FUNDAMENTAL FERMIONS AS NONCOMMUTATIVE DIFFERENTIAL FORMS

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(joint work with Francesco d'Andrea and Ludwik Dąbrowski)

1. Introduction: the idea of spectral triples
2. The finite spectral triples: the basics
3. The spinors
4. The finite spectral triple of the SM
5. The Hodge condition
6. Conclusions

Geometry through spectral triples

Definition: the spectral triple

Algebra $\mathcal{A}$, its faithful representation π on a Hilbert space $\mathcal{H}$, a selfadjoint operator D, satisfying several conditions:

1. $\forall a \in \mathcal{A}\ [D, \pi(a)] \in B(\mathcal{H})$, D^{-1} is compact

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3. $\exists\, J$, antilinear $J^2 = \pm 1, JJ^\dagger = 1$
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5. ...+ conditions of „analysis” type

Theorem [Connes]

If $\mathcal{A} = C^\infty(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

EXAMPLES OF SPECTRAL GEOMETRIES

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  Dirac operator the same as on the torus [Connes]

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The application to physics ?

So far: *the finite spectral triple of the Standard Model*

The general structure

Let (A, H, D) be a finite-dimensional spectral triple, J an antilinear isometry and call $A_{\mathbb{C}}$ the complex $*$-subalgebra of $\mathrm{End}_{\mathbb{C}}(H)$ generated by A (so $A = A_{\mathbb{C}}$ if A is already complex). Then:

$$A_{\mathbb{C}} \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) \; .$$

If P_i is the unit of the summand $M_{n_i}(\mathbb{C})$, then $P_1, \ldots, P_N$ are (represented by) orthogonal projections on H whose sum is 1. The operators $Q_i := JP_iJ^{-1}$ form a set of orthogonal projections as well, commuting with the projections P_i's, and whose sum is also 1.

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THE DIRAC OPERATOR:

$$D_{ij,kl} := P_i Q_j D P_k Q_l \; ,$$

The general structure of Dirac operator

Lemma

We can decompose the Dirac operator into four pieces

$$D = D_0 + D_1 + D_2 + D_R \ ,$$

$$D_0 := \sum_{i,j,k\,:\,i\neq k} D_{ij,kj} \ , \qquad D_1 := \sum_{i,j,l\,:\,j\neq l} D_{ij,il} \ ,$$

$$D_2 := \sum_{\substack{i,j,l,k \\ i\neq k, j\neq l}} D_{ij,kl} \ , \qquad D_R := \sum_{i,j} D_{ij,ij} \ .$$

so that:

$$JD_0J^{-1} = \varepsilon' \, D_1 \ , \qquad JD_2J^{-1} = \varepsilon' \, D_2 \ , \qquad JD_RJ^{-1} = \varepsilon' \, D_R \ .$$

The general structure of Dirac operator

Lemma (Dirac and differential forms)

$D_0 + D_2 \in \Omega^1_D(A)$.

Proof.

An explicit computation gives
$D_0 + D_2 = \sum_{i \neq k} P_i D P_k = \sum_{i \neq k} P_i [D, P_k]$, where we used the fact that $P_i P_k = 0$ for $i \neq k$. □

Lemma (1st order)

D satisfies the 1st order condition if and only if: $D_2 = 0$, $D_1 \in A'$, and D_R satisfies the 1st order condition.

Is that Hilbert space same as spinors ?

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Morita equivalence

Recall what is Morita equivalence...

Are spectral triples **ONLY** based on spinors ?

The Hodge-Dirac operator

Take a Riemannian manifold M, the space of differential forms $\Omega^*(M)$ and then consider: $\left(C^\infty(M), d + \delta, L^2(\Omega^*(M))\right)$.
(*Exercise:* with additional grading and reality !)

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We obtain a spectral triple, which satisfies all the conditions for a spectral geometry (and indeed it could be used for index computations for instance).
So, what is the difference ?

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- First of all, it may be *reducible*

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The difference

- First of all, it may be *reducible*
- The "spinors" are a bimodule over the Clifford algebra

The finite spectral triple of the SM

The SM Hilbert space

We arrange particles in a 4×4 matrix in the following way:

$$F = \begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix} .$$

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The grading and the real structure

The real structure is given by

$$J \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} w^* \\ v^* \end{bmatrix}$$

The grading γ on F is the operator of left multiplication by the diagonal matrix $\mathrm{diag}(+1, +1, -1, -1)$.

The finite spectral triple of the SM

The SM algebra and its representation

We identify $\mathrm{End}_{\mathbb{C}}(H)$ with the algebra $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$, represented on H as follows:

$$\pi(\alpha \otimes 1 \otimes \beta) \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} \alpha v \beta^t \\ \alpha w \beta^t \end{bmatrix}, \quad \pi\left(1 \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes 1\right) \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} av + bw \\ cv + dw \end{bmatrix}$$

for all $\alpha, \beta, v, w \in M_4(\mathbb{C})$, $a, b, c, d \in \mathbb{C}$.

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The SM algebra and its representation

The algebra $A \simeq \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ has elements

$$\left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{q}} \\ 0 & 0 & & \end{array}\right] \otimes e_{11} \otimes 1 + \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array}\right] \otimes e_{22} \otimes 1$$

with $\lambda \in \mathbb{C}$, $q \in \mathbb{Q}$ a quaternion and $m \in M_3(\mathbb{C})$.

What are possible Dirac operators ?

Are there true "spinorial" spectral triples ?

To rephrase the question: are there Dirac operators so that the Hilbert space is equivalence bimodule between the complexified algebra of the Standard Model and Clifford algebra ?

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Lemma (FdA & LD)

No Dirac operators for the Hilbert space of the SM can possibly guarantee the Morita equivalence condition.

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Is there a way out ?

The *next best* idea: the Hodge-Dirac operator and Hodge condition.

The Hodge condition - the basics

The necessary condition

We say that a real spectral triple (A, H, D, J) satisfies the second-order condition if

$$\textit{(i)} \quad \mathcal{C}\ell_D(A) \quad \text{and} \quad \mathcal{C}\ell_D(A)^\circ \quad \text{commute.}$$

We say that the Hodge condition holds if,

$$\textit{(ii)} \quad \mathcal{C}\ell_D(A)' = \mathcal{C}\ell_D(A)^\circ.$$

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Theorem

There are four classes of Dirac operators for the spectral triple of the Standard model so that the Hodge condition is satisfied.

What are the possible Hodge-duality SM triples?

The Dirac operators

$$D_0 = \left[\begin{array}{cc|cc} & & \alpha_{13} & \alpha_{14} \\ & & \alpha_{23} & \alpha_{24} \\ \hline \overline{\alpha}_{13} & \overline{\alpha}_{23} & & \\ \overline{\alpha}_{14} & \overline{\alpha}_{24} & & \end{array}\right] \otimes e_{11} \otimes e_{11} +$$

$$+ \left[\begin{array}{cc|cc} & & \beta_{13} & \beta_{14} \\ & & \beta_{23} & \beta_{24} \\ \hline \overline{\beta}_{13} & \overline{\beta}_{23} & & \\ \overline{\beta}_{14} & \overline{\beta}_{24} & & \end{array}\right] \otimes e_{11} \otimes (1 - e_{11})$$

$$+ \left[\begin{array}{cc|cc} & \delta_{12} & \delta_{13} & \delta_{14} \\ \delta_{21} & \delta_{22} & \delta_{23} & \delta_{24} \\ \hline & & & \\ & & & \end{array}\right] \otimes e_{12} \otimes e_{11} + \left[\begin{array}{cc|cc} & \overline{\delta}_{21} & & \\ \overline{\delta}_{12} & \overline{\delta}_{22} & & \\ \hline \overline{\delta}_{13} & \overline{\delta}_{23} & & \\ \overline{\delta}_{14} & \overline{\delta}_{24} & & \end{array}\right] \otimes e_{21} \otimes e_{11}$$

where $\alpha_{ij}, \beta_{ij}, \delta_{ij} \in \mathbb{C}$ and zeroes are omitted.

First case

Theorem

Let all δ vanish. Then the Hodge duality is satisfied if and only if: (i) the matrices

$$\alpha := \begin{bmatrix} \alpha_{13} & \alpha_{14} \\ \alpha_{23} & \alpha_{24} \end{bmatrix} \qquad \beta := \begin{bmatrix} \beta_{13} & \beta_{14} \\ \beta_{23} & \beta_{24} \end{bmatrix}$$

have no zero rows and (ii) there are no $\phi, \psi \in \mathbb{R}$ such that

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The Clifford algebra

The complex algebra generated by A and D_0 is:
$B' \simeq M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \simeq B$ and we have $B^\circ = B'$.

SECOND CASE

THEOREM

Let D_0 as above with $\delta_{21} = 0$ and $\beta_{13} = \beta_{14} = 0$ the Hodge duality is satisfied if and only if $(\beta_{23}, \beta_{24}) \neq (0, 0)$ and of the four vectors

$$(\alpha_{13}, \alpha_{14}) , \qquad (\alpha_{23}, \alpha_{24}) , \qquad (\delta_{12}, \delta_{13}, \delta_{14}) , \qquad (\delta_{22}, \delta_{23}, \delta_{24}) ,$$

at least three are not zero.

Second case

Theorem

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The Clofford algebra

The complex algebra generated by A and D_0 is:
$B' \simeq M_7(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus \mathbb{C} \simeq B$, and so $B^\circ = B'$.

The other two cases

The symmetry

Let us also define:

$$U := 1 \otimes 1 \otimes 1 + e_{11} \otimes (e_{12} + e_{21} - 1) \otimes e_{11} \ .$$

This is a permutation matrix: $U = U^*$ and $U^2 = 1$ (hence a unitary). It's action on the Hilbert space is to exchange the basis vectors ν_R and $J(\nu_R)$.

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Lemma

U commutes with A and J.

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Lemma

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The symmetry applied

Since the symmetry does not change anything in the spectral triple apart from the Dirac operator, if D_0 is such that Hidge duality holds, same is true for UD_0U. This is how the other two cases are recovered.

THE PROOFD

A STRAIGHTFORWARD PROOF

Compute explicitly the commutant and check whether it is exactly $\mathcal{C}\ell_D(A)^o$.

The proofD

A straightforward proof

Compute explicitly the commutant and check whether it is exactly $\mathcal{C}\ell_D(A)^o$.

A lazy proof

Use Burnside's theorem.

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THEOREM

Burnside 1905 A set of matrices generates a full matrix algebra iff they have no common invariant subspace.

How to apply Burnside's theorem ?

Example - case 1

The algebra is:

$$A_{SM} = (\mathbb{C}_1 \oplus \mathbb{C}_2 \oplus M_2(\mathbb{C}))^{(4)} \oplus (\mathbb{C}_1 \oplus M_3(\mathbb{C}))^{(4)}$$

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The algebra generated by A_{SM} and D_0 is a subalgebra of A_H:

$$M_4(\mathbb{C}) \oplus (M_4(\mathbb{C}))^{(3)} \oplus (\mathbb{C}_1 \oplus M_3(\mathbb{C}))^{(4)}.$$

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Then we verify that its commutant is the same (using basic algebra)
Then we check conditions under which D_0 and A_{SM} generate this algebra (using Burnside's theorem)

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Still, a Dirac operator that allows for $SU(3)$ breaking and leptoquarks is possible *see: Paschke, M., F. Scheck, and A. Sitarz (1999). Can (noncommutative) geometry accommodate leptoquarks? Phys. Rev. D59 035003*)

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More good news:

There are no relations between masses *but* there should be no degeneracies.

Conclusions

Good news:

It appears that indeed, SM spectral triple is of Hodge-Dirac type (that is, the Hodge duality is satisfied)

Bad news:

Still, a Dirac operator that allows for $SU(3)$ breaking and leptoquarks is possible *see: Paschke, M., F. Scheck, and A. Sitarz (1999). Can (noncommutative) geometry accommodate leptoquarks? Phys. Rev. D59 035003*)

More good news:

There are no relations between masses *but* there should be no degeneracies.

More bad news:

Possibly introduction of families does not keep Hodge duality.

Outlook

Next

There are variations of that duality to be checked, for example: *Twisted Reality Condition for Dirac Operators, Brzeziński, Ciccoli, Dąbrowski, Sitarz. Math Phys Anal Geom (2016) 19: 16.*

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Reference

The Standard Model in noncommutative geometry: fundamental fermions as internal forms
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THANK YOU !