Braided Hopf algebra from twisting

Andrzej Sitarz
(joint work with Arkadiusz Bochniak)

Institute of Physics
Jagiellonian University
Kraków

Institute of Mathematics,
Polish Academy of Sciences
Warsaw

April 22, 2018

Motivation: $SU_q(2), q \in \mathbb{R}$

Let q be a real number with $0 < q < 1$, and let $\mathcal{A} = \mathcal{A}(SU_q(2))$ be the $*$-algebra generated by a and b, subject to the following commutation rules:

$$ab = qba, \qquad ab^* = qb^*a, \qquad bb^* = b^*b,$$
$$a^*a + b^*b = 1, \qquad aa^* + q^2b^*b = 1.$$

As a consequence, $a^*b = q^{-1}ba^*$ and $a^*b^* = q^{-1}b^*a^*$. This becomes a Hopf $*$-algebra under the coproduct

$$\Delta a := a \otimes a - q\, b^* \otimes b,$$
$$\Delta b := b \otimes a + a^* \otimes b,$$

counit $\epsilon(a) = 1$, $\epsilon(b) = 0$, and the antipode

$$Sa = a^*, \quad Sb = -qb, \quad Sb^* = -q^{-1}b^*, \quad Sa^* = a.$$

Motivation: $SU_q(2), q \in \mathbb{C}$

Let q be a complex number with $0 < |q| < 1$, and let $\mathcal{A} = \mathcal{A}(SU_q(2))$ be the $*$-algebra generated by a and b, subject to the following commutation rules:

$$ab = qba, \qquad ab^* = qb^*a, \qquad bb^* = b^*b,$$
$$a^*a + b^*b = 1, \qquad aa^* + |q|^2 b^*b = 1.$$

As a consequence, $a^*b = (q^*)^{-1} ba^*$ and $a^*b^* = (q^*)^{-1} b^*a^*$. This is **NOT** a Hopf $*$-algebra but with the coproduct:

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it is a *braided Hopf algebra* [Kasprzak,Meyer,Roy,Woronowicz]

Geometry of $SU_q(2)$

Whad do we know about the geometry (differential calculi, spectral triples etc) of these objects ?

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3. cocycle twist of $A(S^3)$ [S,V]

Hopf algebra in braided monoidal category

Let H be an algebra in a monoidal category that is equipped with a *braiding* $\Psi : H \otimes H \to H \otimes H$. The trivial braiding is $\Psi(x \otimes y) = y \otimes x$.

Twisting Hopf algebras

Let H be a Hopf algebra with the coproduct Δ, the counit ϵ and the antipode S. Let $\chi : H \to \mathbb{C}[z, z^{-1}]$ be a Hopf algebra homomorphism, where $\mathbb{C}[z, z^{-1}]$ is the Hopf algebra of the group algebra of $\mathbb{Z}$.

Lemma

The following maps define left and right coactions of $\mathbb{C}[z, z^{-1}]$ on H,

$$\Delta_L = (\chi \otimes \mathrm{id})\Delta, \qquad \Delta_R = (\mathrm{id} \otimes \chi)\Delta,$$

which are equivariant under the coproduct in H:

$$(\mathrm{id} \otimes \Delta)\Delta_L(x) = (\Delta_L \otimes \mathrm{id})\Delta(x).$$

Definition

We say that $x \in H$ is a homogeneous element of degrees $\mu(x), \nu(x)$, where $\mu(x), \nu(x) \in \mathbb{Z}$ if:

$$\Delta_L(x) = z^{\mu(x)} \otimes x, \qquad \Delta_R(x) = x \otimes z^{\nu(x)}.$$

We also define the index

$$\delta(x) = \mu(x) - \nu(x).$$

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Lemma

We have the following identities for any $x \in H$ and $\Delta x = \sum_i x^i_{(1)} \otimes x^i_{(2)}$, where $x^i_{(1)}$ and $x^i_{(2)}$ are homogeneous:

$$\mu(x) = \mu(x^i_{(1)}), \qquad \nu(x) = \nu(x^i_{(2)}), \qquad \text{for any } i,$$

and

$$\mu(x^i_{(2)}) = \nu(x^i_{(1)}), \qquad \text{for any } i.$$

Braiding with bigrading

We take H as a vector space and define for x, y bihomogeneous elements:

$$x * y = e^{i\phi(\mu(x)\nu(y)-\nu(x)\mu(y))}\, x \cdot y$$

$$\Delta_\phi(x) = \sum_j e^{i\phi\delta(x^j_{(1)})\delta(x^j_{(2)})} x^j_{(1)} \otimes x^j_{(2)},$$

$$\Psi\,(x \otimes y) = e^{2i\phi\delta(x)\delta(y)}\,(y \otimes x)\,.$$

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Proposition

H with the product $*_\phi$, coproduct Δ_ϕ, braiding Ψ and the antipode $S_\phi(x) = e^{i\phi\delta(x)^2} S(x)$ is a braided Hopf algebra.

Examples

The algebra $C_\lambda(a,b)$

Consider a unital Hopf algebra generated by a, a^{-1}, b with relation $a\,b = \lambda b\,a$, and coproduct:

$$\Delta(a) = a \otimes a,\ \Delta(b) = a \otimes b + b \otimes 1.$$

We take the map $\chi : C_\lambda(a,b) \to \mathbb{C}[z, z^{-1}]$ as $\chi(a) = z$ and $\chi(b) = 0$, which is a morphism of Hopf algebras.
The bigrading is

$$\mu(a) = \nu(a) = \mu(b) = 1,\ \nu(b) = 0.$$

The twisting procedure gives $a * b = e^{-2i\phi}\lambda b * a$, with the same coproduct, the antipode $S_\phi(a) = S(a)$, $S_\phi(b) = e^{i\phi}S(b)$ and the only nontrivial braiding $\Psi(b,b) = e^{2i\phi} b \otimes b$.

Quantum double torus

Let $\mathcal{A} = C(\mathbb{T}^2) \oplus C(\mathbb{T}_q^2)$ with generators u, v and U, V of $C(\mathbb{T}^2)$ and $C(\mathbb{T}_q^2)$, respectively.

$$\Delta(u) = u \otimes u + V \otimes U, \qquad \Delta(v) = v \otimes v + U \otimes V,$$
$$\Delta(U) = U \otimes u + v \otimes U, \qquad \Delta(V) = V \otimes v + u \otimes V,$$

Define $\chi : \mathcal{A} \to \mathbb{C}[z, z^{-1}]$ by $\chi(u) = z$, $\chi(v) = z^{-1}$, $\chi(U) = \chi(V) = 0$. Then, using the twisting procedure the algebra structure does not change while the coproduct,

$$\Delta_\phi(u) = u \otimes u + e^{-4i\phi} V \otimes U,$$
$$\Delta_\phi(v) = v \otimes v + e^{-4i\phi} U \otimes V,$$

and the braiding change.

Quantum $SU_q(n)$

The $\mathcal{A}(SU_q(n))$ algebra is generated by a unitary matrix elements $u_{ij}, 1 \leq i, j \leq n$, with relations:

$$u_{ik}u_{jk} = qu_{jk}u_{ik}, (i < j), \qquad u_{ki}u_{kj} = qu_{kj}u_{ki}, (i < j),$$

$$u_{il}u_{jk} = u_{jk}u_{il}, (i < j; k < l), \qquad u_{ik}u_{jl} - u_{jl}u_{ik} = (q - q^{-1})u_{jk}u_{il}, (i < j; k < l),$$

$$\sum_{\sigma}(-q)^{|\sigma|}u_{1\sigma(1)} \cdots u_{n\sigma(n)} = 1,$$

Let $p : \{1, 2, \ldots, n\} \to \mathbb{Z}$ be a function such that $\sum_{k=1}^{n} p(k) = 0$.

Lemma

The following map:

$$\chi_p : u_{ij} \mapsto z^{p(i)}\delta_{ij},$$

is a $*$-algebra homomorphism from $\mathcal{A}(SU_q(n))$ to $\mathbb{C}[z, z^{-1}]$.

The $SU_q(2)$ case

For $\mathcal{A}(SU_q(2))$ we have:

$$\delta(a) = 0 = \delta(a^*), \qquad \delta(b^*) = 2 = -\delta(b).$$

The product and coproduct for $\mathcal{A}_\phi(SU_q(2))$:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \qquad \alpha\alpha^* + q^2\gamma^*\gamma = 1, \qquad \gamma\gamma^* = \gamma^*\gamma$$

$$\alpha\gamma = qe^{4i\phi}\gamma\alpha, \qquad \alpha\gamma^* = qe^{-4i\phi}\gamma^*\alpha.$$

$$\Delta_\phi(\alpha) = \alpha \otimes \alpha - qe^{-4i\phi}\gamma^* \otimes \gamma \qquad \Delta_\phi(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

In the $\mathcal{A}(SU_q(2))$ case the only nontrivial braiding phase factors ψ between the generators α, γ are:

$$\psi(\gamma,\gamma) = e^{8i\phi} = \psi(\gamma^*,\gamma^*), \qquad \psi(\gamma^*,\gamma) = e^{-8i\phi} = \psi(\gamma,\gamma^*).$$

The antipode $S_\phi(\alpha) = \alpha^*, S_\phi(\gamma) = -qe^{4i\phi}\gamma$.

The differential calculi

Recall that a left action of a (braided) Hopf algebra H on vector space Γ is a linear map $\triangleright : H \otimes \Gamma \to \Gamma$ s.th. $(hg) \triangleright x = h \triangleright (g \triangleright x)$ and $1 \triangleright x = x$ for all $g, h \in H, x \in \Gamma$. Similarly we define right action. Then we say that Γ is a left (resp. right) H-module.

A left coaction of H on Γ is a linear map $\delta_L : \Gamma \to H \otimes \Gamma$ satisfying

$$(\Delta \otimes \mathrm{id}) \circ \delta_L = (\mathrm{id} \otimes \delta_L) \circ \delta_L,$$

and

$$(\epsilon \otimes \mathrm{id})\, \delta_L = \mathrm{id}.$$

If Γ is both left and right H-module, with the actions that are commutative with each other:

$$(h \triangleright x) \triangleleft g = h \triangleright (x \triangleleft g),$$

then we call H-bimodule. Simiralry we define H-bicomodule.

Bicovariant differential calculi

Definiton

We say that Γ is a left covariant bimodule if it is a left H-bimodule which is a left comodule over H with coaction δ_L such that for any $a, b \in H,\ \rho \in \Gamma$ the following are satisfied

$$\delta_L(a\,\rho) = \Delta(a)\delta_L(\rho), \qquad \delta_L(\rho\,a) = \delta_L(\rho)\Delta(a).$$

Remark

Note that this requires *braiding* on the algebra as well as braiding between Γ and H:

$$\delta_L(\rho)\Delta(a) = (m \otimes \triangleleft)(\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\delta_L \otimes \Delta)(\rho \otimes a).$$

Bicovariant differential calculi

Definition

We say that Γ is a bicovariant bimodule or Hopf bimodule over H if it is both H-bicomodule, left and right covariant bimodule.

Definition

We say that the first order differential calculus (Γ, d) over a braided Hopf algebra H is *braided left covariant* if Γ is a Hopf bimodule over H and

$$\delta_L(dh) = (\mathrm{id} \otimes d)\Delta(h), \quad \forall h \in H.$$

Aim: classify all bicovariant differential calculi over $SU_q(2)$.

Braided bicovariant calculi over twisted braided Hopf algebras

The universal calculus

The first order universal differential calculus is *braided bicovariant* for H_ϕ.

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Twisting of modules and comodules

If Γ is a left (right) module which is bigraded then $\Gamma_\phi = \Gamma$ as a vector space, the following defines a left and right module structure on Γ_ϕ over H_ϕ:

$$x * \omega = e^{i\phi(\mu(x)\hat{\nu}(\omega)-\nu(x)\hat{\mu}(\omega))}\, x\,\omega,$$

$$\omega * x = e^{i\phi(\hat{\mu}(\omega))\nu(x)-\hat{\nu}(\omega)\mu(x)}\, \omega\, x.$$

Braided bicovariant calculi over twisted braided Hopf algebras

Lemma

The following defines the left (right) coaction of H_ϕ on Γ_ϕ on homogeneous elements as:

$$\delta_{L^\phi}(\omega) = e^{i\phi\delta(\omega_{(-1)})\hat{\delta}(\omega_{(0)})}\omega_{(-1)} \otimes \omega_{(0)},$$

and

$$\delta_{R^\phi}(\omega) = e^{i\phi\hat{\delta}(\omega_{(0)})\delta(\omega_{(1)})}\omega_{(0)} \otimes \omega_{(1)}.$$

with the braiding between the bimodule Γ_ϕ and H_ϕ defined in the same way:

$$\Psi(\omega \otimes x) = e^{2i\phi\hat{\delta}(\omega)\delta(x)} x \otimes \omega,$$
$$\Psi(x \otimes \omega) = e^{2i\phi\hat{\delta}(\omega)\delta(x)} \omega \otimes x$$

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Theorem

With the above definitions (Γ_ϕ, d) is a braided bicovariant differential calculus.

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Corrolary

If we have a Hopf algebra H which can be twisted to H_ϕ then if Γ, d is a left (right,bi)-covariant first order differential calculus then (Γ_ϕ, d) is a braided bicovariant differential calculus.

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Corrolary

If (Γ, d) is a first order differential calculus over $SU_q(2)$ braided Hopf algebra (q complex) then using *untwisting* we find a bicovariant differential calculus over $SU_{|q|}(2)$ Hopf algebra.

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