Hopf algebroids over quantum projective spaces

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Noncommutative geometry: metric and spectral aspects

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Hopf-Galois extensions are the (noncommutative) algebraic version of principal bundles. In the classical theory one associates to a them a groupoid known as the gauge groupoid, which allows one to characterize connections.
Its algebraic counterpart is known as the Ehresmann-Schauenburg bialgebroid and recently has been proven to be a Hopf algebroid (in the sense of Schauenburg). It is not clear yet if this Hopf algebroid is always full, i.e. if it posses an antipode map. In my project I am studying conditions under which this map exists and also the concept of twist of an antipode.

Recall that a **bialgebra** over a field $\mathbb{K}$ is the datum of an algebra (H, m, ν) together with two maps $\Delta : H \otimes H \longrightarrow H$ and $\epsilon : H \longrightarrow \mathbb{K}$ satisfying

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \quad (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} = (\epsilon \otimes \mathrm{id}) \circ \Delta$$

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called **coproduct** and **counit** such that are algebra morphism.
A **Hopf algebra** is a bialgebra H as above endowed with a linear endomorphism $S : H \longrightarrow H$ satisfying

$$m \circ (S \otimes \mathrm{id}) \circ \Delta = \nu \circ \epsilon = m \circ (\mathrm{id} \circ S) \circ \Delta$$

such a map is called the **antipode**.

Example (Coordinate algebra of a group)

Let G be a finite group and $O(G)$ the $\mathbb{K}$-linear space of functions $f : G \longrightarrow \mathbb{K}$. The latter is a unital algebra if equipped with

$$(ff')(g) = f(g)f'(g)$$

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where $f, f' \in O(G)$ and $g \in G$, unit is the constant function 1. It becomes a Hopf algebra with the $\mathbb{K}$-linear extensions of

$$\Delta(f)(g, g') = f(gg'), \quad \epsilon(f) = f(e), \quad S(f)(g) = f(g^{-1})$$

where e is the identity in G, here we identify $O(G \times G)$ with $O(G) \otimes O(G)$ for the coproduct.

Let now V be a $\mathbb{K}$-linear space. We say that is a (right) H-**comodule** when there is a linear map $\rho_V : V \longrightarrow V \otimes H$ such that

$$(\mathrm{id} \otimes \Delta) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho, \quad (\mathrm{id} \otimes \epsilon) \circ \rho = \mathrm{id}$$

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The space of coinvariant elements in a H-comodule algebra

$$A^{coH} := \{b \in A | \rho_A(b) = b \otimes 1_H\}$$

is a sub-algebra of A.

Denote by $B = A^{coH}$. The algebra inclusion $B \subseteq A$ is called a H-**Hopf-Galois extension** if the canonical map

$$\chi : A \otimes_B A \longrightarrow A \otimes H, \quad a \otimes_B a' \longmapsto (a \otimes 1_H)\rho_A(a')$$

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Hopf-Galois extension are non-commutative principal bundles in the following sense: consider a G-space P and a projection $\pi : P \longrightarrow X$, we say that this set of data is a principal G-bundle if and only if the following map is bijective

$$\alpha : P \times G \longrightarrow P \times_X P, \quad (p, g) \longmapsto (p, p \cdot g)$$

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If Now we take $A = O(P)$, $B = O(X)$ and $H = O(G)$ we have that H coacts on A and $A^{coH} = B$. One checks that χ is the pull-back of α, so principality of the bundle is equivalent to the Hopf-Galois condition.

Let now B be an algebra. A B-**coring** is the datum of a B-bimodule $\mathcal{C}$ together with B-bimodule morphism $\underline{\Delta} : \mathcal{C} \longrightarrow \mathcal{C} \otimes_B \mathcal{C}$ and $\underline{\epsilon} : \mathcal{C} \longrightarrow B$ fulfilling the same axioms of coproduct and counit in a coalgebra.

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If $B^e := B \otimes B^{op}$ is the enveloping algebra of B then we define a B^e-**ring** to be a triple $(\mathcal{R}, s, t)$ where $\mathcal{R}$ is an algebra and

$$s : B \longrightarrow \mathcal{R}, \quad t : B^{op} \longrightarrow \mathcal{R}$$

are algebra morphism with commuting images. The latter induce a B-bimodule structure on $\mathcal{R}$ via

$$brb' := s(b)t(b')r \quad b, b' \in B, r \in \mathcal{R}$$

Now a **bialgebroid** over B is the datum of $(\mathcal{H}, \underline{\Delta}, \underline{\epsilon}, s, t)$ where $(\mathcal{H}, s, t)$ is a B^e-ring and $(\mathcal{H}, \underline{\Delta}, \underline{\epsilon})$ a B-coring with the B-bimodule structure given before, plus some compatibility conditions.

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Definition (Antipode)

An anti-algebra isomorphism $\underline{S} : \mathcal{H} \longrightarrow \mathcal{H}$ satisfying

$$\underline{S} \circ t = s$$

$$(\underline{S}^{-1}(h_{(2)}))_{(1')} \otimes_B (\underline{S}^{-1}(h_{(2)}))_{(2')} h_{(1)} = \underline{S}^{-1}(h) \otimes_B 1_{\mathcal{H}}$$

$$(\underline{S}(h_{(1)}))_{(1')} h_{(2)} \otimes_B (\underline{S}(h_{(1)}))_{(2')} = 1_{\mathcal{H}} \otimes_B \underline{S}(h)$$

where $\underline{\Delta}(h) = h_{(1)} \otimes_B h_{(2)}$, is called an **antipode** for $\mathcal{H}$.

We refer to a bialgebroid with a (bijective) antipode as a **full Hopf algebroid**.

A B-bialgebroid $\mathcal{H}$ is a **Hopf algebroid** (in the sense of Schauenburg [7]) if

$$\beta : \mathcal{H} \otimes_{B^{op}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_B \mathcal{H}, \quad h \otimes_{B^{op}} h' \longmapsto h_{(1)} \otimes_B h_{(2)} h'$$

is bijective. Any full Hopf algebroid is a Hopf algebroid.

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Example (Erhesmann-Schauenburg bialgebroid)

Let $B \subseteq A$ be a H-Hopf-Galois extension, the algebra $\mathcal{C}(A, H) := (A \otimes A)^{coH} \subseteq A^e$ is a bialgebroid over B if endowed with

$$s(b) = b \otimes 1, \quad t(b) = 1 \otimes b$$
$$\underline{\Delta}(a \otimes a') = a_{(0)} \otimes \chi^{-1}(1 \otimes a_{(1)}) \otimes a'$$
$$\underline{\epsilon}(a \otimes a') = aa'$$

where $a, a' \in A$, $b \in B$. Recently has been proved that this is a Hopf algebroid [5], a natural question is: When is it full?

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Thus the quotient $\Omega := (P \times P)/G$ is a Lie groupoid over X with source and target

$$s([p_1, p_2]) = \pi(p_2), \quad t([p_1, p_2]) = \pi(p_1)$$

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While the composition is

$$[p, r] \circ [r, q] = [p, q]$$

where $p, r, q \in P$.

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So in the classical case the E.-S. bialgebroid is actually a full Hopf algebroid. For a general noncommutative Hopf-Galois extension this is no longer true.

We now study in details an example. Let n be a positive integer and $q \in (0,1)$, we denote by $A(S_q^{2n-1})$ the $*$-algebra generated by $\{z_i, z_i^*\}$ for $i = 1, \ldots, n$ with commutation relations

$$z_i z_j = q z_j z_i \quad \forall i < j, \quad z_i^* z_j = q z_j z_i^* \quad \forall i \neq j$$

$$[z_1^*, z_1] = 0, \quad [z_k^*, z_k] = (1 - q^2) \sum_{j=1}^{k-1} z_j z_j^* \quad \forall 1 < k \leq n$$

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For $q = 1$ one gets back the algebra of functions on the sphere S_q^{2n-1}, so we refer to $A(S_q^{2n-1})$ as the quantum odd-dimensional spheres.

If one takes the sub-algebra $A(\mathbb{C}P_q^{n-1})$ generated by $P_{ij} = z_i^* z_j$ with $i, j = 1, \ldots, n$, finds a deformation of the function algebra of the projective space $\mathbb{C}P^{n-1}$.

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This sub-algebra can be realized as the coinvariants with respect to the $O(U(1))$-coaction

$$\rho : A(S_q^{2n-1}) \longrightarrow A(S_q^{2n-1}) \otimes O(U(1)), \quad z_i \longmapsto z_i \otimes t, \quad z_i^* \longmapsto z_i^* \otimes t^{-1}$$

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It is proved that $A(S_q^{2n-1}) \subseteq A(\mathbb{C}P_q^{n-1})$ is a $O(U(1))$-Hopf-Galois extension.

Now take the free module $A(S_q^{2n-1})^n \simeq A(S_q^{2n-1}) \otimes \mathbb{C}^n$ and the elements

$$v = \begin{pmatrix} z_1^* \\ z_2^* \\ \vdots \\ z_n^* \end{pmatrix}, \quad w = \begin{pmatrix} q^{(n-1)} z_1 \\ q^{(n-2)} z_2 \\ \vdots \\ z_n \end{pmatrix}$$

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Using the commutation relations in $A(S_q^{2n-1})$ one proves that $v^\dagger v = 1 = w^\dagger w$, thus the two matrices

$$P = vv^\dagger, \quad Q = ww^\dagger$$

are projections that take value in $A(\mathbb{C}P_q^{n-1})$.

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Then they define two elements in $K_0(A(\mathbb{C}P_q^{n-1}))$ with topological charges -1 and 1 respectively.

Proposition

The E.-S. bialgebroid associated to $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$ *is generated by* $V_{ij} = z_i^* \otimes z_j$, $W_{ij} = q^{(2n-i-j)} z_i \otimes z_j^*$ *and moreover the map*

$$\underline{S} : V_{ij} \longmapsto q^{(j-i)} W_{ji}, \quad W_{ij} \longmapsto q^{(i-j)} V_{ji}$$

is an antipode for $\mathcal{C}(A, H)$ *with inverse*

$$\underline{S}^{-1} : V_{ij} \longmapsto q^{(i-j)} W_{ji}, \quad W_{ij} \longmapsto q^{(j-i)} V_{ji}$$

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Being the Hopf algebra $O(U(1))$ commutative, also the flip σ is an antipode. It is straightforward to see that they are different

$$\underline{S}(V_{11}) = W_{11}, \quad \sigma(V_{11}) = q^{2(1-n)} W_{11}$$

What is the relationship between different antipodes on a given (left) bialgebroid?

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Let $\mathcal{H}$ be a full B-Hopf algebroid and denote by $\mathcal{H}_*$ the set of maps $\phi_* : \mathcal{H} \longrightarrow B$ that are right B-module morphism. They are a ring with respect to

$$(\phi_*\psi_*)(h) = \psi_*(s(\phi_*(h_{(1)}))h_{(2)}), \quad h \in \mathcal{H}, \quad \phi_*, \psi_* \in \mathcal{H}_*$$

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Moreover $\mathcal{H}$ becomes a right $\mathcal{H}_*$-module if endowed with

$$h \triangleleft \phi_* := s(\phi_*(h_{(1)}))h_{(2)}, \quad h \in \mathcal{H}, \quad \phi_* \in \mathcal{H}_*$$

The group of **twists** is the set of invertible elements $\phi_* \in \mathcal{H}_*$ satisfying

$$1_{\mathcal{H}} \triangleleft \phi_* = 1_{\mathcal{H}}, \quad (h \triangleleft \phi_*)(h' \triangleleft \phi_*) = (hh') \triangleleft \phi_*$$

$$\underline{S}(h_{(1)}) \triangleleft \phi_* \otimes h_{(2)} = \underline{S}(h_{(1)}) \otimes h_{(2)} \triangleleft \phi_*^{-1}, \quad h, h' \in \mathcal{H}$$

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Theorem ([1])

Let $(\mathcal{H}, \underline{S})$ be a full Hopf algebroid, then $(\mathcal{H}, \underline{S}')$ is a full Hopf algebroid iff there exists a twist ϕ_* such that

$$\underline{S}'(h) := \underline{S}(h \triangleleft \phi_*), \quad h \in \mathcal{H}$$

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In our case where $\mathcal{H} = \mathcal{C}(A, H)$ for the $O(U(1))$-extension $A(\mathbb{C}P_q^{n-1}) \subseteq A(S_q^{2n-1})$, we are in the situation where both $\underline{S}$ and the flip σ are antipodes. The twist connecting them is given by

$$\phi_* : V_{ij} \longmapsto q^{2(i-n)} P_{ij}, \quad W_{ij} \longmapsto q^{2(n-j)} Q_{ij}$$

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Thank you!