Conformally rescaled noncommutative geometries

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Abstract. Noncommutative geometry aims to provide a set of mathematical tools to describe spacetime, gravity and quantum field theory at small scales. The paper reviews the idea that noncommutative spaces are described in terms algebras and their geometry is encoded in spectral triples, which are basic ingredients of the new notion of Riemannian spin geometry adapted to the language of operator algebras. Using this background we propose a new idea of conformally rescaled and curved spectral triples, which are obtained from a real spectral triple by a nontrivial scaling of the Dirac operator. The obtained family is shown to share many properties with the original spectral triple. We compute the Wodzicki residue and the Einstein-Hilbert functional for such family on the four-dimensional noncommutative torus.

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1. Introduction

It is commonly believed that when approaching the smallest scale of physics, Planck length, current image of space (or space-time) as a differentiable manifold should break down. Still, is not clear whether this would call for a new notion of space or whether we will only need a better consistent description of quantum theory including the theory of quantum gravity. The latter, quantum theory of gravity, which is the long-awaited dream of theoretical physics, is still unattainable despite various attempts and huge efforts.

One of possible hints where to look for solutions is coming from simple physical considerations. Even though at the moment our limits of measurement are still quite low when compared to the Planck length – we measure

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distances of $10^{-15}m$, which is roughly the size of a proton, and time difference of the order of $10^{-18}s$, while the Planck length is $10^{-35}m$ – the question about possible limits of our measurement accuracy remains valid. Already in 1958 Salecker and Wigner [29] suggested that quantum mechanics implies:

$$\delta l \geq \left(\frac{\hbar l}{mc}\right)^{\frac{1}{2}},$$

which combined with the general relativity and Schwarzschild radius gives a rough estimate

$$\delta l \geq (l l_P^2)^{\frac{1}{3}},$$

where l_p is Planck length.

As the uncertainty relations are linked to noncommutativity of the observables in quantum theory we should expect that positions itself should be a noncommutative algebra. To link such description with the classical tools of Riemannian geometry we need to look for a more general mathematical theory, which would imply both *geometry* as well a *noncommutativity* like in a quantum theory. The *Noncommutative Geometry* is a proposition, which goes into this direction.

The paper is organised as follows: we briefly review basic ideas and dictionary of noncommutative geometry and spectral triples (for a more comprehensive introduction with details, examples and references see [26, 27]). Based on the introduced notion we then propose a family of conformally rescaled geometries and study their fundamental properties. As a particular example we present the result of pure gravity functional (Einstein-Hilbert action) for the four-dimensional noncommutative torus, computed first with the use of the Wodzicki residue on the algebra of pseudodifferential symbols as well as using a "naive" approach based on the formalism of moving frames adapted to the noncommutative setting.

2. From Spaces and Algebras

Classical geometry is based on the principle of describing spaces, which are sets of points equipped with some additional structures. However, the notion of a function (in particular a continuous function if we have a topological space) appears to be more fundamental. In quantum theory this is even more important, since the classical phase space (space of possible positions and momenta) of a physical object is no longer a space. Moreover, what we usually describe as a state of a physical object corresponds to the expectations values of these operators for a given state (a normalized vector) in the Hilbert space. However, the above picture lacks one significant ingredient, the metric, the ability to *measure* the noncommutative space. Noncommutative Geometry is the first sound mathematical concept, which proposes a consistent way of creating *a geometry* of quantum-like spaces. Its long term goal is to provide a meaningful definition of geometry, which would describe both the fundamental interactions as we know them together with the notion of quantised space (for some arguments and models see for instance [13]).

2.1. The theorems behind it: Gelfand-Naimark

The basic ideas of noncommutative geometry lie in the theorems, which demonstrated that one can describe topological spaces using the algebra of continuous functions. Such functions form an algebra, more precisely a C^*-algebra. The latte is an involutive Banach algebra, that is, a complex normed algebra, which is complete as a topological space in the norm, and for every element $a \in A$:

$$||aa^*|| = ||a||^2.$$

It is easy to see that with the supremum norm on the space of continuous functions $C(X)$ for some topological space we have:

Remark 2.1. *If X is a (locally) compact Hausdorff space and $C(X)$ is the algebra of continuous functions on X, then $C(X)$ is a commutative (non) unital C^*-algebra.*

However, a typical example of a C^*-algebra comes from linear bounded operators on a Hilbert space:

Remark 2.2. *Take a separable Hilbert space $\mathcal{H}$ and $B(\mathcal{H})$, the algebra of all bounded operators on $\mathcal{H}$ (with the operator norm). It is a C^* algebra. Any norm closed subalgebra of $B(\mathcal{H})$ is a C^*-algebra.*

What makes these two remarks interesting is the following couple of theorems:

Theorem 2.3 (Gelfand-Naimark-Segal, [20]). *Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^* algebra of operators on a Hilbert space $\mathcal{H}$. If the algebra $\mathcal{A}$ is separable then we can take $\mathcal{H}$ to be separable.*

Theorem 2.4 (Gelfand-Naimark [19]). *If a C^* algebra is commutative then it is an algebra of continuous functions on some (locally compact, Hausdorff) topological space.*

So, shortly speaking – all C^* algebras are subalgebras of bounded operators on a Hilbert space and the commutative ones correspond $1:1$ to locally compact Hausdorff spaces. This makes all noncommutative C^*-algebras perfect candidates for noncommutative spaces, or spaces with singularities.

2.2. Dictionaries and examples

So far we had just given an idea that there exists a natural way to consider some *objects*, which have no counterparts as topological spaces yet still share a lot of common features with them. A simple example are just finite dimensional matrices (like $M_n(\mathbb{C})$), which for $n > 1$ form a noncommutative algebra, also, their direct sums. A different, more sophisticated example (one of the best known ones), is, for instance the so-called noncommutative torus.

Example 2.5 (Irrational Rotation Algebra aka Noncommutative Torus). *Consider the Hilbert space $L^2(S^1)$ and the following operators:*

$$(Uf)(z) = zf(z), \quad (Vf)(z) = f(e^{2\pi i\theta}z),$$

where $0 < \theta < 1$ is an irrational real number. We define $\mathbb{T}^2_\theta$ as a C^*-algebra generated by the unitary operators U, V, U^*, V^*. We easily check that:

$$UV = e^{2\pi i\theta}VU.$$

In fact one just take the above relation as the defining relation of the noncommutative torus. Although there is no geometric picture what this algebra corresponds to (as there is no space) a good intuition is that the algebra describes the space of all possible leaves of Kronecker foliation (with the parameter θ) of the usual torus. If θ is irrational then all leaves are homeomorphic to the real line and the set of all leaves is not even Hausdorff. Yet passing to the algebra (one can understand it as a certain groupoid algebra) we have a much better description and can study it as a noncommutative manifold.

Remark 2.6. *Let us note that although many of the "noncommutative spaces" (like the noncommutative torus above) are described in terms of deformations of manifolds (families of algebras, which for a certain value of a parameter give a commutative algebra of functions on a manifold) this is not always the case.*

In the previous sections we indicated an equivalence between commutative C^*-algebras and spaces. Following the standard literature we just want to point out that this correspondence could be promoted to other topological constructions, like continuous maps between spaces, Cartesian products etc. The following dictionary provides the necessary links:

TOPOLOGY	ALGEBRA
locally compact Hausdorff topological space	nonunital C^*-algebra
homeomorphism	automorphism
continuous proper map	morphism
compact Hausdorff topological space	unital C^*-algebra
open (dense) subset	(essential) ideal
compactification	unitization
Stone-Čech compactification	multiplier algebra
Cartesian product	tensor product (completed)

Of course, the above notions are (almost) purely topological and we would like to extend them to more geometric objects. The noncommutative geometry is a programme to establish such correspondence and use it to study objects in the same way differential geometry is used to study spaces. Below is an approximate version of the extended version of the dictionary of noncommutative geometry.

DIFFERENTIAL GEOMETRY	NONCOMMUTATIVE GEOMETRY
vector bundle	finitely generated projective module
differential forms	differential forms
differential forms	Hochschild homology
de Rham cohomology	cyclic cohomology
vector fields	operators
group	Hopf algebra
Lie algebra	Hopf algebra
principal fibre bundle	Hopf-Galois extension
measurable functions	von Neumann algebra
infinitesimals	compact operators
metric	Dirac operator
spin^c geometry	spectral triple
spin geometry	real spectral triple
integrals	exotic traces

3. Spectral triples and the Dirac operator

In differential geometry the recipe to construct the Dirac operator over a spin manifold is rather easy. You start with a compact, closed Riemannian manifold with a fixed metric g. Then you find the Clifford algebra bundle, choose your favourite spinor bundle, then lift the Levi-Civita metric connection to the spinor bundle. If you compose it with the Clifford map then you obtain a first order differential operator on smooth sections of the spinor bundle. A nontrivial statements can then be proven - that D is, in fact, an elliptic operator, extends to selfadjoint operator on the square-summable sections of the spinor bundle, has compact resolvent and hence a discrete spectrum.

However, a different approach is to use the operational definition. Take an algebra of smooth functions $C^\infty(M)$ represented on a Hilbert space of some sections of a suitable vector bundle over M and look for operators, which behave like the Dirac operators. The crucial point is, of course, in the work "like". What we require is that D needs to be a first order differential operator with compact resolvent. Having that assured, one recovers the differential calculus (the bimodule of differential one-forms) by setting:

$$df := [D, f], \quad f \in C^\infty(M),$$

understood further as an operator on the Hilbert space. An arbitrary one-form will be $\sum f[D, g]$. Moreover, the following formula gives a nice way to recover the metric on your manifold:

$$d(x, y) = \sup_{||[D,f]|| \le 1, f \in C^\infty(M)} |f(x) - f(y)|, \quad \forall x, y \in M.$$

These are just the examples – as from the spectral information about the Dirac operator we can recover a lot of information about the additional structures on the manifold. Apart from the differential calculus and the metric we can construct the measure and discover the dimension of the manifold.

3.1. The noncommutative generalisation

We are ready to define what is expected to replace Riemannian spin geometry in the realm of noncommutative algebras. The idea of spectral triples is based on the properties of Dirac operators and constructions we discussed earlier.

Definition 3.1 (see [3, 4]). *A real (even) spectral triple is given by the data $(\mathcal{A}, \pi, \mathcal{H}, D, J, (\gamma))$, where $\mathcal{A}$ is an involutive algebra, π its faithful bounded star representation on a Hilbert space $\mathcal{H}$, D an selfadjoint operator with compact resolvent, such that $[D, \pi(a)]$ is bounded for every $a \in \mathcal{A}$, γ is (in the even case) a hermitian $\mathbb{Z}_2$ grading, $D\gamma = -\gamma D$, and J is an antilinear isometry such that:*

$$[J\pi(a)J^{-1}, \pi(b)] = 0, \quad \forall a, b \in \mathcal{A},$$

and

$$\left[J\pi(a)J^{-1}, [D, \pi(b)]\right] = 0, \quad \forall a, b \in \mathcal{A}.$$

The latter requirement is called the order-one condition. *The dimension of the real spectral triple is defined as the integer n, such that there exists an n-Hochschild cycle with coefficients in the bimodule $\mathcal{A} \otimes \mathcal{A}^{op}$,*

$$a_0 \otimes b_0 \otimes a_1 \otimes \cdots \otimes a_n = c \in Z_0(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^{op}),$$

for which

$$\pi(c) = \pi(a_0)\left(J\pi(b_0)J^{-1}\right)[D, \pi(a_1)] \cdots [D, \pi(a_n)] = \gamma.$$

Moreover, one assumes further relations:

$$DJ = \epsilon JD, \quad J^2 = \epsilon', \quad J\gamma = \epsilon'' \gamma J.$$

where $\epsilon, \epsilon', \epsilon''$ are ± 1 depending on n modulo 8 according to the following rules:

n mod 8	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
ϵ	+	−	+	+	+	−	+	+
ϵ'	+	+	−	−	−	−	+	+
ϵ''	+		−		+		−	

If we do not assume existence of J, we have a spectral triple without real structure. If the spectral triple is odd then γ as described above does not exist and the cycle condition reduces to $\pi_D(c) = 1$.

It is reasonable to assume always that the subalgebra of elements of $\mathcal{A}$ which commute with D is $\mathbb{C}$ (in case of the unital algebra $\mathcal{A}$). Otherwise the differential algebra defined by D shall be degenerate, that is there shall be a nontrivial kernel of d in $\mathcal{A}$. We call spectral triples such that $[D, \pi(a)] = 0$ implies $a \in \mathbb{C}$ *non-degenerate*, we always consider only such triples.

The following tells us that the motivating example of spin geometry with Dirac operator is indeed described in this language:

Remark 3.2. *If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ is the Hilbert space of summable sections of the spinor bundle and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).*

The above definition (which was shown more or less in this form) was proposed by Connes in [2] then developed later by many authors. Details of the proof of the above theorem could be found in [1].

In fact, spectral triples over commutative algebras (which satisfy some additional requirements) are only of that type, thanks to Connes' reconstruction theorem [8]. In other words, commutative spectral triples are equivalent (in the sense of 1:1 correspondence) to compact spin manifolds.

3.2. Examples of spectral triples

Several examples of genuinely noncommutative spectral geometries have already been constructed. The list includes the noncommutative torus [2], [23], more general θ-deformations of manifolds (of which the NC Torus is a special case) [5], Moyal deformation [18], finite matrix algebras: $\oplus_i M_{n_i}(\mathbb{C})$ [22] as well as some specific examples of quantum groups and quantum spaces [9, 12].

We shall review here very briefly the example of the spectral triple over the noncommutative tori, which shall be later used to modify the Dirac operator and introduce a new family of noncommutative metrics.

Example 3.3. *We use the standard presentation of the algebra of d-dimensional noncommutative torus as generated by d unitary elements U_i, $i = 1, \ldots, d$, with the relations*

$$U_j U_k = e^{2\pi i \theta_{jk}} U_k U_j, \quad 1 \leq j, k \leq d,$$

where $0 < \theta_{jk} < 1$ is real and antisymmetric. The smooth algebra $\mathcal{A}(\mathbb{T}^d_\theta)$ is then taken as an algebra of elements

$$a = \sum_{\beta \in \mathbb{Z}^d} a_\beta U^\beta, ,$$

where a_β is a rapidly decreasing sequence and

$$U^\beta = U_1^{\beta_1} \cdots U_d^{\beta_d}, \quad \beta \in \mathbb{Z}^d.$$

The natural action of $U(1)^d$ by automorphisms, gives, in its infinitesimal form, d linearly independent derivations on the algebra, which are determined by the action on the generators:

$$\delta_k(U_j) = \delta_{jk} U_j, \quad \forall j, k = 1, \ldots, d,$$

here δ_{jk} denotes the Kronecker delta.

The algebra of the noncommutative torus $\mathcal{A}(\mathbb{T}^d_\theta)$ has a canonical trace:

$$\mathfrak{t}(a) = a_{\mathbf{0}},$$

where $\mathbf{0} = \{0, 0, \ldots, 0\} \in \mathbb{Z}^d$. The trace is invariant with respect to the action of $U(1)^d$, hence it is closed,

$$\mathfrak{t}(\delta_j(a)) = 0, \quad \forall j = 1, \ldots, d, \forall a \in \mathcal{A}(\mathbb{T}^d_\theta).$$

By $\mathcal{H}$ we denote the Hilbert space of the GNS construction with respect to the trace $\mathfrak{t}$ on the C^* completion of $\mathcal{A}(\mathbb{T}^d_\theta)$ and π the associated faithful representation. The elements of the smooth algebra $\mathcal{A}(\mathbb{T}^d_\theta)$ act on $\mathcal{H}$ as bounded *operators by left multiplication, whereas the derivations δ_i extend to densely defined selfadjoint operators on $\mathcal{H}$ with the smooth elements of the Hilbert space, $\mathcal{A}(\mathbb{T}^d_\theta)$, in their common domain.*

To construct a Dirac operator one usually restrict to the equivariant case [25] *postulating that the spectral triple has $U(1)^d$ as the global isometry group. The equivariant Dirac operator (which we can also call flat) is defined over $\mathcal{H} \otimes \mathbb{C}^r$, where $r = 2^{[\frac{n}{2}]}$, as:*

$$D = \sum_{i=1}^{n} \gamma_i \delta_i,$$

and γ^i are selfadjoint generators of the Clifford algebra:

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}.$$

3.3. Getting numbers out of spectral triples

Having a spectral triple we have very little information on its geometry apart from the data hidden in the properties of the Dirac operator. To recover this information we use the spectrum of D.

Let T be a compact positive operator on a separable Hilbert space such that for sufficiently large $r > 0$ the operator T^r is trace class. Therefore, the function:

$$\zeta_T(z) := \mathrm{Tr}\ |T|^z,$$

is well defined and holomorphic for $\Re(z) > r$. Taking the analytic continuation of $\zeta_T(z)$ to the rest of the complex plane we obtain a function, which has (possibly) some poles. We may then set for any $d \in \mathbb{R}$:

$$\tau(T) := \mathrm{Res}_{z=d} \zeta_T(z).$$

It appears that for a genuine Dirac-type or Laplace-type operator and most of the known operators for spectral triples the residue is nonzero only for some discrete subset of $\mathbb{R}$. In fact, if D is the Dirac operator on a spin manifold of dimension n then the function $\zeta_{|D|^{-1}}$ (if D has a kernel it is certainly finite dimensional and we can neglect it) may have only first order poles only at integers on the real axis not exceeding n and, in particular, has a nonzero residue at $z = n$ (which is proportional to the volume of the manifold). One usually shortens the notation writing ζ_D (meaning $\zeta_{|D|^{-1}}$).

Note, that the zeta function we may look at its poles which are located generally in a half of the complex plane and are not necessarily real. The collection of all points, which are the poles of the zeta function of the operator D from a given spectral triple we call *the dimension spectrum.* So, the dimension is not a number - it is a discrete set in a complex plane !

Remark 3.4. *The dimension spectrum of a compact spin manifold M, given by its spectral triple $(C^\infty(M), L^2(S), D)$ is contained in a set: $\{n, n-1, n-2, \ldots\}$ where n is the classical dimension of M. In fact, $z = n$ is always in the dimension spectrum, whereas not all of other the points of the set may belong to the dimension spectrum.*

Remark 3.5. *The dimension spectrum may contain complex numbers (with nonzero imaginary part) and any real numbers (for instance, if one considers fractals) see* [14]*, for an example.*

3.4. Families of Dirac operators

A single Dirac operator is an interesting object in itself but it corresponds exclusively (in the classical case) to one fixed metric and one chosen spin structure. However, once we have such Dirac operator for a given spectral triple, we might construct an entire family of them by taking all *inner fluctuations* of Dirac operators:

$$\mathsf{D}_A = \{D' : D' = D + A\},$$

where A is a self-adjoint one-form $A = \sum_i a_i[D, b_i]$ and $A = A^*$. Classically this corresponds to the twisting of the Dirac operator by a (trivial in this case) complex line bundle, or – using physics terminology – adding the $U(1)$ gauge field. A generalisation, which involves twisting by nontrivial line bundle is also possible.

Of course, one could ask a question whether the family we get depends on the starting point (that is whether the family is the same if we start with the Dirac already perturbed by a one-form) and it is very convenient that indeed the inner fluctuation of inner fluctuation are inner fluctuations so the family we obtain is not dependent on the initial choice. If we restrict ourselves to real spectral triples then there is a huge difference between the classical (commutative situation) when we have:

Lemma 3.6. *Commutative real spectral triples (Dirac-type operators over spin manifolds) admit no fluctuations of the type $A = \sum_i a_i[D, b_i]$, however, might admit higher-order fluctuations if their dimension $d > 2$.*

The proof is based on the relations from the definition 3.1 and the fact, that the real structure for commutative spectral triples over spin manifolds is related to complex conjugation: $JfJ^{-1} = f^*$. Therefore on one hand side, a real fluctuation of the Dirac operator must be:

$$D_A = D + A + \varepsilon JAJ^{-1},$$

as only then $JD_A = \varepsilon D_A J$. But since A is a selfadjoint one-form and the algebra is commutative:

$$Ja[D, b]J^{-1} = \varepsilon a^*[D, b^*] = -\varepsilon(a[D, b])^* = -\varepsilon a[D, b],$$

hence the fluctuation term identically vanishes. However, observe that this shall be different once the 'fluctuations" are allowed to be (more generally) higher-order forms - as then the nontrivial commutation between one-forms will be significant. In particular, in any odd dimensions one can "fluctuate" the Dirac operator of a commutative real spectral triple by a scalar term Φ:

$$D_\Phi = D + \Phi, \quad \Phi = \Phi^* \in \mathcal{A},$$

but only in dimension 3 this has a geometric interpretation of a torsion.

4. Conformally rescaled spectral triples

A completely different family of Dirac operators and spectral triples has been suggested recently for noncommutative tori [6]. While originally the proposed setup used twisted spectral triples, it has a natural formulation in the language of spectral triples. In fact, the rephrasing of the original construction in the language, which we present below fits amazingly well into the entire picture of spectral geometry.

Our starting point is a real spectral triple $(\mathcal{A}, D, \mathcal{H}, J)$ and a positive element $h > 0$, $h \in \mathcal{A}$.

Definition 4.1. *A conformally rescaled Dirac operator* $D_h = h^o D h^o$ *where* $h^o = JhJ^{-1}$ *defines a conformally rescaled spectral triple over* $\mathcal{A}$*:* $(\mathcal{A}, \mathcal{H}, D_h)$*.*

Note that the triple is not real. Below we verify that all crucial conditions are satisfied. First of all, since h^o is in the commutant of $\mathcal{A}$, for every $a \in \mathcal{A}$:

$$[D_h, \pi(a)] = h^o[D, \pi(a)]h^o \in B(\mathcal{H}).$$

Since h commutes with γ so does H^o and therefore if the spectral triple was even γ still provides the $\mathbb{Z}_2$ grading for the conformally rescaled triple. The cocycle condition is also satisfied. If $c = a_0 \otimes b_0 \otimes a_1 \otimes \cdots \otimes a_n = c$ is the desired cycle for D then $c_h = a_0 \otimes b_0 (h^o)^{-2n} \otimes a_1 \otimes \cdots \otimes a_n$ is good for D_h:

$$\begin{aligned}\pi_{D_h}(c_h) &= \pi(a_0)\left(J\pi(b_0)J^{-1}\right)\left(J\pi(h^{2n})J^{-1}\right)[D_h, \pi(a_1)]\cdots[D_h, \pi(a_n)] \\ &= \pi(a_0)\left(J\pi(b_0)J^{-1}\right)(h^o)^{-2n}(h^o[D, \pi(a_1)]h^o)\cdots(h^o[D, \pi(a_n)]h^o) \\ &= \pi_D(c).\end{aligned}$$

Furthermore let us compute the resolvent:

$$(D_h - \lambda)^{-1} = (h^o)^{-1}(D - \lambda(h^o)^{-2})^{-1}(h^o)^{-1}.$$

But $(h^o)^{-2}$ is also a positive bounded operator so the entire expression is compact for $\lambda = \pm i$, for instance (which is sufficient).

It is more complicated to check specific spectral properties of the conformally rescaled Dirac operator, in particular the dimension spectrum. One may only state the following:

Lemma 4.2. *Let* $(\mathcal{H}, A, D, J)$ *be a real spectral triple of metric dimension* n*, that is* $|D|^{-(n+\epsilon)}$ *is trace class for any* $\epsilon > 0$*. Then the conformally rescaled spectra triple* $(\mathcal{A}, D_h, \mathcal{H})$ *has the same metric dimension.*

The proof follows from a simple inequality between positive operators (we assumed that h^o is bounded positive and has a bounded inverse):

$$||(h^o)^{-1}||^{-2}|D| \leq |h^o D h^o| \leq ||h^o||^2 |D|.$$

Extending it to respective powers and taking trace we see that trace of $(|h^o D h^o|)^\alpha$ will be estimated by a multiple of trace of $|D|^\alpha$ from both sides.

4.1. The Fredholm module and K-homology class

A spectral triple is an object, which has a significant topological importance when considered as an unbounded Fredholm module. Let us recall the definition of a Fredholm module ofer an algebra $\mathcal{A}$:

Definition 4.3. *A triple $(\mathcal{A}, \mathcal{H}, F)$ is a Fredholm module iff $F = F^*$, $F^2 = 1$ on $\mathcal{H}$ and for every $a \in \mathcal{A}$ the commutator $[F, \pi(a)]$ is compact on $\mathcal{H}$. If there exists a grading $\gamma = \gamma*$ such that $\gamma^2 = 1$ and $F\gamma = -\gamma F$ on $\mathcal{H}$ then we have an even Fredholm module, otherwise we have an odd Fredholm module.*

A properly defined relation based on homotopy between Fredholm modules allows to introduce equivalence classes and show that these classes form an abelian group with respect to natural operations. These groups are, in a sense, corresponding dual objects to K-theory groups of the algebra $\mathcal{A}$: $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$. The Chern character (expressed easily for finitely summable Fredholm modules) provides the standard pairing between the K-theory and K-homology groups and factorizes through the classes in cyclic cohomology of the algebra $\mathcal{A}$.

An unbounded Fredholm module (a spectral triple) immediately gives a Fredholm module by an assignment $F = \text{sign}(D)$. Having constructed a family of conformally rescaled triples we might want to check how it affects the topological properties of the triple. Certainly the Fredholm module might not be the same, however what matters is its class in K-homology. We have:

Lemma 4.4. *The K-homology class of the Fredholm module obtained from the spectral triple of a conformally rescaled Dirac operator D_h does not depend on h.*

As $h > 0$ we define $s = \log h$ by continuous functional calculus. Then $h(t) = e^{ts}$ is a continuous path in $B(\mathcal{H})$ and $F_t = \text{sign}(h(t)^o D h(t)^o)$ will be a continuous path of operators giving us the homotopy between the $(\mathcal{A}, \mathcal{H}, \text{sign}(D))$ and $(\mathcal{A}, \mathcal{H}, \text{sign}(D_h))$.

4.2. The differential calculus

Assume that we have a real spectral triple and a conformally rescaled one. Sind D establishes the first order differential calculus we may ask a question whether the calculus defined by D_h is isomorphic to the original one.

Lemma 4.5. *Let $(\mathcal{A}, D, \mathcal{H}, J)$ be a real spectral triple and $D_h = h^o D h^o$ be a conformally rescaled Dirac operator. Then $\Omega^1_D(\mathcal{A}) \cong \Omega^1_{D_h}(\mathcal{A})$.*

We define for any one-form ω in $\Omega^1_D(\mathcal{A})$ the map ϕ_h:

$$\phi_h(\omega) = h^o \omega h^o.$$

Since h is invertible it is a bijective, and since h^o is in the commutant of $\mathcal{A}$ it clearly is a bimodule isomorphism. It remains to verify that:

$$\phi_h(da) = d_h a, \quad \forall a \in \mathcal{A},$$

where $d_h(a) = [D_h, \pi(a)]$. But:

$$\phi_h(da) = h^o[D, \pi(a)]h^o = [h^o D h^o, \pi(a)] = [D_h, \pi(a)] = d_h a.$$

4.3. Partial conformal rescaling

In special cases, where the Dirac operator can be presented as a sum of two (or more) operators, which alone satisfy most of the spectral triple conditions we can repeat the conformal rescaling but only partially. A typical example will be the case of the product of two spectral triples.

Let us assume that $(\mathcal{A}, D, \mathcal{H}, J)$ is a real spectral triple and $D = D_1 + D_2$ and D_1, D_2 are Dirac operators for real spectral triples for the largest subalgebras of $\mathcal{A}$, which is not annihilated (respectively) by them.

Then we can have for $h, k \in \mathcal{A}$ positive and such that inverses are bounded, $D_{h,k} = h^o D_1 h^o + k^0 D_2 k^o$. This is an operator, which has again bounded commutators and compact resolvent. Similar arguments as in the conformal case show that again the metric dimension does not change.

An example of partial conformal rescaling with h arbitrary positive and $k = 1$ was studied for the noncommutative torus in [11].

Remark 4.6. *Note that to obtain the isomorphisms between the respective bimodules of one-forms one needs some additional requirement that the bimodule of one-forms split as a direct sum of two bimodules.*

4.4. Fluctuations of conformally rescaled geometries

As a next problem we look into the fluctuations - of the type describe earlier but this time with the operator D_h. We have:

Lemma 4.7. *Fluctuations of the conformally rescaled Dirac operator are conformally rescaled fluctuation of the original Dirac operator.*

To prove it, let us take $D + A$, where $A = \sum_i \pi(a_i)[D, \pi(b_i)]$. Then:

$$h^o(D+A)h^o = D_h + h^o \left(\sum_i \pi(a_i)[D, \pi(b_i)] \right) h^0 = D_h + \sum_i \pi(a_i)[D_h, \pi(b_i)].$$

So, conformal rescaling does not change the family of possible fluctuactions.

5. The curvature and Einstein-Hilbert functional

One of the most interesting problems in all these examples appers to be the computation of some "local" geometric objects like scalar curvature. So far, the only approach that allowed to have an insight into such objects depends on the specific form of spectral triple for the noncommutative torus and explicit heat-trace computations using the generalized version of the pseudodifferential calculus for th noncommutative torus.

We shall review here some alternative approach, which is adapted to the case of dimension 4 (most interesting from the point of view of physical applications) and based on Wodzicki residue. We sketch the basic definitions and results below.

5.1. Wodzicki Residue on noncommutative tori.

In a series of papers first [6, 7] and [15, 16] studied a conformally rescaled metric for the noncommutative two and four-tori. This led to the expressions of Gauss-Bonnet theorem and formulae for the noncommutative counterpart of "dressed" scalar curvature.

The computations used explicitly the possibility to write the Laplace-type operators as pseudodifferential operators on the noncommutative torus with their symbol expansion and the possibility to construct a parametrix for a given elliptic operator.

As it has been shown [17] and more generally in [21] Wodzicki residue exists also in the case of the pseudodifferential calculus over noncommutative tori. This has been shown in full generality (in an explicit way, which follows directly from the classical situation) to the d-dimensional case [28]. The symbol calculus defined in [6] and developed further in [7] (see also [21]) is easily generalised to the d-dimensional case and to the operators defined above. Let us recall that a differential operator of order at most n is of the form

$$P = \sum_{0\leq k\leq n} \sum_{|\beta_k|=k} a_{\beta_k} \delta^{\beta_k},$$

where a_{β_k} are assumed to be in the algebra $\mathcal{A}(\mathbb{T}^d_\theta)$, $\beta_k \in \mathbb{Z}^d$ and:

$$|\beta_k| = \beta_1 + \cdots + \beta_d, \qquad \delta^{\beta_k} = \delta_1^{\beta_{k,1}} \cdots \delta_d^{\beta_{k,d}}.$$

Its symbol is:

$$\rho(P) = \sum_{0\leq k\leq n} \sum_{|\beta_k|=k} a_{\beta_k} \xi^{\beta_k},$$

where

$$\xi^\beta = \xi_1^{\beta_1} \cdots \xi_d^{\beta_d}.$$

On the other hand, let ρ be a symbol of order n, which is assumed to be a C^∞ function from $\mathbb{R}^d$ to $\mathcal{A}(\mathbb{T}^d_\theta)$, which is homogeneous of order n, satisfying certain bounds (see [6] for details). With every such symbol ρ there is associated an operator P_ρ on a dense subset of $\mathbb{H}$ spanned by elements $a \in \mathcal{A}(\mathbb{T}^d_\theta)$:

$$P_\rho(a) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-i\sigma\cdot\xi} \rho(\xi) \alpha_\sigma(a)\, d\sigma d\xi,$$

where

$$\alpha_\sigma(U^\alpha) = e^{i\sigma\cdot\alpha} U^\alpha, \quad \sigma \in \mathbb{R}^d, \alpha \in \mathbb{Z}^d.$$

For two operators P, Q with symbols:

$$\rho(P) = \sum p_\alpha \xi^\alpha, \quad \rho(Q) = \sum q_\beta \xi^\beta,$$

we use the formula, which follows directly from the same computations as in the case of classical calculus of pseudodifferential operators:

$$\rho(PQ) = \sum_{\gamma\in\mathbb{N}^d} \frac{1}{\gamma!} \partial_\xi^\gamma (\rho(P)) \delta^\gamma (\rho(Q)), \tag{1}$$

where $\gamma! = \gamma_1! \cdots \gamma_d!$. In [28] we have shown that

Proposition 5.1. *Let* $\rho = \sum_{j \leq k} \rho_j(\xi)$ *be a symbol over the noncommutative torus* $\mathcal{A}(\mathbb{T}^d_\theta)$. *Then the functional:*

$$\rho \mapsto Wres(\rho) = \int_{S^{d-1}} \mathfrak{t}\left(\rho_{-d}(\xi)\right) d\xi,$$

is a trace over the algebra of symbols.

Then for family of conformally rescaled Laplace-type operators we have computed the following functional:

$$S(h) = \Lambda \, \mathrm{Wres}(D_h^{-4}) + \mathrm{Wres}(D_h^{-2}).$$

and demonstrated that it is not a minimal operator. That signifies that there is no single operator, which minimizes for a fixed h the second term (Einstein-Hilbert functional). Classically the minimal point corresponds to the Laplace operator obtained from the Levi-Civita (torsion free) connection.

5.2. Einstein-Hilbert functional for conformally rescaled Dirac in 4D.

Let $h \in J\mathcal{A}(\mathbb{T}^4_\theta)J$, $h > 0$ from the commutatnt of the algebra. We know that for the standard Dirac operator D the conformally rescaled Dirac $D_h = h^{-1} D h^{-1}$ defines a spectral triple with the same metric dimension. For simplicity we denote by h already the element from the commutant. Fixing the dimension of the NC torus to be $d = 4$ and using the above defined calculus of symbols of pseudodifferential operators on the NC Torus we obtain:

Lemma 5.2. *The action functional for the conformally rescaled Dirac over* 4*-dimensional noncommutative torus is*

$$S(h) = \Lambda \, \mathfrak{t}(h^8) + \mathfrak{t}\left(h\delta_i(h)\delta_i(h)h + h\delta_i(h)h\delta_i(h)\right).$$

The proof is a straightforward but tedious computation, which is a part of computation made in [28]. It is interesting to compare it with the classical result. Since the Dirac operator is rescaled by h^{-1} from both sides in the commutative case this means that its principal symbol is rescaled by h^{-2} and the metric rescaled by h^4. The curvature scalar of such metric is:

$$R(h) = -12h^{-6}\left((\partial_i h)(\partial_i h) + h(\Delta h)\right),$$

whereas the volume form is h^8. It is easy to see that if h and its derivations commute with each other the result is the classical one, as:

$$h^3(\Delta h) = \partial_a(h^3 \partial_a h) - 3h^2(\partial_a h)(\partial_a h),$$

and since on the torus the integral is a closed trace with respect to derivations, we have:

$$\int_{\mathbb{T}^4} \sqrt{g} R = 24h^2(\partial_i h)(\partial_i h).$$

Hence we might consider the operator D_h as truly the correct Dirac for a conformally rescaled noncommutative geometry.

5.3. Derivations and moving frame formalism

Apart from the classical limit there is also another possibility to check whether the above result makes sense. In [24] Rosenberg observed that conformal rescaling of the metric could be translated into the rescaling of derivations, since on can write the conformally rescaled metric tensor in the basis of dual space to derivations (forms) as:

$$g_h = \eta_{ab}(ke^a) \otimes (ke^b).$$

In his paper he studies the geometry and curvature tensors following standard recipe, which can be naturally adapted to this case. Reformulating slightly his approach and using the spin connection rather than Levi-Civita connection one can repeat the computations in arbitrary dimensions.

We introducing, similarly as in the classical case, the spin connection:

$$\omega^a_b = \omega^a_{bc}(ke^c).$$

Assuming metric compatibility and vanishing of the torsion:

$$0 = d(ke^a) + \omega^a_b(ke^b) = (\delta_i k)e^i e^a + \omega^a_{bc} k^2 e^c e^b.$$

we obtain the solution,

$$\omega^a_{bc} = \delta^a_c \delta_b(k)k^{-2}.$$

As the difference from the classical situation is only in the order of terms (as they do not commute with each other) one can easily compute the two-form of the curvature tensor:

$$R^a_b = \delta^a_c \delta_{br}(k)k^{-1}e^r e^c + \delta^a_c \delta_b(k)\delta_r(k^{-1})e^r e^c + \delta^a_c \delta_p(k)k^{-1}\delta_b(k)k^{-1}e^c e^p,$$

and its contraction to the Ricci tensor:

$$\mathrm{Ric}_{bc} = -\delta_{bc}(k)h^{-1} - \delta_b(k)\delta_c(k^{-1}) + \delta_c(k)k^{-1}\delta_b(k)k^{-1}.$$

Finally, one obtains a "naive" expression for the scalar curvature:

$$r = k^{-2}\left(-\delta_{aa}(k)k^{-1} + 2\delta_a(k)k^{-1}\delta_a(k)k^{-1}\right).$$

We call the expression "naive generalization" of the classical scalar curvature as we multiply the Ricci tensor by the conformal factor from the left when contracting it with the metric. We could have done it symmetrically or we could have multiplied from the right. Since later we compute the functional, which involves the trace, this does not play any significant role. On the other hand we always need to remember that this is not a curvature in the classical sense.

To compare with the result for the Dirac operator we need to set $k = h^{-2}$, then,

$$\delta_a(k) = -h^{-1}\delta_a(h)h^{-2} - h^{-2}\delta_a(h)h^{-1}$$

and

$$\begin{aligned}\delta_{aa}(k) = & - h^{-1}\delta_{aa}(h)h^{-2} - h^{-2}\delta_{aa}(h)h^{-1} \\ & + h^{-1}\delta_a(h)h^{-1}\delta_a(h)h^{-2} + h^{-2}\delta_a(h)h^{-1}\delta_a(h)h^{-1} \\ & + h^{-1}\delta_a(h)\left(h^{-1}\delta_a(h)h^{-2} + h^{-2}\delta_a(h)h^{-1}\right) \\ & + \left(h^{-1}\delta_a(h)h^{-2} + h^{-2}\delta_a(h)h^{-1}\right)\delta_a(h)h^{-1} \\ = & - h^{-1}\delta_{aa}(h)h^{-2} - h^{-2}\delta_{aa}(h)h^{-1} + 2\left(h^{-1}\delta_a(h)h^{-1}\delta_a(h)h^{-2}\right. \\ & \left. + h^{-1}\delta_a(h)h^{-2}\delta_a(h)h^{-1} + h^{-2}\delta_a(h)h^{-1}\delta_a(h)h^{-1}\right).\end{aligned}$$

Therefore, in the end we have:

$$r(h) = 2h^{-6}\delta_a(h)\delta_a(h) + h^{-6}\delta_{aa}(h)h + h^{-5}\delta_{aa}(h).$$

Finally, we can compute the Einstein-Hilbert functional,

$$\mathfrak{t}(h^8 r) = -2\mathfrak{t}\left(h^2\delta_a(h)\delta_a(h) + h\delta_a(h)h\delta_a(h)\right).$$

where we have used the cyclicity and closedeness of the trace:

$$\mathfrak{t}\left(h^3\delta_{aa}(h)\right) = \mathfrak{t}\left(\delta_a\left(h^3\delta_a(h)\right) - 2h^2\delta_a(h)\delta_a(h) - h\delta_a(h)h\delta_a(h)\right).$$

It is surprising that (up to trivial rescaling) we obtain the same formula as this arising from the Wodzicki residue of the Dirac operator.

6. Conclusions

Noncommutative geometry is a sound and well-motivated theory, which can provide excellent tools to study and describe the geometry of the world. At its current stage, it still is focusing on some simple examples. The presented class of conformally rescaled spectral triples is one of the first steps to go beyond single Dirac operator or fluctuations of Dirac operator and study geometries in a similar manner as we study classical manifold.

As we have mentioned there are different approaches, which should merge to provide a comprehensive picture of the geometries we study. First, we have purely algebraic approach, when we could work with algebraic objects (at least in some cases) like symmetries (also in the Hopf algebra sense), derivations or twisted derivations, differential calculi etc. The Second approach is based purely on spectral properties of the Dirac and computation of some geometric quantities using heat-trace expansion or natural traces like Wodzicki residue on an appropriate algebra of (generalized) pseudodifferential operators.

Surprisingly, as shown in the above paper, there might be a link, even in the noncommutative case between these two approaches. It is worth mentioning that a spectacular link between the notion of connection for noncommutative $U(1)$ principal bundles and a new families of Dirac operators was established by the author and L.Dabrowski in [10]. All these examples and further might be a sound basis for a better understanding of geometry of quantum spaces.

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