Spectral Triples and AF algebras

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Based on arXiv:2207.04466:

T.Masson, G.N: Lifting Bratteli Diagrams between Krajewski Diagrams

Talk for the workshop Noncommutative geometry:
metric and spectral aspects

General context

- Real spectral triple $(\mathcal{A}, \mathcal{H}_{\mathcal{A}}, D_{\mathcal{A}}, J_{\mathcal{A}}, \gamma_{\mathcal{A}})$.
- NCGFT which reproduces the standard model Lagrangian coupled to Gravity (Connes, Chamseddine, Marcolli, Suijlekom.. 1996 $\rightarrow$ 2012)
- Based on the model of Almost-Commutative Manifold (A-C Manifold)

$$\widehat{\mathcal{A}} := C^{\infty}(M) \otimes \mathcal{A} \quad \xrightarrow{S_{\mathcal{A}}} \quad \mathrm{NCGFT}_{\mathcal{A}}$$

$\mathcal{A}$: finite dimensional complex algebra.

$S_{\mathcal{A}}$: spectral action.

Purpose: Set up a general formalism to build Grand Unified Theories (GUTs) beyond the Standard Model of particle physics (SMPP) in the NCGFT framework using AF-algebras for $\mathcal{A}$.

Non-Commutative Gauge Field Theory (NCGFT) beyond the Standard Model of Particle Physics (SMPP)

- AF-algebra $\mathcal{A}$: inductive limit of a sequence of finite-dimensional algebras:

$$\mathcal{A}_1 \hookrightarrow \mathcal{A}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{A}_n \hookrightarrow \cdots$$

- Embedding structure encoded by ϕ: $\mathcal{A}_n \overset{\phi_n}{\hookrightarrow} \mathcal{A}_{n+1}$
- Given $\widehat{\mathcal{A}} := C^\infty(M) \otimes \mathcal{A}$ and $\widehat{\mathcal{B}} := C^\infty(M) \otimes \mathcal{B}$ with $\mathcal{A} \overset{\phi}{\hookrightarrow} \mathcal{B}$

  $\rightsquigarrow$ How $\text{NCGFT}_{\mathcal{A}}$ and $\text{NCGFT}_{\mathcal{B}}$ are connected?
- If $\text{NCGFT}_{\mathcal{A}}$ gives the SMPP, how $\text{NCGFT}_{\mathcal{B}}$ can represent a GUT extension of the SMPP.

$$\text{NCGFT}_{\mathcal{A}_1} \hookrightarrow \text{NCGFT}_{\mathcal{A}_2} \hookrightarrow \cdots \hookrightarrow \text{NCGFT}_{\mathcal{A}_n} \hookrightarrow \cdots$$

- $\rightarrow$ New way to create models beyond the SMPP.
- T.Masson, G.N: Lifting Bratteli Diagrams between Krajewski Diagrams: Spectral Triples, Spectral Actions, and AF algebras (arXiv:2207.04466)

Essentials on Krajewski diagram

Notations for Algebras:

- $\mathcal{A} = \oplus_{i=1}^{r} M_{n_i}(\mathbb{C}) \qquad \rightsquigarrow \qquad \mathcal{A}^e = \oplus_{i,j=1}^{r} M_{n_i} \otimes M_{n_j}^{\circ}$
- $\iota^i(\mathcal{A}_i)$ the inclusion map

Notations for Modules and Bimodules:

- $M_{n_i}(\mathbb{C})$ act on the irrep $\mathbb{C}^{\mathsf{n}_i}$.
- $\Lambda := \{\mathsf{n}_1, \ldots, \mathsf{n}_r\}$ each element n_i corresponding to the irrep $\mathbb{C}^{\mathsf{n}_i}$
- $\widehat{\mathcal{H}}_{\mathsf{n}_i\mathsf{n}_j} := \iota^i(\mathcal{A}_i)\iota^j(\mathcal{A}_j)^{\circ}\mathcal{H} \qquad \mathcal{H} = \oplus_{i,j=1}^{r} \widehat{\mathcal{H}}_{\mathsf{n}_i\mathsf{n}_j}$
- $\mathcal{H}_{\mathsf{n}_i\mathsf{n}_j} = \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\circ} \quad \rightsquigarrow \quad \widehat{\mathcal{H}}_{\mathsf{n}_i\mathsf{n}_j} \simeq \mathcal{H}_{\mathsf{n}_i\mathsf{n}_j} \otimes \mathbb{C}^{\mu_{ij}} \simeq \mathbb{C}^{n_i} \otimes \mathbb{C}^{\mu_{ij}} \otimes \mathbb{C}^{n_j\circ}$
- μ_{ij} is the multiplicity of the irrep $\mathcal{H}_{\mathsf{n}_i\mathsf{n}_j}$.
- orthonormal basis $\{\sigma_{ij}^{p}\}_{1\leq p\leq\mu_{ij}}$ of $\mathbb{C}^{\mu_{ij}}$ with p for the irreps

Essentials on Krajewski diagram : vertex

- The set of vertex $\Gamma^{(0)}$ of the graph is equipped with a map $\pi_{\lambda\rho} : \Gamma^{(0)} \to \Lambda \times \Lambda$ such that $\pi_{\lambda\rho}(v) = (\mathsf{n}_i, \mathsf{n}_j)$

- $\Gamma^{(0)}_{\mathsf{n}_i\mathsf{n}_j} := \{v \in \Gamma^{(0)} \mid \pi_{\lambda\rho}(v) = (\mathsf{n}_i, \mathsf{n}_j)\} \quad \rightsquigarrow \quad \Gamma^{(0)} := \cup_{i,j=1}^{r} \Gamma^{(0)}_{\mathsf{n}_i\mathsf{n}_j}$

- $\sigma^p_{ij} \in \mathbb{C}^{\mu_{ij}} \to \mathcal{H}_v := \mathrm{Span}\{\xi_i \otimes \sigma^p_{ij} \otimes \eta^\circ_j \mid \xi_i \in \mathbb{C}^{n_i},\ \eta^\circ_j \in \mathbb{C}^{n_j\circ}\} \simeq \mathcal{H}_{\mathsf{n}_i\mathsf{n}_j}$

  $\rightsquigarrow \quad \mathcal{H} = \oplus_{v \in \Gamma^{(0)}} \mathcal{H}_v$

- The element $\pi_{\lambda\rho}(v) \in \Lambda \times \Lambda$ is a decoration of the vertex v.

- γ induce sign decoration $s(v) = \pm 1$ of $v \quad \rightsquigarrow \quad \mathcal{H}_{\mathcal{A}} = \mathcal{H}^+_{\mathcal{A}} \oplus \mathcal{H}^-_{\mathcal{A}}$

- $J : \mathcal{H}_v \to \mathcal{H}_{\kappa(v)} \qquad \kappa : \Gamma^{(0)}_{\mathsf{n}_i\mathsf{n}_j} \to \Gamma^{(0)}_{\mathsf{n}_j\mathsf{n}_i} \qquad \widehat{\kappa}_v : \mathcal{H}_v \to \mathcal{H}_{\kappa(v)}$

Essentials on Krajewski diagram : edges

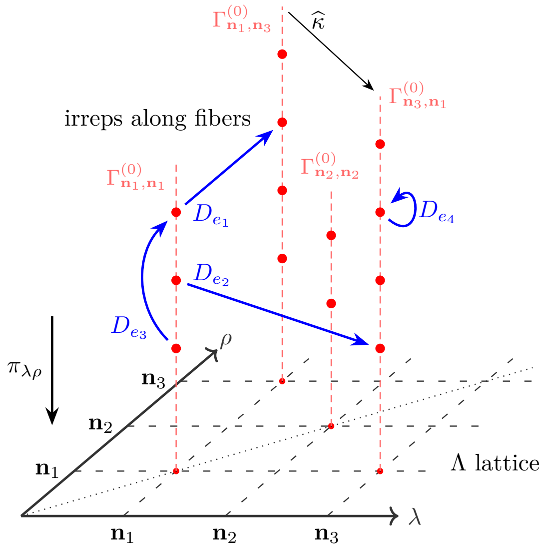
- The space $\Gamma^{(1)} \subset \Gamma^{(0)} \times \Gamma^{(0)}$ of edges $\equiv$ couples $e = (v_1, v_2)$
- Given $e = (v_1, v_2) \in \Gamma^{(0)} \times \Gamma^{(0)}$ we have $D_e : \mathcal{H}_{v_1} \to \mathcal{H}_{v_2}$ such that:

$$\widehat{D}_e(\xi \otimes \sigma_v \otimes \eta^\circ) = \begin{cases} 0 & \text{if } v \neq v_1 \\ (D_{L,e}^{(1)}\xi) \otimes \sigma_{v_2} \otimes (D_{R,e}^{(2)}\eta^\circ) & \text{if } v = v_1 \end{cases}$$

$(1, 2) \equiv$ notation for finite summation

- $D_e : \mathcal{H}_{v_1} \to \mathcal{H}_{v_2}$ defines a decoration of e.
- $\gamma D = -\gamma D \quad \to \quad s(v_2) = -s(v_1)$
- $D^\dagger = D$ is equivalent to $D_{\bar{e}} = D_e^\dagger$ with $\bar{e} := (v_2, v_1)$

The Krajewski diagram



$\lambda : \Gamma^{(0)}_{\mathsf{n}_i} \to \Lambda$

$\lambda(v) := \mathsf{n}_i$

$\rho(v) := n_j{}^{\circ}$

$\mathbb{C}^{\lambda(v)} \otimes \mathbb{C}^{\rho(v)} = \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j{}^{\circ}}$

AF-algebras

- $\mathcal{A} = \oplus_{i=1}^{r} M_{n_i}$ $\qquad$ $\mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$
- $\forall a \in \mathcal{A} : a = \oplus_{i=1}^{r} a_i$ $\qquad$ $a_i \in M_{n_i}$

$$\phi_k(a) := \pi_k^{\mathcal{B}} \circ \phi(a) = \begin{pmatrix} a_1 \otimes \mathbb{1}_{\alpha_{k1}} & 0 & \cdots & 0 & 0 \\ 0 & a_2 \otimes \mathbb{1}_{\alpha_{k2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_r \otimes \mathbb{1}_{\alpha_{kr}} & 0 \\ 0 & 0 & \cdots & 0 & \mathbb{0}_{n_{0,k}} \end{pmatrix} \subset M_{m_k}$$

$$a_i \otimes \mathbb{1}_{\alpha_{ki}} = \left.\begin{pmatrix} a_i & 0 & 0 & 0 \\ 0 & a_i & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_i \end{pmatrix}\right\} \alpha_{ki} \text{ times.}$$

- α_{ki} is the multiplicity of the embedding.

AF-algebras

$$\phi_k^i(a_i) = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_i \otimes \mathbb{1}_{\alpha_{ki}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \sum_{\alpha=1}^{\alpha_{ki}} \phi_{k,\alpha}^i(a_i) : M_{n_i} \to M_{m_k}$$

- The n_i's designate vertices in Bratteli $\rightsquigarrow$ matrix blocs M_{n_i}.
- Edges (n_i, m_k) between two vertices of $(\mathcal{A},\mathcal{B})$ exist iff $\alpha_{ki} \neq 0$.
- **The multiplicities α_{ki} define the Bratteli diagram of the AF-algebra.**

ϕ-compatibility condition

- ϕ-compatibility condition $\rightarrow$ link between NCGFT$_{\mathcal{A}}$'s and NCGFT$_{\mathcal{B}}$'s structure at the level of their spectral triples.

  $\rightsquigarrow$ Physical meaning: conservation of actions on embedded Hilberts spaces

- ϕ-compatibility of Hilbert spaces: $\phi_{\mathcal{H}} : \mathcal{H}_{\mathcal{A}} \rightarrow \mathcal{H}_{\mathcal{B}}$ is ϕ-compatible if $\phi_{\mathcal{H}}(a\psi) = \phi(a)\phi_{\mathcal{H}}(\psi)$ for any $(a, \psi) \in (\mathcal{A}, \mathcal{H}_{\mathcal{A}})$

- $\mathcal{H}_{\mathcal{B}} = \phi_{\mathcal{H}}(\mathcal{H}_{\mathcal{A}}) \oplus \phi_{\mathcal{H}}(\mathcal{H}_{\mathcal{A}})^{\perp} \quad \rightarrow \quad \forall B$ on $\mathcal{H}_{\mathcal{B}}$: $B = \begin{pmatrix} B_{\phi}^{\phi} & B_{\perp}^{\phi} \\ B_{\phi}^{\perp} & B_{\perp}^{\perp} \end{pmatrix}$.

- Given A on $\mathcal{H}_{\mathcal{A}}$ and B on $\mathcal{H}_{\mathcal{B}}$ we said that they are:
  - ϕ-compatible if $\forall \psi \in \mathcal{H}_{\mathcal{A}}$: $\phi_{\mathcal{H}}(A\psi) = B_{\phi}^{\phi}\phi_{\mathcal{H}}(\psi)$
    $\equiv$ equality in $\phi_{\mathcal{H}}(\mathcal{H}_{\mathcal{A}})$
  - Strong ϕ-compatible if $\forall \psi \in \mathcal{H}_{\mathcal{A}}$: $\phi_{\mathcal{H}}(A\psi) = B\phi_{\mathcal{H}}(\psi)$
    $\equiv$ equality in $\mathcal{H}_{\mathcal{B}}$

- ϕ-compatibility looks more natural for physics since it is based on constraints on inherited degrees of freedom (dofs) only.

Consequences of Phi-compatibility on real spectral Triples

- Strong ϕ-compatibility implies ϕ-compatibility.
- $(\mathcal{A}, \mathcal{H}_\mathcal{A}, D_\mathcal{A}, J_\mathcal{A}, \gamma_\mathcal{A})$ and $(\mathcal{B}, \mathcal{H}_\mathcal{B}, D_\mathcal{B}, J_\mathcal{B}, \gamma_\mathcal{B})$ are said to be ϕ-compatible if each of it's structure are.

  $\rightarrow$ lift of arrows in a Bratteli diagram to arrows between Krajewski diagrams.
- If two real spectral triples are strong ϕ-compatible $\rightarrow$ same KO-dimension (mod 8).
- If two real spectral triples are ϕ-compatible and $J_\mathcal{B}$ is strong ϕ-compatible with $J_\mathcal{A}$ $\rightarrow$ same KO-dimension (mod 8).
- If A on $\mathcal{H}_\mathcal{A}$ and B on $\mathcal{H}_\mathcal{B}$ are strong ϕ-compatible then $B_\perp^\phi = 0$.
- $\forall a \in \mathcal{A}$, $\pi_\mathcal{B} \circ \phi(a)$ on $\mathcal{H}_\mathcal{B}$ reduces to $\begin{pmatrix} \pi_\mathcal{B}\circ\phi(a)_\phi^\phi & 0 \\ 0 & \pi_\mathcal{B}\circ\phi(a)_\perp^\perp \end{pmatrix}$
- Gauge transformations are preserved under certain circumstances.

Embedding structure of AF algebras

- Let $\mathcal{A} = \mathcal{A}_n$ and $\mathcal{B} = \mathcal{A}_{n+1}$, and (v, w) be nodes in $(\Gamma^{(0)}_{\mathcal{A}}, \Gamma^{(0)}_{\mathcal{B}})$.
- $\phi_{\mathcal{H}}$ is injective.
- $\phi_{\mathcal{H}}$ is totaly specified by the $\phi^{v}_{\mathcal{H},w}$'s.
- Taking $\pi_{\lambda\rho}(v) = (n_i, n_j)$ and $\pi_{\lambda\rho}(w) = (m_k, m_l)$.

Embedding structure of AF algebras

$\phi^{v}_{\mathcal{H},w}$ reduces to $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j \circ} \to \mathbb{C}^{n_i} \otimes \mathbb{C}^{\alpha_{ki}} \otimes \mathbb{C}^{\alpha_{\ell j}} \otimes \mathbb{C}^{n_j \circ}$.

Then it reduces to a linear map $\mathbb{C} \to \mathbb{C}^{\alpha_{ki}} \otimes \mathbb{C}^{\alpha_{\ell j}}$.

Therefore to an element $\mathbf{u(v,w)} \in \mathbb{C}^{\alpha_{ki}} \otimes \mathbb{C}^{\alpha_{\ell j}}$, such that we have:

$$\phi^{v}_{\mathcal{H},w}(\xi_i \otimes \eta_j^{\circ}) = I^{i,j}_{k,\ell}(\xi_i \otimes \mathbf{u(v,w)} \otimes \eta_j^{\circ}) \quad \text{for any } \xi_i \otimes \eta_j^{\circ} \in \mathcal{H}_{\mathcal{A},v}.$$

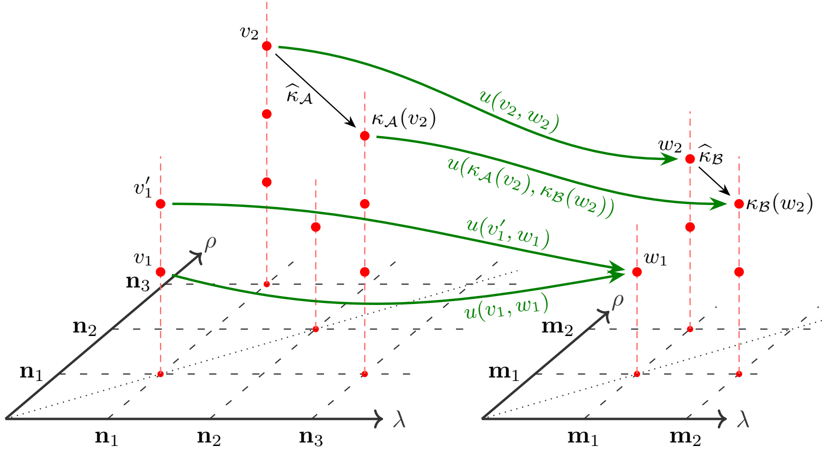
with the inclusion:

$$I^{i,j}_{k,\ell} : \mathbb{C}^{n_i} \otimes \mathbb{C}^{\alpha_{ki}} \otimes \mathbb{C}^{\alpha_{\ell j}} \otimes \mathbb{C}^{n_j \circ} \hookrightarrow \mathbb{C}^{m_k} \otimes \mathbb{C}^{m_\ell \circ}$$

Then $\phi^{v}_{\mathcal{H},w}$ is completely determined by $\mathbf{u(v,w)} \in M_{\alpha_{ki} \times \alpha_{\ell j}} \simeq \mathbb{C}^{\alpha_{ki}} \otimes \mathbb{C}^{\alpha_{\ell j}}$

Embedding structure of AF algebras

The $u(v, w)$ contain all the data of edges in the Bratteli Diagram, thus linking Krajewski and Bratteli diagram structures:



Link between scalar products

Scalar products on $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ are linked by the relation:

$$\langle \phi_\mathcal{H}^{v_1}(\psi_{v_1}), \phi_\mathcal{H}^{v_2}(\psi'_{v_2})\rangle_{\mathcal{H}_\mathcal{B}} = \langle \psi_{v_1}, \iota_{v_1}^{v_2}(\psi'_{v_2})\rangle_{\mathcal{H}_{\mathcal{A},v_1}} \mathrm{T}^{v_1,v_2}$$

$$\text{with} \qquad \mathrm{T}^{v_1,v_2} := \begin{cases} 0 & \text{if } \pi_{\lambda\rho}(v_1) \neq \pi_{\lambda\rho}(v_2) \\ \sum_{w \in \Gamma_\mathcal{B}^{(0)}} \mathrm{tr}(u(v_1,w)^* u(v_2,w)) & \text{if } \pi_{\lambda\rho}(v_1) = \pi_{\lambda\rho}(v_2) \end{cases}$$

There are orthonormal bases $\{\sigma_{ij}^p\}_{1\leq p\leq \mu_{ij}}$ of $\mathbb{C}^{\mu_{ij}}$ such that:

- $\langle \phi_\mathcal{H}^{v_1}(\psi_{v_1}), \phi_\mathcal{H}^{v_2}(\psi'_{v_2})\rangle_{\mathcal{H}_\mathcal{B}} = 0$ if $v_1 \neq v_2$.
- $\langle \phi_\mathcal{H}^{v}(\psi_{v}), \phi_\mathcal{H}^{v}(\psi'_{v})\rangle_{\mathcal{H}_\mathcal{B}} = \mathrm{t}_v \, \langle \psi_v, \psi'_v\rangle_{\mathcal{H}_{\mathcal{A},v}} \qquad \forall v.$
- $\mathrm{t}_{\kappa_\mathcal{A}(v)} = \mathrm{t}_v$ is a real number $\forall v.$

Normalized $\phi_\mathcal{H}$ map: $\phi_\mathcal{H}(\oplus_{v\in\Gamma_\mathcal{A}^{(0)}} \psi_v) := \sum_{v\in\Gamma_\mathcal{A}^{(0)}} \mathrm{t}_{v(\tilde{v})}^{-1/2} \phi_\mathcal{H}^{0,v}(\psi_v)$

$\rightsquigarrow$ More natural because it preserve scalar products.

$\rightsquigarrow$ Idea of dilution of the dofs.

The Corresponding AC-Manifold

- $\widehat{\mathcal{A}} := C^\infty(M) \otimes \mathcal{A}$ and $\widehat{\mathcal{B}} := C^\infty(M) \otimes \mathcal{B}$ are said ϕ-compatible if $\mathcal{A}$ and $\mathcal{B}$ are.
- $ST_{\mathcal{A}} = (\widehat{\mathcal{A}} := C^\infty(M) \otimes \mathcal{A}, \mathcal{H}_{\widehat{\mathcal{A}}} := L^2(S) \otimes \mathcal{H}_{\mathcal{A}}, D_{\widehat{\mathcal{A}}} := D_M \otimes 1 + J_M \otimes D_{\mathcal{A}}, J_{\widehat{\mathcal{A}}} := J_M \otimes J_{\mathcal{A}}, \gamma_{\widehat{\mathcal{A}}} := \gamma_M \otimes \gamma_{\mathcal{A}})$
- $ST_{\mathcal{B}} = (\widehat{\mathcal{B}}, \mathcal{H}_{\widehat{\mathcal{B}}}, D_{\widehat{\mathcal{B}}} := D_M \otimes 1 + J_M \otimes D_{\mathcal{B}}, J_M \otimes J_{\mathcal{B}}, \gamma_M \otimes \gamma_{\mathcal{B}})$
- Fluctuated Dirac operator:

$$D_{\widehat{\mathcal{A}},\omega} = D_M \otimes 1 + \gamma^\mu \otimes B_\mu + \gamma_M \otimes \Phi$$

- B_μ and Φ are the usual gauge connections and Higgs fields.
- ϕ-compatibility condition is taken on ω (it's equivalent to take it on B_μ and Φ).

Bosonic and Fermionic's Lagrangian

- Given the spectral triple $ST_{\mathcal{A}}$ the associated action is:

$$S_{\mathcal{A}}[\omega, \widetilde{\psi}] = S_{b,\mathcal{A}}[\omega] + S_{f,\mathcal{A}}[\omega, \widetilde{\psi}]$$

with the Bosonic spectral action and Fermionic action given by:

$$S_{b,\mathcal{A}}[\omega] := \mathrm{Tr}\, f(D_{\widehat{\mathcal{A}},\omega}/\Lambda) \qquad S_{f,\mathcal{A}}[\omega, \widetilde{\psi}] := \langle J_{\widehat{\mathcal{A}}}\widetilde{\psi}, D_{\widehat{\mathcal{A}},\omega}\widetilde{\psi}\rangle_{\widetilde{\mathcal{H}}_{\widehat{\mathcal{A}}}}$$

Remark

NCSMPP: $dim(M) = 4$, and KO dimension of $\mathcal{A}$ (then $\mathcal{B}$) is 6.

Comparison between the Lagrangian of $\mathcal{A}$ and $\mathcal{B}$

If $(\omega, \widetilde{\psi})$ and $(\omega', \widetilde{\psi}')$ are ϕ-compatible, the action of $ST_{\mathcal{B}}$ is:

$$S_{\mathcal{B}}[\omega', \widetilde{\psi}'] = S_{\mathcal{A}}[\omega, \widetilde{\psi}] + \mathsf{TNIC}$$

TNIC $\equiv$ Terms with Non-Inherited Components.

The Spectral action of $\mathcal{A}$ is recovered, new terms that correspond to the mixing with new dofs appear.

This structural result works in general KO dimension for $\mathcal{A}$ (then $\mathcal{B}$).

$\equiv$ Constraints of a NCGFT on $\mathcal{B}$ with another on $\mathcal{A}$

Perspective to create models beyond the SMPP

- General framework for building phenomenological models.
- Explore the way in which the interplay between old and new dofs permits to extend the NCSMPP.
- In arXiv:2106.08358 (Derivation-based noncommutative field theories on AF algebras) we study the phenomenology of these embeddings:
  - $M_2 \hookrightarrow M_3$
  - $M_2 \oplus M_2 \hookrightarrow M_4$
  - $M_2 \oplus M_2 \hookrightarrow M_5$
  - $M_2 \oplus M_3 \hookrightarrow M_5$ $\quad \rightsquigarrow \quad$ Georgi-Glashow model
- Many possibilities to do unified theories in different ways.

Thank you

Suggested readings:

- WD Van Suijlekom: Noncommutative geometry and particle physics.
- KR Davidson: C^*-algebras by example.
- T Masson, G Nieuviarts: Derivation-based noncommutative field theories on algebras.
- T Masson, G Nieuviarts: Lifting Bratteli Diagrams between Krajewski Diagrams: Spectral Triples, Spectral Actions, and AF algebras.

Backup slides

The operator J

- $J_v := \epsilon(v,d)\, J_0 \widehat{\kappa}_v = \epsilon(v,d)\, \widehat{\kappa}_v J_0 : \mathcal{H}_v \to \mathcal{H}_{\kappa(v)}$

$$\epsilon(v,d) := \begin{cases} 1 & \text{for } i(v) < j(v), \\ \epsilon & \text{for } i(v) > j(v), \\ 1 & \text{for } i(v) = j(v) \text{ and } d = 0,1,7, \\ \epsilon^{\chi(v)} & \text{for } i(v) = j(v) \text{ and } d = 2,3,4,5,6. \end{cases} \quad (1)$$

- $J\gamma = \epsilon'' \gamma J \qquad \rightsquigarrow \qquad s \circ \kappa = \epsilon'' s \qquad\qquad J^2 = \epsilon$
- $\phi^e : \mathcal{A}^e \to \mathcal{B}^e$ as $\phi^e := \phi \otimes \phi^\circ$, *i.e.* $\phi^e(a_1 \otimes a_2^\circ) = \phi(a_1) \otimes \phi^\circ(a_2^\circ)$
- $\phi_{\mathcal{H}}$ restricts to maps $\mathcal{H}^{\pm}_{\mathcal{A}} \to \mathcal{H}^{\pm}_{\mathcal{B}}$.
- $J_{\mathcal{B}}$ is strong ϕ-compatible with $J_{\mathcal{A}}$ iff $\forall (v,w) \in (\Gamma^{(0)}_{\mathcal{A}}, \Gamma^{(0)}_{\mathcal{B}})$:

$$u(\kappa_{\mathcal{A}}(v), \kappa_{\mathcal{B}}(w)) = \frac{\epsilon_{\mathcal{A}}(v, d_{\mathcal{A}})}{\epsilon_{\mathcal{B}}(w, d_{\mathcal{B}})} u(v,w)^* \quad (2)$$

where $d_{\mathcal{A}}$ (resp. $d_{\mathcal{B}}$) is the KO-dimension of $\mathcal{A}$ (resp. $\mathcal{B}$).

ϕ-compatibility condition

- ϕ-compatibility for Hilbert spaces :

$$\phi^{v}_{\mathcal{H},w}(a_i b_j^{\circ} \psi_v) = \phi^{i}_{k}(a_i)\phi^{j}_{\ell}(b_j)^{\circ}\phi^{v}_{\mathcal{H},w}(\psi_v)$$

- ϕ-compatibility for operators :

$$\sum_{v_2 \in \Gamma^{(0)}_{\mathcal{A}}} \phi^{v_2}_{\mathcal{H},w_2}(A^{v_1}_{v_2}\psi_{v_1}) = \sum_{w_1 \in \Gamma^{(0)}_{\mathcal{B}}} B^{\phi,w_1}_{\phi,w_2}\phi^{v_1}_{\mathcal{H},w_1}(\psi_{v_1})$$

Bosonic and Fermionic's Lagrangian

- $S_{b,\mathcal{A}}[\omega] \sim \int_M \mathcal{L}(B_\mu, \Phi), \mathrm{d}^4 x + \mathcal{O}(\Lambda^{-1})$
- Bosonic Lagrangian : $\mathcal{L}(B_\mu, \Phi) = \mathcal{L}_B(B_\mu) + \mathcal{L}_\varphi(B_\mu, \Phi)$

  $\rightarrow \mathcal{L}_B(B_\mu) = \frac{f(0)}{24\pi^2} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu})$

  $\rightarrow \mathcal{L}_\varphi(B_\mu, \Phi) = -\frac{2 f_2 \Lambda^2}{4\pi^2} \operatorname{tr}(\Phi^2) + \frac{f(0)}{8\pi^2} \operatorname{tr}(\Phi^4) + \frac{f(0)}{8\pi^2} \operatorname{tr}\left((D_\mu \Phi)(D^\mu \Phi)\right)$