordinates are unbounded operators).

$$(f * g)(x) = (\pi\theta)^{-2} \int \int f(x+s) g(x+t) e^{-2i\Theta(s,t)},$$

where

$$\Theta(s,t) = \theta^{-1}(s_1 t_2 - s_2 t_1),$$

$$a = \frac{1}{\sqrt{2}}(x_1 + i x_2), \quad a^* = \frac{1}{\sqrt{2}}(x_1 - i x_2)$$

Moyal Facts

Then we define:

$$f_{00} = 2e^{-\frac{1}{\theta}(x_2^2+x_2^2)},$$

and the algebra A_θ has a natural basis consisting of:

$$f_{mn} = \frac{1}{\sqrt{m!n!\theta^{m+n}}}(a^*)^m * f_{00} * (a)^n,$$

or more explicitly:

$$f_{mn}(r,\phi) = 2(-1)^m 2\sqrt{\frac{m!}{n!}}e^{i\phi(m-n)}\left(\sqrt{\frac{2}{\theta}}r\right)^{(n-m)} L_m^{n-m}\left(\frac{2r^2}{\theta}\right)e^{-\frac{r^2}{\theta}}.$$

More Moyal Facts

We have for each $m, n, k, l \geq 0$:

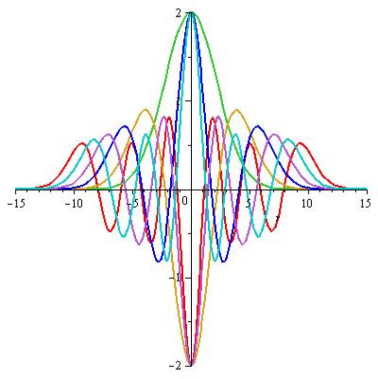
$$f_{mn} * f_{kl} = \delta_{kn} f_{ml}, \quad f_{mn}^* = f_{nm}.$$

In particular for all $k \geq 0$ f_{kk} are pairwise orthogonal projections of rank one.

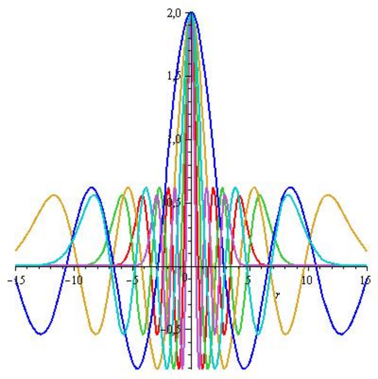
The natural (not normalized) trace on the Moyal algebra is:

$$\tau(f) = \int_{\mathbb{R}^2} f(x), \quad \tau(f_{mn}) = 2\pi\theta.$$

EXAMPLES



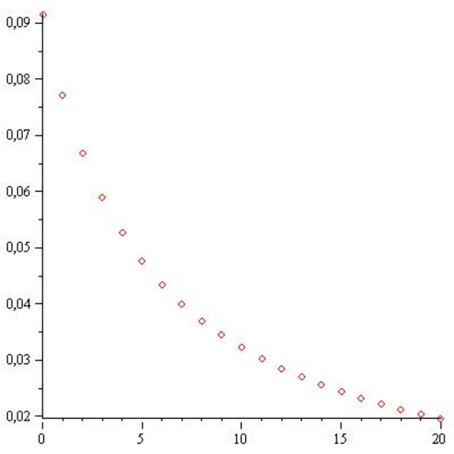
f_{00}



f_{11}

EXAMPLES

Fubini-Study metric:



coefficients in the expansion in f_{mn} basis.

The curvature: orthonormal basis formalism

Let us have a basis of orthonormal vector fields with the following structure of the commutators:

$$[e_i, e_j] = c_{ijk} e_k.$$

Then one can express the scalar curvature as:

$$R = 2\, e_i(c_{ijj}) - c_{kii} c_{kjj} - \frac{1}{4} c_{ijk} c_{ijk} - \frac{1}{2} c_{ijk} c_{kji}\,,$$

THE CURVATURE: ORTHONORMAL BASIS FORMALISM

Let the orthonormal basis of frames be $e_i = h\delta_i$ for h in the commutant and δ_i the standard derivations ($i = 1, 2$).
We have:

$$[e_1, e_2] = h\delta_1(h)h^{-1}e_2 - h\delta_2(h)h^{-1}e_1.$$

so that

$$c_{122} = h\delta_1(h)h^{-1}, \quad c_{121} = -h\delta_2(h)h^{-1}.$$

and

$$c_{212} = -h\delta_1(h)h^{-1}, \quad c_{211} = h\delta_2(h)h^{-1}.$$

We compute the scalar curvature:

$$R = 2h\delta_i(h\delta_i(h)h^{-1}) - (h\delta_i(h)h^{-1})^2 - \frac{1}{4}2(h\delta_i(h)h^{-1})^2 - \frac{1}{2}(h\delta_i(h)h^{-1})^2$$

Orthonormal basis formalism for conformally rescaled Moyal

Finally:

$$\begin{aligned} R &= 2h^2(\delta_{ii}h)h^{-1} + 2h\delta_i(h)\delta_i(h)h^{-1} \\ &\qquad - 2h^2\delta_i(h)h^{-1}\delta_i(h)h^{-1} - 2h\delta_i(h)\delta_i(h)h^{-1} \\ &= 2h^2(\delta_{ii}h)h^{-1} - 2h^2\delta_i(h)h^{-1}\delta_i(h)h^{-1}. \end{aligned}$$

So, to solve the equation $R(h) = \mathrm{const}$ we need to solve:

$$(\Delta h) - \delta_i(h)h^{-1}\delta_i(h) = Ch^{-1}$$

Question

Does the above equation admit nontrivial solutions within the Moyal algebra (or its multiplier) ?

Solution in matrix basis

Ansatz

First we put $C = 1$ then we assume we look for a radial solution:

$$h = \sum_{n=0}^{\infty} \phi_n f_{nn}.$$

What we need to know is the action of partial derivatives on the basis:

$$\partial = \frac{1}{\sqrt{2}}(\partial_{x_1} - i\partial_{x_2}), \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{x_2}).$$

Then:

$$\partial f_{m,n} = \sqrt{\frac{n}{\theta}} f_{m,n-1} - \sqrt{\frac{m+1}{\theta}} f_{m+1,n},$$

$$\bar{\partial} f_{m,n} = \sqrt{\frac{m}{\theta}} f_{m-1,n} - \sqrt{\frac{n+1}{\theta}} f_{m,n+1},$$

Solution

$$\Delta f_{m,n} = \frac{2}{\theta}\left(-(m+n+1)f_{m,n} + \sqrt{(m+1)(n+1)}f_{m+1,n+1} + \sqrt{mn}f_{m-1,n-1}\right).$$

So, we obtain the following equation:

$$\sum_n \left(-(2n+1)\phi_n - n\phi_n^2\phi_{n-1}^{-1} - (n+1)\phi_n^2\phi_{n+1}^{-1}\right) f_{n,n}$$
$$+ \sum_n (n+1)(\phi_{n+1} + \phi_n) f_{n+1,n+1} + \sum_n n(\phi_{n-1} + \phi_n) f_{n-1,n-1}$$
$$= \sum_n c(\phi_n)^{-1} f_{n,n},$$

SOLUTION

The recurrence relation:

$$\frac{n+1}{\phi_{n+1}}(\phi_{n+1}^2 - \phi_n^2) + \frac{n}{\phi_{n-1}}(\phi_{n-1}^2 - \phi_n^2) = c\phi_n^{-1},$$

Is there a solution ?

Of course, it is a recurrence relation but do we have a solution within a class of admissible funtions ?

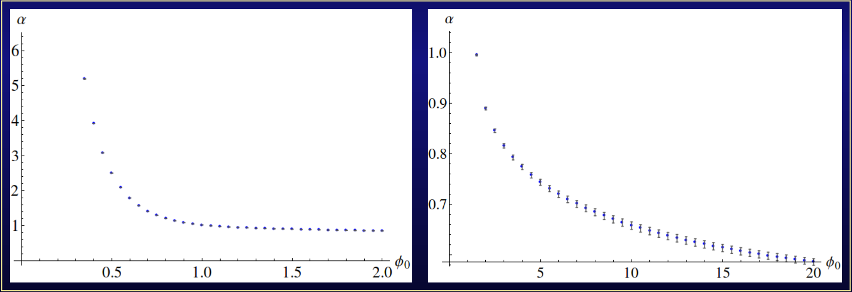
The recurrence relation is a quadratic one of the form:

$$(n+1)x^2 + x\left(\frac{n}{\phi_{n-1}}(\phi_{n-1}^2 - \phi_n^2) - \frac{c}{\phi_n}\right) - (n+1)\phi_n^2 = 0,$$

and it is easy to see that it has only one positive root. It could be also easily seen (by induction) that the sum of roots is positive and since their products is $-\phi_n^2$ then the positive root must be bigger than ϕ_n. Hence the solution will be an increasing positive sequence.

NUMERICAL SOLUTIONS

Depending on the initial value $\phi_0 > 0$ one can get any power-law asymptotics $\phi^n \sim An^\alpha$ for large n. A numerical simulation shows the dependence $\alpha(\phi_0)$ for $n = 5000 \pm 1000$.



Gauss–Bonnet Theorem for Moyal

In the ONB formalism we have $\sqrt{g} = h^{-2}$ and we can compute the Gauss–Bonnet term:

$$\int_{\mathbb{R}^2} \sqrt{g} R = \int_{\mathbb{R}^2} 2 \Big((\delta_{ii} h) - \delta_i(h) h^{-1} \delta_i(h) \Big) h^{-1}.$$

With the radial ansatz $h = \sum \phi_n f_{nn}$, we obtain

$$\int_{\mathbb{R}^2} \sqrt{g} R = 4\pi \lim_{N \to \infty} (N+1) \Big(\phi_{N+1} \phi_N^{-1} - \phi_N \phi_{N+1}^{-1} \Big).$$

Assuming that $\phi_n \sim A n^\alpha$ as $n \to \infty$, we have

$$\int_{\mathbb{R}^2} \sqrt{g} R = 8\pi\alpha.$$

On the other hand, since the curvature is constant, we have

$$\int_{\mathbb{R}^2} \sqrt{g} R = c \int_{\mathbb{R}^2} \sqrt{g} = 2\pi c \theta \sum_{n=0}^{\infty} \phi_n^{-2},$$

so we need to impose $\alpha > 1/2$ to have the volume finite.

Perturbative expansion in θ

In the physics literature the deformation parameter θ is frequently treated as a small perturbation parameter and all possible physical quantities and fields are computed up to certain order in θ.

Using the formal expression for the Moyal product of two functions on $\mathbb{R}^2$:

$$(f * g)(p) = \left(e^{\frac{i}{2}\partial_p \partial_r} f(p) g(r) \right)_{|p=q} ,$$

we obtain the following perturbative expansion of the Moyal product of two radial functions:

$$(f * g)(r) = f(r)\, g(r) - \frac{\theta^2}{8r} \left(f''(r) g'(r) + f'(r) g''(r) \right) ,$$

where derivatives are with respect to the radius r.

More perturbative...

As a consequence the following gives the perturbative formula for the Moyal inverse of the radial function:

$$f_*^{-1} = f^{-1} + \frac{\theta^2}{4r}\left((f')^3 f^{-4} - f'' f' f^{-3}\right).$$

We shall also need the formula for the perturbative expansion of the following product:

$$\delta_i(f) * f_*^{-1} * \delta_i(f),$$

$$\begin{aligned}
\delta_i(f) * f_*^{-1} * \delta_i(f) &= +(f')^2 f_*^{-1} \\
+\theta^2 \Bigg(\frac{1}{4}\frac{1}{r^4} f'^2\, f^{-1} &+ \frac{1}{2}\frac{1}{r^3} f'^3\, f^{-2} - \frac{1}{2}\frac{1}{r^3} f'\, f''\, f^{-1} + \frac{1}{r^2} f'^4 f^{-3} - \frac{3}{2}\frac{1}{r^2} f'^2\, f''\, f^{-2} \\
&+ \frac{1}{4}\frac{1}{r^2} f'\, f'''\, f^{-1} + \frac{1}{4}\frac{1}{r^2} f''^2\, f^{-1} - \frac{1}{2}\frac{1}{r} f'^3\, f''\, f^{-3} \\
&+ \frac{1}{4}\frac{1}{r} f'^2\, f'''\, f^{-2} + \frac{1}{2}\frac{1}{r} f'\, f''^2\, f^{-2} - \frac{1}{4}\frac{1}{r} f''\, f'''\, f^{-1} \Bigg)
\end{aligned}$$

QUESTIONS

WARNING

Numerical solutions are just a *hint* not a proof.

Questions ?

- what is the *good* range of initial parameters ?
- what does it mean *a regular* solution ?
- what does it mean *regular at infinity* ?
- what is the *Moyal Fubini-Study* ?
- what is the class of metrics that defines Moyal spheres ?

Spectral triples over Moyal plane

There are (at least) two spectral triples over Moyal plane and its higher-dimensional versions: a locally compact triple with the *flat* - "isospectral" Dirac operator and the compact "harmonic oscillator" triple.
The difference is in the action of the $\Delta = D^2$ operator:

$$\frac{\theta}{s}\Delta f_{m,n} = -(m+n+1) f_{m,n} + \sqrt{(m+1)(n+1)}\, f_{m+1,n+1} + \sqrt{mn}\, f_{m-1,n-1}$$

standard

$$\Delta f_{m,n} = -(m+n+1) f_{m,n}$$

harmonic

Rescaling the Laplace operator

Consider the harmonic oscillator spectral triple and the conformally rescaled operator

$$\Delta_H = H \, \Delta \, H = D_H^2.$$

where again we restrict ourselves to the situation where h is depending only on r so:

$$H = \sum_{n=0}^{\infty} a_n f_{n,n}.$$

The problem

Can we compute the ζ function of the conformally rescaled Laplace operator and determine its value at $z = 0$?

Remark

Certainly some assumptions about H are necessary. First of all $H > 0$ then we need to be sure that the spectral dimension of Δ_H is indeed 2.

THE COMPUTATION

$$\begin{aligned} D_H^2 f_{k,l} &= \sum_{m,n} a_m a_n f^o_{mm} D^2 f^o_{nn} f_{k,l} \\ &= \sum_{m,n} a_m a_n (n+k+1) f_{km} \delta_{ln} \delta_{nm} \\ &= \sum_{n} (a_n)^2 (n+k+1) f_{kn} \delta_{ln} = (a_l)^2 (l+k+1) f_{k,l}. \end{aligned}$$

Compute $\zeta_{D_H^2}$:

$$\zeta_{D_H^2}(s) = \sum_{m,n} |a_m|^{-2s} \frac{1}{(m+n+1)^s},$$

The volume...

Assume that $a_k \sim k^\alpha$ we have:

$$\zeta_{D_H^2}(s) = \sum_{m,n} \frac{1}{m^{2\alpha s}} \frac{1}{(m+n+1)^s},$$

which is an example of the multiple zeta function, in particular:

$$\zeta_{D_H^2}(s) = \zeta(2\alpha s, s),$$

which has possible poles at $s = 1$ and $2\alpha s + s = 3 - p$ for $p \in \mathbb{Z}_+$. The sum converges absolutely if $s > 1$ and $s(2\alpha + 1) > 2$.

Lemma

For $\alpha > 0$ the function $\zeta_{D_H^2}(s)$ has no pole at $s = 2$ and the value at $s = 1$ is $\zeta(2\alpha)$.

Gauss-Bonnet ?

Lemma

The value

$$\zeta_{D_H^2}(0) = \frac{8\alpha + 5}{12(2\alpha + 1)}.$$

Interpretation

There are several remarks to be made:

- the result is not necessarily a rational number and depends on the way the series decays.
- the integral value n can be attained for

$$\alpha = \frac{1}{8}\frac{12n - 5}{1 - 3n}$$

- there is no way to tell about the regularity of the solution

Conclusions

1 Remark and 3 Questions.

There are interesting conformally rescaled metrics on the Moyal plane ! *BUT: are they regular ?*

Questions ?

Is this possible for *harmonic oscillator* spectral triple ?

Sphere ?

Is there a *Moyal sphere* ? – Moyal plane with a *regular* constant curvature metric ?

Spheres as homogeneous spaces ?

Can one have Moyal sphere as a homogeneous space ?

More questions ?

- what is the distance on the space of states they define ?
- how can we identify the metric (in general) ?
- can one relate Gauss-Bonnet term to some index-type quantity ?