Sequences of Noncommutative Gauge Field Theories on AF-algebras

(Join work with Gaston Nieuviarts)

Noncommutative geometry: metric and spectral aspects, September 2022

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Why?

GUT:

- "big group" ➞ "smaller group" by Spontaneous Symmetry Breaking Mechanism(s) (SSBM).
- Sequence of Gauge Field Theories (GFT) with decreasing numbers of degrees of freedom.

Known Facts:

1. A finite-dimensional C^*-algebra: $\mathcal{A} = M_{n_1} \oplus \cdots \oplus M_{n_r}$ up to isomorphism.
   $M_n := M_n(\mathbb{C})$ is the space of $n \times n$ matrices over $\mathbb{C}$.
   - NCGFT have been investigated on "almost commutative" algebras $C^\infty(M) \otimes \mathcal{A}$ (manifold M).
   - These NCGFT are of Yang-Mills-Higgs types with symmetry group related to the automorphisms of $\mathcal{A}$.
   - Propositions for NC versions of the Standard Model of Particle Physics.

2. *AF* C^*-algebra: control the approximation of the algebra by finite-dimensional C^*-algebras.
   - Defining inductive sequence: $\mathcal{A} = \varinjlim \mathcal{A}_n$ where $\mathcal{A}_n$ is finite-dimensional.
   - K_0-group of $\mathcal{A}$ is "approximated" by the limit of the K_0-groups of the $\mathcal{A}_n$.

Motivation:

- Embed a "small algebra" into a "larger algebra" and try to relate some NCGFT on them.
- *AF*-algebra ➞ sequence of NCGFT with increasing numbers of degrees of freedom.

How?

- Defining sequence of an *AF*-algebra: $\mathcal{A} = \varinjlim \mathcal{A}_n$, $\mathcal{A}_n$ finite-dimensional.
  $\{(\mathcal{A}_n, \phi_{n,m}) \,/\, 0 \leq n < m\}$ with $\phi_{n,m} : \mathcal{A}_n \to \mathcal{A}_m$ and $\phi_{m,p} \circ \phi_{n,m} = \phi_{n,p}$ for any $0 \leq n < m < p$.
- To any $\mathcal{A}_n$, associate a NCGFT denoted by $\text{NCGFT}_{\mathcal{A}_n}$.
  ➠ Sequence $\{\text{NCGFT}_{\mathcal{A}_n}\}_{n\geq 0}$ on top of $\{\mathcal{A}_n\}_{n\geq 0}$.
- Find a good definition for the relation between $\text{NCGFT}_{\mathcal{A}_n}$ and $\text{NCGFT}_{\mathcal{A}_{n+1}}$.
  This definition must depend on $\phi_{n,n+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$.
- This needs some "ϕ-compatibility" conditions between elements in $\text{NCGFT}_{\mathcal{A}_n}$ and elements in $\text{NCGFT}_{\mathcal{A}_{n+1}}$.
  ➠ This depends on the way one defines $\text{NCGFT}_{\mathcal{A}}$ for an algebra $\mathcal{A}$.

This is what we propose in our two papers:

- Masson, T. and Nieuviarts, G. (2021). Derivation-based Noncommutative Field Theories on *AF* algebras. *International Journal of Geometric Methods in Modern Physics* 18.13, p. 2150213
- Masson, T. and Nieuviarts, G. (July 2022). Lifting Bratteli Diagrams between Krajewski Diagrams: Spectral Triples, Spectral Actions, and *AF* algebras. eprint: 2207.04466

Outline

How to model a Gauge Field Theory?

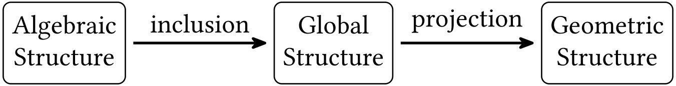
Gauge field theories are based on physical ideas...
... and these ideas require some mathematical structures in order to be implemented:

1. A space of local symmetries (local = they depend on points in space-time).
   gauge group (finite gauge transformations) or **Lie algebra** (infinitesimal gauge transformations)...
2. An implementation of the symmetry on matter fields.
   Use the natural **representation theory** from the mathematical framework.
3. A **differential structure** with which equations of motion are written.
4. A kind of **covariant derivative** ➞ "minimal coupling" between matter fields and gauge fields.
5. A way to write a gauge invariant Lagrangian density.
   The **action functional** from which the equations of motion are deduced.

Many (mathematical) frameworks described in
François, J., Lazzarini, S., and Masson, T. (2014). "Gauge field theories: various mathematical approaches". In: *Mathematical Structures of the Universe.* Ed. by Eckstein, M., Heller, M., and Szybka, S. J. Kraków, Poland: Copernicus Center Press, pp. 177–225

How to model a GFT? One Pattern to rule them all...

A common pattern to all known mathematical frameworks (fiber bundles, NCG, Lie algebroids...):



In the present situation (NCG):

- "Algebraic Structure": a finite dimensional algebras $\mathcal{A}_F$ (defining an AF-algebra).
- "Geometric Structure": an ordinary space-time M.
- "Global Structure": the almost commutative algebras $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$.

- AF-algebras will only concern the "Algebraic Structure" of these NCGFT's.
- The underlying geometry is "constant" (relative to the inductive limit defining the AF-algebra).

Open question: can we use some AF-algebra also for the "Geometric Structure"?

How to model a NCGFT?

The basic ingredient is an associative algebra $\mathcal{A}$. Then:

Representation theory: a left (projective finitely generated) module $\mathcal{M}$ over $\mathcal{A}$.

Gauge group: $\mathrm{Aut}(\mathcal{M})$ or $\mathcal{U}(\mathcal{A})$.

Differential structure: any differential calculus defined on top of $\mathcal{A}$.

There is no canonical construction here: explicit choice to be made.

- The derivation-based differential calculus canonically associated to the algebra $\mathcal{A}$.
- Spectral triple $(\mathcal{A}, \mathcal{H}, D)$: need to add supplementary structures, and $\mathcal{M} = \mathcal{H}$.

Covariant derivative: a NC connection defined on $\mathcal{M}$ relative to the chosen differential calculus.

In general it is described by a "connection 1-form" in the chosen space of forms.

Action functional: depends on the choice of the differential calculus.

- Derivation-based differential calculus ➞ integration and Hodge star operator may be defined...
- Spectral triple ➞ Spectral action and Fermionic action...

Objective: Make all these structures compatible with the imbeddings $\phi_{n,n+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$.

Derivation-based NCG

Dubois-Violette, M. (1988). Dérivations et calcul differentiel non commutatif. *C.R. Acad. Sci. Paris, Série I* 307, pp. 403–408

Consider an associative algebra $\mathcal{A}$.

- Let $\mathcal{Z}(\mathcal{A}) = \{a \in \mathcal{A} \,/\, ab = ba, \forall b \in \mathcal{A}\}$ be its center.
- $\text{Der}(\mathcal{A}) = \{\mathfrak{X} : \mathcal{A} \to \mathcal{A} \,/\, \mathfrak{X} \text{ linear}, \mathfrak{X}{\cdot}(ab) = (\mathfrak{X}{\cdot}a)b + a(\mathfrak{X}{\cdot}b), \forall a, b \in \mathcal{A}\}$.
  ➞ Lie algebra and $\mathcal{Z}(\mathcal{A})$-module.
- $\Omega^p_{\text{Der}}(\mathcal{A})$ the vector space of $\mathcal{Z}(\mathcal{A})$-multilinear antisymmetric maps from $\text{Der}(\mathcal{A})^p$ to $\mathcal{A}$.
  Convention: $\Omega^0_{\text{Der}}(\mathcal{A}) = \mathcal{A}$.
- $\Omega^\bullet_{\text{Der}}(\mathcal{A}) = \oplus_{p\geq 0} \Omega^p_{\text{Der}}(\mathcal{A})$ is a $\mathbb{N}$-graded differential algebra:
  - $(\omega \wedge \eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) := \frac{1}{p!q!} \sum_{\sigma\in\mathfrak{S}_{p+q}} (-1)^{|\sigma|} \omega(\mathfrak{X}_{\sigma(1)}, \dots, \mathfrak{X}_{\sigma(p)}) \eta(\mathfrak{X}_{\sigma(p+1)}, \dots, \mathfrak{X}_{\sigma(p+q)})$
  - $d\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+1}) := \sum_{i=1}^{p+1} (-1)^{i+1} \mathfrak{X}_i{\cdot}\omega(\mathfrak{X}_1, \dots \overset{i}{\vee} \dots, \mathfrak{X}_{p+1}) + \sum_{1\leq i<j\leq p+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \dots \overset{i}{\vee} \dots \overset{j}{\vee} \dots, \mathfrak{X}_{p+1})$

Derivation-based NCG: matrix algebra

Dubois-Violette, M., Kerner, R., and Madore, J. (1990b). Noncommutative Differential Geometry of Matrix Algebras. *J. Math. Phys.* 31, p. 316

Let $\mathcal{A} = M_n(\mathbb{C})$.

- $\mathcal{Z}(M_n) = \mathbb{C}\mathbb{1}_n$ where $\mathbb{1}_n$ is the unit matrix in M_n.
- $\mathrm{Der}(M_n) = \mathrm{Int}(M_n) \simeq \mathfrak{sl}_n$ for $\mathfrak{sl}_n \ni a \mapsto \mathrm{ad}_a \in \mathrm{Int}(M_n)$.
- $\Omega^\bullet_{\mathrm{Der}}(M_n) = M_n \otimes \bigwedge^\bullet \mathfrak{sl}_n^*$ and d is the Chevalley-Eilenberg differential.
- $\{E_\alpha\}_{\alpha\in I_n}$ be a basis of $\mathfrak{sl}_n$, where I_n is a totally ordered set with $\mathrm{card}(I_n) = n^2 - 1 = \dim \mathfrak{sl}_n$;
  $\{\theta^\alpha\}_{\alpha\in I_n}$ be its dual basis in $\mathfrak{sl}_n^*$;
  $\{\partial_\alpha := \mathrm{ad}_{E_\alpha}\}_{\alpha\in I_n}$ the associated basis of $\mathrm{Der}(M_n) = \mathrm{Int}(M_n)$.
- Canonical metric $g : \mathrm{Der}(M_n) \times \mathrm{Der}(M_n) \to \mathcal{Z}(M_n) \simeq \mathbb{C}$ defined by $g(\mathrm{ad}_a, \mathrm{ad}_b) := \mathrm{tr}(ab)$ for $a, b \in \mathfrak{sl}_n$.
- Noncommutative integral $\int_{M_n}$ on $\Omega^\bullet_{\mathrm{Der}}(M_n)$: zero on $\Omega^p_{\mathrm{Der}}(M_n)$ for $p < n^2 - 1$ and $\int_{M_n} \omega := \mathrm{tr}(a)$ for $\omega \in \Omega^{n^2-1}_{\mathrm{Der}}(M_n)$ written as $\omega = a\sqrt{|g|}\theta^{\alpha_1^0} \wedge \cdots \wedge \theta^{\alpha^0_{n^2-1}}$ for a unique $a \in M_n$ (where $\alpha_1^0 < \cdots < \alpha^0_{n^2-1}$).
- Hodge star operator $\star : \Omega^p_{\mathrm{Der}}(M_n) \to \Omega^{n^2-1-p}_{\mathrm{Der}}(M_n)$ (using g).

⚠ We differ from the original paper which uses a convention with extra factor $\frac{1}{n}$ for g and $\int_{M_n}$.

Derivation-based NCGFT: matrix algebra

- NC Connection: $\nabla_{\mathfrak{X}} : \mathcal{M} \to \mathcal{M}$ defined for any $\mathfrak{X} \in \mathrm{Der}(\mathcal{A})$.
  $\nabla_{f\mathfrak{X}} = f\nabla_{\mathfrak{X}}, \quad \nabla_{\mathfrak{X}+\mathfrak{Y}} = \nabla_{\mathfrak{X}} + \nabla_{\mathfrak{Y}}, \quad \nabla_{\mathfrak{X}}(ae) = (\mathfrak{X}{\cdot}a)e + a\nabla_{\mathfrak{X}}e.$
- Curvature: $R(\mathfrak{X},\mathfrak{Y})e := (\nabla_{\mathfrak{X}}\nabla_{\mathfrak{Y}} - \nabla_{\mathfrak{Y}}\nabla_{\mathfrak{X}} - \nabla_{[\mathfrak{X},\mathfrak{Y}]})e$ for any $e \in \mathcal{M}$ and $\mathfrak{X},\mathfrak{Y} \in \mathrm{Der}(\mathcal{A})$.
- Action of the gauge group $\mathcal{G} = \mathrm{Aut}(\mathcal{M})$ is well-defined...

- Simplified situation: left module $\mathcal{M} = \mathcal{A}$.
  - NC Connection 1-form: $\omega \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$ such that $\nabla_{\mathfrak{X}}e = (\mathfrak{X}{\cdot}e) + e\omega(\mathfrak{X})$ for any $e \in \mathcal{M} = \mathcal{A}$.
  - NC Curvature 2-form: $R(\mathfrak{X},\mathfrak{Y})e = e\Omega(\mathfrak{X},\mathfrak{Y})$ with $\Omega(\mathfrak{X},\mathfrak{Y}) := (\mathrm{d}\omega)(\mathfrak{X},\mathfrak{Y}) - [\omega(\mathfrak{X}),\omega(\mathfrak{Y})]$
  - Suppose E_α are anti-Hermitean (traceless) matrices in $\mathfrak{sl}_n$ (and define a basis).
  - Canonical connection : $\mathring{\nabla}_{\partial_\alpha}e := E_\alpha e$ for any $\alpha \in I_n$ and $e \in \mathcal{M} = \mathcal{A}$ ➡ $\mathring{\omega} = E_\alpha\theta^\alpha$.
    ➡ Define $\omega = \omega_\alpha\theta^\alpha = \mathring{\omega} - B_\alpha\theta^\alpha = (E_\alpha - B_\alpha)\theta^\alpha$.
  - Action functional: $S[\omega] = -\int_{M_n} \Omega \wedge \star\Omega = -\frac{1}{2}\sum_{\alpha,\beta} \mathrm{tr}([B_\alpha, B_\beta] - C^\gamma_{\alpha\beta}B_\gamma)^2$.

Derivation-based NCGFT: $\widehat{\mathcal{A}} := C^\infty(M) \otimes M_n$

Dubois-Violette, M., Kerner, R., and Madore, J. (1990a). Noncommutative Differential Geometry and New Models of Gauge Theory. *J. Math. Phys.* 31, p. 323

- $\mathcal{Z}(\widehat{\mathcal{A}}) = C^\infty(M)$.
- $\mathrm{Der}(\widehat{\mathcal{A}}) = [\Gamma(M) \otimes \mathbb{1}_n] \oplus [C^\infty(M) \otimes \mathfrak{sl}_n]$ where $\Gamma(M) = \mathrm{Der}(C^\infty(M))$ is the space of vector fields on M.
- $\{\partial_\mu\}_{\mu=1,\dots,\dim M}$ basis of derivations on the geometric part and $\{dx^\mu\}$ its dual basis of 1-forms.
- $\mathcal{M} = \widehat{\mathcal{A}}$, connection 1-form $\omega = \omega_\mu dx^\mu + \omega_\alpha \theta^\alpha = A_\mu dx^\mu + (E_\alpha - B_\alpha)\theta^\alpha$ with $A_\mu, B_\alpha \in \widehat{\mathcal{A}}$.
- Curvature $\Omega = \frac{1}{2}\Omega_{\mu\nu} dx^\mu \wedge dx^\nu + \Omega_{\mu\alpha} dx^\mu \wedge \theta^\alpha + \frac{1}{2}\Omega_{\alpha\beta}\theta^\alpha \wedge \theta^\beta$ with

$$\Omega_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu], \quad \Omega_{\mu k} = -(\partial_\mu B_\alpha - [A_\mu, B_\alpha]), \quad \Omega_{k\ell} = -([B_\alpha, B_\beta] - C^\gamma_{\alpha\beta} B_\gamma).$$

- Lagrangian: $-\frac{1}{2}\,\mathrm{tr}(\Omega_{\mu\nu}\Omega^{\mu\nu}) - \mathrm{tr}(\Omega_{\mu\alpha}\Omega^{\mu\alpha}) - \frac{1}{2}\,\mathrm{tr}(\Omega_{\alpha\beta}\Omega^{\alpha\beta})$.
- $\mathrm{NCGFT}_{\widehat{\mathcal{A}}}$ is of Yang-Mills-Higgs type...

Derivation-based NCG: $\mathcal{A} = \oplus_{i=1}^{n} \mathcal{A}_i$

Masson, T. and Nieuviarts, G. (2021). Derivation-based Noncommutative Field Theories on *AF* algebras. *International Journal of Geometric Methods in Modern Physics* 18.13, p. 2150213

Let $\mathcal{A} = \oplus_{i=1}^{n} \mathcal{A}_i$.

- $\pi^i : \mathcal{A} \to \mathcal{A}_i$ and $\iota_i : \mathcal{A}_i \to \mathcal{A}$.
- Center of $\mathcal{A}$: $\mathcal{Z}(\mathcal{A}) = \oplus_{i=1}^{r} \mathcal{Z}(\mathcal{A}_i)$.
- Derivations on $\mathcal{A}$: $\mathrm{Der}(\mathcal{A}) = \oplus_{i=1}^{r} \mathrm{Der}(\mathcal{A}_i)$ as Lie algebras and $\mathcal{Z}(\mathcal{A})$-modules.
  - $a = \oplus_{i=1}^{r} a_i \in \mathcal{A}$ and $\mathfrak{X} = \oplus_{i=1}^{r} \mathfrak{X}_i \in \mathrm{Der}(\mathcal{A})$ ➞ $\mathfrak{X}(a) = \oplus_{i=1}^{r} \mathfrak{X}_i(a_i)$
  - If $\mathrm{Der}(\mathcal{A}_i) = \mathrm{Int}(\mathcal{A}_i)$ for any $i = 1, \dots, r$, then $\mathrm{Der}(\mathcal{A}) = \mathrm{Int}(\mathcal{A}) = \oplus_{i=1}^{r} \mathrm{Int}(\mathcal{A}_i)$.
- For any $p \geq 0$, $\Omega^p_{\mathrm{Der}}(\mathcal{A}) = \oplus_{i=1}^{r} \Omega^p_{\mathrm{Der}}(\mathcal{A}_i)$.
  - $\omega \in \Omega^p_{\mathrm{Der}}(\mathcal{A})$ decomposes as $\omega = \oplus_{i=1}^{r} \omega_i$ with $\omega_i \in \Omega^p_{\mathrm{Der}}(\mathcal{A}_i)$.
  - $\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_p) = \oplus_{i=1}^{r} \omega_i(\mathfrak{X}_{1,i}, \dots, \mathfrak{X}_{p,i})$ for any $\mathfrak{X}_k = \oplus_{i=1}^{r} \mathfrak{X}_{k,i} \in \mathrm{Der}(\mathcal{A})$.
- d on $\Omega^\bullet_{\mathrm{Der}}(\mathcal{A})$ decomposes along the d_i on $\Omega^\bullet_{\mathrm{Der}}(\mathcal{A}_i)$: $\mathrm{d}\omega = \oplus_{i=1}^{r} \mathrm{d}_i \omega_i$

Derivation-based NCG: $\mathcal{A} = \oplus_{i=1}^{n} \mathcal{A}_i$

- $\mathcal{M}$ left modules on $\mathcal{A}$ such that $\mathcal{M} = \oplus_{i=1}^{r} \mathcal{M}_i$ where $\mathcal{M}_i$ is a left module on $\mathcal{A}_i$.
- ∇ connection on $\mathcal{M}$.
  - There is a unique family of connections ∇^i on the left $\mathcal{A}_i$ modules $\mathcal{M}_i$ s.t. $\nabla_{\mathfrak{X}} e = \oplus_{i=1}^{r} \nabla^i_{\mathfrak{X}_i} e_i$.
    $e = \oplus_{i=1}^{r} e_i \in \mathcal{M}$ and $\mathfrak{X} = \oplus_{i=1}^{r} \mathfrak{X}_i \in \mathrm{Der}(\mathcal{A})$
  - R_i the curvature associated to ∇^i, then $R(\mathfrak{X}, \mathfrak{Y})e = \oplus_{i=1}^{r} R_i(\mathfrak{X}_i, \mathfrak{Y}_i)e_i$.
    $\mathfrak{Y} = \oplus_{i=1}^{r} \mathfrak{Y}_i \in \mathrm{Der}(\mathcal{A})$
- Case $\mathcal{M} = \mathcal{A}$. Then with $\nabla \longrightarrow \omega \in \Omega^1_{\mathrm{Der}}(\mathcal{A})$ and $\nabla^i \longrightarrow \omega_i \in \Omega^1_{\mathrm{Der}}(\mathcal{A}_i)$.
  - $\omega = \oplus_{i=1}^{r} \omega_i$
  - $\Omega = \oplus_{i=1}^{r} \Omega_i$ for the curvatures.

Derivation-based NCGFT: $\mathcal{A} = \oplus_{i=1}^{n} M_{n_i}$

Let $\mathcal{A} = \oplus_{i=1}^{n} M_{n_i}$.

- $\mathcal{Z}(\mathcal{A}) = \oplus_{i=1}^{r} \mathbb{C}$.
- $\mathrm{Der}(\mathcal{A}) = \mathrm{Int}(\mathcal{A}) \simeq \oplus_{i=1}^{r} \mathfrak{sl}_{n_i}$
- $\{E^i_\alpha\}_{\alpha \in I_i}$ basis (anti-Hermitean matrices) of $\mathfrak{sl}_{n_i}$ where I_i is a totally ordered set of cardinal $n_i^2 - 1$.
- $g(\partial^i_\alpha, \partial^{i'}_{\alpha'}) = 0$ for $i \neq i'$ and $g^i_{\alpha\alpha'} := g(\partial^i_\alpha, \partial^i_{\alpha'}) = \mathrm{tr}(E^i_\alpha E^i_{\alpha'})$.
- $\int_{\mathcal{A}} \omega := \sum_{i=1}^{r} \int_i \omega_i$ for any $\omega = \oplus_{i=1}^{r} \omega_i \in \Omega^\bullet_{\mathrm{Der}}(\mathcal{A})$.
- Then for $\omega = \oplus_{i=1}^{r} \omega_i$ and $\omega' = \oplus_{i=1}^{r} \omega'_i \in \Omega^\bullet_{\mathrm{Der}}(\mathcal{A})$, $\omega \wedge \star\omega' = \sum_{i=1}^{r} \omega_i \wedge \star_i \omega'_i$ ($\star_i$ defined on $\Omega^\bullet_{\mathrm{Der}}(M_{n_i})$).

➞ All the structures to define a $\mathrm{NCGFT}_{\mathcal{A}}$ "decompose along i"...

Similar structures to define $\mathrm{NCGFT}_{\widehat{\mathcal{A}}}$ in a natural way ($\widehat{\mathcal{A}} = C^\infty(M) \otimes \big(\oplus_{i=1}^{r} M_{n_i} \big)$)...

This $\mathrm{NCGFT}_{\widehat{\mathcal{A}}}$ requires (almost) no choice (once $\mathcal{M} = \widehat{\mathcal{A}}$).

➞ define compatibilities between modules and derivations along the defining sequence of an AF-algebra...

Spectral triples and spectral action

- $(\mathcal{A}, \mathcal{H}, D)$ spectral triple, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ the representation on the Hilbert space $\mathcal{H}$.
- Even spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$: γ a $\mathbb{Z}_2$-grading on $\mathcal{H}$, $\gamma^\dagger = \gamma, \gamma^2 = 1, \gamma D + D\gamma = 0$ (D is odd), $\gamma\pi(a) = \pi(a)\gamma$ for any $a \in \mathcal{A}$.
- Real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$: J anti-unitary operator, $\langle J\psi_1, J\psi_2\rangle = \langle\psi_2, \psi_1\rangle$,
  - $[a, Jb^* J^{-1}] = 0$ (commutant property) and $[[D, a], Jb^* J^{-1}] = 0$ (first-order condition).
  - $\mathcal{H}$ bimodule for $a^\circ\psi = Ja^* J^{-1}\psi = \psi a$ (a° element in the opposite algebra $\mathcal{A}^\circ$).
- KO-dimension $n \mod 8$ depends on $\epsilon, \epsilon', \epsilon'' = \pm 1$: $J^2 = \epsilon$, $JD = \epsilon' DJ$, and $J\gamma = \epsilon''\gamma J$.
- $u \in \mathcal{U}(\mathcal{A})$ defines the unitary $U = \pi(u)J\pi(u)J^{-1} : \mathcal{H} \to \mathcal{H}$.
  - D modified as $D^u = D + \pi(u)[D, \pi(u)^*] + \epsilon' J\left(\pi(u)[D, \pi(u)^*]\right)J^{-1}$.
  - $\omega \in \Omega^1_U(\mathcal{A})$ (universal differential calculus) ➞ $D_\omega := D + \pi_D(\omega) + \epsilon' J\pi_D(\omega)J^{-1}$ with $\pi_D\left(a^0 \mathrm{d}_U a^1\right) := \pi(a^0)[D, \pi(a^1)]$.
  - $\omega^u := u\omega u^* + u\mathrm{d}_U u^*$ ➞ $(D_\omega)^u = D_{\omega^u}$.
- Action functional = Spectral action + Fermionic action associated to D_ω.

This $\text{NCGFT}_{\mathcal{A}}$ requires some choices.
➞ define compatibilities for representations, Dirac, grading and real operators, 1-forms ω...

AF-algebras

$\mathcal{A} = \varinjlim \mathcal{A}_n$ with:

- $\mathcal{A}_n$ finite dimensional algebra.
- $\{(\mathcal{A}_n, \phi_{n,m}) \,/\, 0 \leq n < m\}$ where $\phi_{n,m} : \mathcal{A}_n \to \mathcal{A}_m$ are one-to-one $*$-homomorphisms.
- $\phi_{m,p} \circ \phi_{n,m} = \phi_{n,p}$ for any $0 \leq n < m < p$

➡ Need only consider "one step" in the sequence: $\phi_{n,n+1} : \mathcal{A}_n \to \mathcal{A}_{n+1}$.

$\phi : \mathcal{A} = \oplus_{i=1}^{r} M_{n_i} \to \mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$ one-to-one.

- ϕ is determined up to unitary equivalence in $\mathcal{B}$ by a $s \times r$ matrix $A = (\alpha_{ki})$.
- $\alpha_{ki} \in \mathbb{N}$ is the multiplicity of the inclusion of M_{n_i} into the diagonal of M_{m_k}.
  ➡ presentation as Bratteli diagrams...
- $\iota_{\mathcal{A}}^{i} : M_{n_i} \hookrightarrow \mathcal{A}$ and $\pi_k^{\mathcal{B}} : \mathcal{B} \to M_{m_k}$ canonical inclusion and projection.

Decomposition of $\phi : \mathcal{A} = \oplus_{i=1}^{n} M_{n_i} \to \mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$

- ϕ is not necessary unital.
- ϕ is normalized such that, for any $a = \oplus_{i=1}^{r} a_i \in \mathcal{A}$,

$$\phi_k(a) := \pi_k^{\mathcal{B}} \circ \phi(a) = \begin{pmatrix} a_1 \otimes \mathbb{1}_{\alpha_{k1}} & 0 & \cdots & 0 & 0 \\ 0 & a_2 \otimes \mathbb{1}_{\alpha_{k2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_r \otimes \mathbb{1}_{\alpha_{kr}} & 0 \\ 0 & 0 & \cdots & 0 & \mathbb{0}_{n_0} \end{pmatrix} \qquad a_i \otimes \mathbb{1}_{\alpha_{ki}} = \underbrace{\begin{pmatrix} a_i & 0 & 0 & 0 \\ 0 & a_i & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_i \end{pmatrix}}_{\alpha_{ki} \text{ times}}$$

$\alpha_{ki} \geq 0$ is the multiplicity of the inclusion of M_{n_i} into M_{m_k}, $\mathbb{1}_{\alpha_{ki}}$ is the unit matrix of size α_{ki}, $\mathbb{0}_{n_0}$ is the $n_0 \times n_0$ zero matrix such that $n_0 \geq 0$ satisfies $m_j = n_0 + \sum_{i=1}^{r} \alpha_{ki} n_i$.

- $\phi_k^i := \phi_k \circ \iota_{\mathcal{A}}^i : M_{n_i} \to M_{m_k}$.
- For $\alpha_{ki} > 0$ and $1 \leq \ell \leq \alpha_{ki}$, let $\boldsymbol{\phi_{k,\ell}^i : M_{n_i} \to M_{m_k}}$ which inserts a_i at the ℓ-th entry on the diagonal of $\mathbb{1}_{\alpha_{ki}}$.
- For any $a_{i_1} \in M_{n_{i_1}}$, $b \in M_{n_{i_2}}$, any $1 \leq i_1, i_2 \leq r$, any $1 \leq \ell_1 \leq \alpha_{ki_1}$, any $1 \leq \ell_2 \leq \alpha_{ki_2}$,

$$\phi_{k,\ell_1}^{i_1}(a_{i_1}) \phi_{k,\ell_2}^{i_2}(b_{i_2}) = \delta_{i_1,i_2} \delta_{\ell_1,\ell_2} \phi_{k,\ell_1}^{i_1}(a_{i_1} b_{i_1}).$$

Requirements for a sequence of NCGFT on top of an *AF*-algebra

$\phi : \mathcal{A} = \oplus_{i=1}^{r} M_{n_i} \to \mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$ one-to-one.

Consider a $\text{NCGFT}_{\mathcal{A}}$ and a $\text{NCGFT}_{\mathcal{B}}$.

What do we require on these NCGFT to be **ϕ-compatible** (and so part of a sequence of NCGFT's)?

1. Possibility to find in $\text{NCGFT}_{\mathcal{B}}$ the degrees of freedom (DOF) defined in $\text{NCGFT}_{\mathcal{A}}$.
   - Keep track of "gauge fields" and "particles" from $\mathcal{A}$ to $\mathcal{B}$ ➞ **"inherited DOF"**.
   - To be able to identify new DOF in $\text{NCGFT}_{\mathcal{B}}$.
   - Similar to GUT where SSBM relate DOF in the opposite direction...

2. Possibility to compare the action (or Lagrangian) defined by $\text{NCGFT}_{\mathcal{B}}$ with the one defined by $\text{NCGFT}_{\mathcal{A}}$.
   - Since we track DOF, we want also to track the equations they satisfy...
   - Understand the mixing between new DOF and inherited ones.

3. Try to define these comparisons in a "natural way" in the chosen framework...

Module structures

$\phi : \mathcal{A} = \oplus_{i=1}^{r} M_{n_i} \to \mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$ one-to-one.

- Both framework use left modules (promoted to bimodules for Real Spectral Triples).
  $\mathcal{M}$ left module on $\mathcal{A}$ and $\mathcal{N}$ left module on $\mathcal{B}$.
- A one-to-one linear map $\phi_{\text{Mod}} : \mathcal{M} \to \mathcal{N}$ is **ϕ-compatible** iff
  $$\phi_{\text{Mod}}(ae) = \phi(a)\phi_{\text{Mod}}(e) \quad \text{for any } a \in \mathcal{A} \text{ and } e \in \mathcal{M}.$$
- Left modules on $\mathcal{A}$ are $\mathcal{M} = \oplus_{i=1}^{r} \mathbb{C}^{n_i} \otimes \mathbb{C}^{\mu_i}$. Left modules on $\mathcal{B}$ are $\mathcal{N} = \oplus_{k=1}^{s} \mathbb{C}^{m_k} \otimes \mathbb{C}^{\nu_k}$.
  μ_i = multiplicity of the irreducible representation of M_{n_i} on $\mathbb{C}^{n_i}$.
- Case $\mathcal{M} = \mathcal{A}$ and $\mathcal{N} = \mathcal{B}$: $\phi_{\text{Mod}} = \phi : \mathcal{M} \to \mathcal{N}$ is ϕ-compatible.
- General case: inject $\mathcal{M}_i := \mathbb{C}^{n_i} \otimes \mathbb{C}^{\alpha_i}$ α_{ki} times (as rows) into $\mathcal{N}_k := \mathbb{C}^{m_k} \otimes \mathbb{C}^{\beta_k}$ (when $\alpha_{ki} > 0$).
  - β_k must be large enough to accept the largest α_i.
  - ϕ_{Mod} decomposes as $\phi^i_{\text{Mod},k} := \pi^{\mathcal{N}}_k \circ \phi_{\text{Mod}} \circ \iota^i_{\mathcal{M}} : \mathcal{M}_i \to \mathcal{N}_k$
  - For any $1 \leq \ell \leq \alpha_{ki}$, let $\boldsymbol{\phi^i_{\text{Mod},k,\ell} : \mathcal{M}_i \to \mathcal{N}_k}$ which inserts $e_i \in \mathcal{M}_i$ at the ℓ-th row.

Operators on Hilbert spaces (Spectral triples case)

$\phi : \mathcal{A} \to \mathcal{B}$ one-to-one, represented on Hilbert spaces $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ by $\pi_\mathcal{A}$ and $\pi_\mathcal{B}$.

- $\phi_\mathcal{H} : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{B}$ is ϕ-compatible iff $\phi_\mathcal{H}(a\psi) = \phi(a)\phi_\mathcal{H}(\psi)$ for any $a \in \mathcal{A}$ and $\psi \in \mathcal{H}_\mathcal{A}$.
- Decompose $\mathcal{H}_\mathcal{B} = \phi_\mathcal{H}(\mathcal{H}_\mathcal{A}) \oplus \phi_\mathcal{H}(\mathcal{H}_\mathcal{A})^\perp$ (in a $\phi_\mathcal{H}$-dependent way).
- Any operator B on $\mathcal{H}_\mathcal{B}$ decomposes as $B = \begin{pmatrix} B^\phi_\phi & B^\perp_\phi \\ B^\phi_\perp & B^\perp_\perp \end{pmatrix}$ (obvious notations).
- Consider two operators A on $\mathcal{H}_\mathcal{A}$ and B on $\mathcal{H}_\mathcal{B}$.
  - They are **ϕ-compatible** iff $\boldsymbol{\phi_\mathcal{H}(A\psi) = B^\phi_\phi \phi_\mathcal{H}(\psi)}$ for any $\psi \in \mathcal{H}_\mathcal{A}$ (equality in $\phi_\mathcal{H}(\mathcal{H}_\mathcal{A})$).
  - They are **strong ϕ-compatible** iff $\boldsymbol{\phi_\mathcal{H}(A\psi) = B\phi_\mathcal{H}(\psi)}$ for any $\psi \in \mathcal{H}_\mathcal{A}$ (equality in $\mathcal{H}_\mathcal{B}$).
- Strong ϕ-compatibility implies ϕ-compatibility.
- Results on the behavior of (strong) ϕ-compatibility under many operations on operators: sum, composition, adjointness…
- $\pi_\mathcal{A}(a)$ and $\pi_\mathcal{B}(\phi(a))$ are strong ϕ-compatible for any $a \in \mathcal{A}$ and $\pi_\mathcal{B}(\phi(a))$ is diagonal.

Bimodule structures (Spectral triples case)

$\phi : \mathcal{A} = \oplus_{i=1}^{r} M_{n_i} \to \mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$ one-to-one.

- $\mathcal{A}^\circ$ and $\mathcal{B}^\circ$ the opposite algebras of $\mathcal{A}$ and $\mathcal{B}$.
  Then $\phi^\circ : \mathcal{A}^\circ \to \mathcal{B}^\circ$ defined by $\phi^\circ(a^\circ) := \phi(a)^\circ$ is a morphism of algebras.
- $\mathcal{A}^e := \mathcal{A} \otimes \mathcal{A}^\circ$ and $\mathcal{B}^e := \mathcal{B} \otimes \mathcal{B}^\circ$ the so-called envelopping algebras of $\mathcal{A}$ and $\mathcal{B}$.
  Then $\phi^e := \phi \otimes \phi^\circ : \mathcal{A}^e \to \mathcal{B}^e$ is a morphism of algebras.
- **Bimodule on $\mathcal{A}$ = left module on $\mathcal{A}^e$.**
- Bimodules on $\mathcal{A}$ are $\mathcal{M} = \oplus_{i,j=1}^{r} \mathbb{C}^{n_i} \otimes \mathbb{C}^{\mu_{ij}} \otimes \mathbb{C}^{n_j\circ}$. Bimodules on $\mathcal{B}$ are $\mathcal{N} = \oplus_{k,\ell=1}^{s} \mathbb{C}^{m_k} \otimes \mathbb{C}^{\nu_{k\ell}} \mathbb{C}^{m_\ell\circ}$.
  μ_{ij} is the multiplicity of the irreducible representation of $M_{n_i} \otimes M_{n_j}^\circ$ on $\mathbb{C}^{n_i} \otimes \mathbb{C}^{n_j\circ}$
  ($\mathbb{C}^{n_j\circ}$ are row vectors on which M_{n_j} acts on the right)
- A one-to-one linear map $\phi_{\text{Mod}} : \mathcal{M} \to \mathcal{N}$ between two $\mathcal{A}$ and $\mathcal{B}$ bimodules is **ϕ-compatible** iff ϕ_{Mod} is ϕ^e-compatible between the corresponding $\mathcal{A}^e$ and $\mathcal{B}^e$ left modules.
- For Real Spectral Triples, bimodule structures on $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ are defined by $J_\mathcal{A}$ and $J_\mathcal{B}$.
  - Suppose $\phi_\mathcal{H} : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{B}$ is ϕ-compatible (left module definition).
  - **Then $\phi_\mathcal{H}$ is ϕ^e-compatible if and only if $J_\mathcal{A}$ and $J_\mathcal{B}$ are strong ϕ-compatible.**

  ➡ Suggest to always consider strong ϕ-compatibility between $J_\mathcal{A}$ and $J_\mathcal{B}$...

ϕ-compatibility and derivations

- ϕ does not relate the centers of $\mathcal{A}$ and $\mathcal{B}$ ➞ no "general" map to inject $\text{Der}(\mathcal{A})$ into $\text{Der}(\mathcal{B})$...
- Strategy: **keep track of the derivations in $\text{Der}(\mathcal{B})$ which "come from" derivations in $\text{Der}(\mathcal{A})$**.
  (These derivations will propagate along the sequence and new derivations will be introduced at each step)
- For any $i = 1, \dots, r$, let $\{\partial^i_{\mathcal{A},\alpha} := \text{ad}_{E^i_{\mathcal{A},\alpha}}\}_{\alpha \in I_i}$ be an basis of $\text{Der}(\mathcal{A}_i) = \text{Int}(M_{n_i})$.
  $E^i_{\mathcal{A},\alpha} \in \mathfrak{sl}_{n_i}$ and I_i is a totally ordered set of cardinal $n_i^2 - 1$.
- For any $k = 1, \dots, s$, introduce a basis of $\text{Der}(\mathcal{B}_k) = \text{Int}(M_{m_k})$ in two steps:

① Let $J^\phi_k := \{(i, \ell, \alpha) \,/\, i \in \{1, \dots, r\},\ \ell \in \{1, \dots, \alpha_{ki}\},\ \alpha \in I_i\}$. J^ϕ_k has a (natural) total order.

- For any $\beta = (i, \ell, \alpha) \in J^\phi_k$, define $E^k_{\mathcal{B},\beta} := \phi^i_{k,\ell}(E^i_{\mathcal{A},\alpha}) \in \mathfrak{sl}_{m_k}$ and $\partial^k_{\mathcal{B},\beta} := \text{ad}_{E^k_{\mathcal{B},\beta}} \in \text{Der}(\mathcal{B}_k)$, **inherited derivations**.
- $g_\mathcal{A}$ and $g_\mathcal{B}$ the metrics on $\mathcal{A}$ and $\mathcal{B}$ as before, for any $\beta = (i, \ell, \alpha)$ and $\beta' = (i', \ell', \alpha')$, one has $g^k_{\mathcal{B},\beta\beta'} = \delta_{ii'}\delta_{\ell\ell'} g^i_{\mathcal{A},\alpha\alpha'}$.
  (reason for the change of normalization...)

② Complete the family $\{\partial^k_{\mathcal{B},\beta}\}_{\beta \in J^\phi_k}$ into a full basis $\{\partial^k_{\mathcal{B},\beta}\}_{\beta \in J_k}$ of $\text{Der}(\mathcal{B}_k)$.

- $J_k = J^\phi_k \cup J^c_k$ where J^c_k is a complementary (total ordered) set to get $\text{card}(J_k) = m_k^2 - 1$.
- Require $g_\mathcal{B}(\partial^k_{\mathcal{B},\beta}, \partial^k_{\mathcal{B},\beta'}) = 0$ for any $\beta \in J^\phi_k$ and $\beta' \in J^c_k$.
  ➞ $g_\mathcal{B}$ is block diagonal and decomposes $\text{Der}(\mathcal{B}_j)$ into two orthogonal summands (inherited *vs* new derivations).

ϕ-compatibility and derivations

- The previous construction can start with an orthogonal basis of $\mathcal{A}$ and end with an orthogonal basis of $\mathcal{B}$.
- Same for orthonormal basis...
- Let $1 \leq j \leq s$, $1 \leq i, i' \leq r$, $1 \leq \ell \leq \alpha_{ki}$, $1 \leq \ell' \leq \alpha_{ki'}$, $\alpha \in I_i$, $\alpha' \in I_{i'}$, $a_{i'} \in \mathcal{A}_{i'}$, one has

$$\partial^k_{\mathcal{B},(i,\ell,\alpha)} \cdot \phi^{i'}_{k,\ell'}(a_{i'}) = \phi^i_{k,\ell}(\partial^i_{\mathcal{A},\alpha}) \cdot \phi^{i'}_{k,\ell'}(a_{i'}) = \delta_{i,i'} \delta_{\ell,\ell'} \phi^i_{k,\ell}(\partial^i_{\mathcal{A},\alpha} \cdot a_{i'})$$

$$[\partial^k_{\mathcal{B},(i,\ell,\alpha)}, \partial^k_{\mathcal{B},(i',\ell',\alpha')}] = [\phi^i_{k,\ell}(\partial^i_{\mathcal{A},\alpha}), \phi^{i'}_{k,\ell'}(\partial^{i'}_{\mathcal{A},\alpha'})] = \delta_{i,i'} \delta_{\ell,\ell'} \phi^i_{k,\ell}([\partial^i_{\mathcal{A},\alpha}, \partial^i_{\mathcal{A},\alpha'}])$$

➞ ϕ-compatibility of the Lie structures on inherited derivations...

ϕ-compatibility and forms

- A form $\omega = \oplus_{i=1}^r \omega_i \in \Omega^\bullet_{\text{Der}}(\mathcal{A})$ is ϕ-compatible with a form $\eta = \oplus_{k=1}^s \eta_k \in \Omega^\bullet_{\text{Der}}(\mathcal{B})$ iff for any $1 \leq i \leq r$, $1 \leq k \leq s$, $1 \leq \ell \leq \alpha_{ki}$, ω_i and η_k have the same degree p and for any $\partial^i_{\mathcal{A},\alpha_1}, \dots, \partial^i_{\mathcal{A},\alpha_p} \in \text{Der}(\mathcal{A}_i)$ $(\alpha_q \in I_i)$, one has

$$\phi^i_{k,\ell}\left(\omega_i(\partial^i_{\mathcal{A},\alpha_1}, \dots, \partial^i_{\mathcal{A},\alpha_p})\right) = \eta_k\left(\phi^i_{k,\ell}(\partial^i_{\mathcal{A},\alpha_1}), \dots, \phi^i_{k,\ell}(\partial^i_{\mathcal{A},\alpha_p})\right)$$

- ϕ-compatibility of forms is compatible with products and differentials.
- ϕ-compatibility for 1-forms.
  - Let $\omega = \oplus_{i=1}^r \omega^i_\alpha \otimes \theta^\alpha_{\mathcal{A},i}$ for $\omega^i_\alpha \in \mathcal{A}_i$ and $\eta = \oplus_{k=1}^s \eta^k_\beta \otimes \theta^\beta_{\mathcal{B},k}$ for $\eta^k_\beta \in \mathcal{B}_k$.
  - ω and η ϕ-compatible then $\phi^i_{k,\ell}(\omega^i_\alpha) = \eta^k_{(i,\ell,\alpha)}$.
  - The components η^k_β of η in the "inherited directions" $\phi^i_{k,\ell}(\partial^i_{\mathcal{A},\alpha})$'s are inherited from components in ω.
  - Control on the inherited degrees of freedom for forms, and so for (NC) connections...

ϕ-compatibility and connections

- $\mathcal{A}$-module $\mathcal{M}$ and a $\mathcal{B}$-module $\mathcal{N}$ with one-to-one ϕ-compatible map $\phi_{\text{Mod}} : \mathcal{M} \to \mathcal{N}$.
- $\nabla^{\mathcal{M}} = \oplus_{i=1}^{r} \nabla^{\mathcal{M},i}$ and $\nabla^{\mathcal{N}} = \oplus_{k=1}^{s} \nabla^{\mathcal{N},k}$ connections on $\mathcal{M}$ and $\mathcal{N}$.
- $\nabla^{\mathcal{M}}$ and $\nabla^{\mathcal{N}}$ are ϕ-compatible iff, for any $1 \leq i \leq r$, $1 \leq k \leq s$, $1 \leq \ell \leq \alpha_{ki}$, $\alpha \in I_i$, one has

$$\phi^{i}_{\text{Mod},k,\ell}\left(\nabla^{\mathcal{M},i}_{\partial^{i}_{\mathcal{A},\alpha}} e_i\right) = \nabla^{\mathcal{N},k}_{\phi^{i}_{k,\ell}(\partial^{i}_{\mathcal{A},\alpha})} \phi^{i}_{\text{Mod},k,\ell}(e_i).$$

- Case $\mathcal{M} = \mathcal{A}$ and $\mathcal{N} = \mathcal{B}$. Introduce the connection 1-forms $\omega_{\mathcal{M}}$ and $\omega_{\mathcal{N}}$ for $\nabla^{\mathcal{M}}$ and $\nabla^{\mathcal{N}}$.
  If $\omega_{\mathcal{M}}$ and $\omega_{\mathcal{N}}$ are ϕ-compatible, then $\nabla^{\mathcal{M}}$ and $\nabla^{\mathcal{N}}$ are ϕ-compatible.

  ⚠ The opposite result is false...

ϕ-compatibility and Lagrangians

What about Lagrangians?

- Consider $\mathcal{A} = \oplus_{i=1}^{r} M_{n_i}$ and $\mathcal{B} = M_m$.
- Suppose that $\phi : \mathcal{A} \to \mathcal{B}$ includes α_i times M_{n_i} on the diagonal of M_m.
- Let $\omega = \oplus_{i=1}^{r} \omega_i \in \Omega^{\bullet}_{\text{Der}}(\mathcal{A})$ and $\eta \in \Omega^{\bullet}_{\text{Der}}(\mathcal{B})$ be ϕ-compatible.
- Suppose η vanishes on every derivation $\partial_{\mathcal{B},\beta}$ with $\beta \in J^c$. Then

$$\int_{\mathcal{B}} \eta \wedge \star_{\mathcal{B}} \eta = \textstyle\sum_{i=1}^{r} \alpha_i \displaystyle\int_i \omega_i \wedge \star_i \omega_i$$

- The Lagrangian on $\mathcal{B}$ decomposes along 3 kinds of terms: “inherited” + “inherited + new” + “new”.
- “inherited” = all the terms (with possible weights) of the Lagrangian on $\mathcal{A}$.

Sequence of NCGFT

- Consider a sequence of finite dimensional algebras $\mathcal{A} = \oplus_{i=1}^{r} M_{n_i}$ (defining an AF-algebra).
- Consider the sequence of almost commutative algebras $\widehat{\mathcal{A}} = C^\infty(M) \otimes \mathcal{A}$ (M is fixed).
  - One can extend $\phi : \mathcal{A} \to \mathcal{B}$ to $\hat{\phi} : \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$.
  - One can extend all the definitions of ϕ-compatibility to $\hat{\phi}$-compatibility.

  ➞ $C^\infty(M)$ is quite "passive" in this extension...
- Construct a sequence of $\hat{\phi}$-compatible NCGFT$_{\widehat{\mathcal{A}}}$.
- One can follow the degrees of freedom from NCGFT$_{\widehat{\mathcal{A}}}$ to NCGFT$_{\widehat{\mathcal{B}}}$.
- The Lagrangian in NCGFT$_{\widehat{\mathcal{B}}}$ contains weighted terms of the Lagrangian in NCGFT$_{\widehat{\mathcal{A}}}$.
- Gauge transformations in NCGFT$_{\widehat{\mathcal{A}}}$ and NCGFT$_{\widehat{\mathcal{B}}}$ are also $\hat{\phi}$-compatible...

ϕ-compatibility of spectral triples

Let $\phi : \mathcal{A} \to \mathcal{B}$ be one-to-one and $\phi_{\mathcal{H}} : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{B}}$ be a ϕ-compatible map.
$\mathcal{A}$ and $\mathcal{B}$ finite dimensional ➡ ignore analytical properties...
(see Floricel, R. and Ghorbanpour, A. (2019). On inductive limit spectral triples. *Proceedings of the American Mathematical Society* 147.8, pp. 3611–3619)

- Two odd spectral triples $(\mathcal{A}, \mathcal{H}_{\mathcal{A}}, D_{\mathcal{A}})$ and $(\mathcal{B}, \mathcal{H}_{\mathcal{B}}, D_{\mathcal{B}})$ are ϕ-compatible iff $D_{\mathcal{A}}$ is ϕ-compatible with $D_{\mathcal{B}}$.
- Two real spectral triples $(\mathcal{A}, \mathcal{H}_{\mathcal{A}}, D_{\mathcal{A}}, J_{\mathcal{A}})$ and $(\mathcal{B}, \mathcal{H}_{\mathcal{B}}, D_{\mathcal{B}}, J_{\mathcal{B}})$ are ϕ-compatible iff $D_{\mathcal{A}}$ (resp. $J_{\mathcal{A}}$) is ϕ-compatible with $D_{\mathcal{B}}$ (resp. $J_{\mathcal{B}}$).
- Two even spectral triples $(\mathcal{A}, \mathcal{H}_{\mathcal{A}}, D_{\mathcal{A}}, \gamma_{\mathcal{A}})$ and $(\mathcal{B}, \mathcal{H}_{\mathcal{B}}, D_{\mathcal{B}}, \gamma_{\mathcal{B}})$ are ϕ-compatible iff $D_{\mathcal{A}}$ (resp. $\gamma_{\mathcal{A}}$) is ϕ-compatible with $D_{\mathcal{B}}$ (resp. $\gamma_{\mathcal{B}}$).
- Strong ϕ-compatibility of spectral triples can be defined in an similar way.

- If two (odd/even) real spectral triples are strong ϕ-compatible, then they have the same KO-dimension (mod 8).
- If two (odd/even) real spectral triples are ϕ-compatible and $J_{\mathcal{A}}$ and $J_{\mathcal{B}}$ are strong ϕ-compatible, then they have the same KO-dimension (mod 8).

Sequence of NCGFT constructed on spectral triples

- Suppose that $D_{\mathcal{B}}$ is ϕ-compatible with $D_{\mathcal{A}}$.
  1. For any $\omega \in \Omega^1_U(\mathcal{A})$, $\pi_{D_{\mathcal{B}}} \circ \phi(\omega)$ is ϕ-compatible with $\pi_{D_{\mathcal{A}}}(\omega)$.
  2. Suppose that $J_{\mathcal{B}}$ is strong ϕ-compatible with $J_{\mathcal{A}}$.
     For any unitaries $u_{\mathcal{A}} \in \mathcal{A}$ and $u_{\mathcal{B}} \in \mathcal{B}$ such that $\pi_{\mathcal{A}}(u_{\mathcal{A}})$ and $\pi_{\mathcal{B}}(u_{\mathcal{B}})$ are ϕ-compatible and $\pi_{\mathcal{B}}(u_{\mathcal{B}})$ is diagonal in the matrix decomposition, $D^{u_{\mathcal{B}}}_{\mathcal{B}}$ is ϕ-compatible with $D^{u_{\mathcal{A}}}_{\mathcal{A}}$.
  3. Using the hypothesis of the previous points, $D^{u_{\mathcal{B}}}_{\mathcal{B},\phi(\omega)}$ is ϕ-compatible with $D^{u_{\mathcal{A}}}_{\mathcal{A},\omega}$.
- Similar result for strong ϕ-compatibility…
- We have all the tools to define sequences of NCGFT on spectral triples…

- Case $\phi : \mathcal{A} = \oplus_{i=1}^{n} M_{n_i} \to \mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$
  - Spectral triples are classified (described) by Krajewski diagrams.
  - (strong) ϕ-compatibility is then implemented between Krajewski diagrams.
  - Gaston Nieuviarts will describe this situation in his talk.

General considerations

$$\mathcal{A} = \oplus_{i=1}^{r} M_{n_i}, \quad \widehat{\mathcal{A}} := C^\infty(M) \otimes \mathcal{A} = \oplus_{i=1}^{r} C^\infty(M) \otimes M_{n_i}, \quad \mathcal{M} = \widehat{\mathcal{A}}.$$

- Connection 1-form: $\omega = \oplus_{i=1}^{r} \omega_i$ with $\omega_i = A^i_{\mathcal{A},\mu} \mathrm{d}x^\mu + (E^i_{\mathcal{A},\alpha} - B^i_{\mathcal{A},\alpha})\theta^\alpha_{\mathcal{A},i}$
- Curvature: $\Omega_i = \frac{1}{2}\Omega^i_{\mu_1\mu_2} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} + \Omega^i_{\mu\alpha} \mathrm{d}x^\mu \wedge \theta^\alpha_{\mathcal{A},i} + \frac{1}{2}\Omega^i_{\alpha_1\alpha_2} \theta^{\alpha_1}_{\mathcal{A},i} \wedge \theta^{\alpha_2}_{\mathcal{A},i}$ with
$$\Omega^i_{\mu_1\mu_2} = \partial_{\mu_1} A^i_{\mathcal{A},\mu_2} - \partial_{\mu_2} A^i_{\mathcal{A},\mu_1} - [A^i_{\mathcal{A},\mu_1}, A^i_{\mathcal{A},\mu_2}], \qquad \Omega^i_{\mu\alpha} = -(\partial_\mu B^i_{\mathcal{A},\alpha} - [A^i_{\mathcal{A},\mu}, B^i_{\mathcal{A},\alpha}]),$$
$$\Omega^i_{\alpha_1\alpha_2} = -([B^i_{\mathcal{A},\alpha_1}, B^i_{\mathcal{A},\alpha_2}] - C(n_i)^{\alpha_3}_{\alpha_1\alpha_2} B^i_{\mathcal{A},\alpha_3}).$$
- Action: $S = -\sum_{i=1}^{r} \int_M \left(\frac{1}{2} \operatorname{tr}(\Omega^i_{\mu_1\mu_2}\Omega^{i,\mu_1\mu_2}) + \operatorname{tr}(\Omega^i_{\mu\alpha}\Omega^{i,\mu\alpha}) + \frac{1}{2} \operatorname{tr}(\Omega^i_{\alpha_1\alpha_2}\Omega^{i,\alpha_1\alpha_2}) \right) \sqrt{|g_M|}\mathrm{d}x$
- Similar for $\mathcal{B} = \oplus_{k=1}^{s} M_{m_k}$ and assume $\hat{\phi}$-compatibility between connection 1-forms on $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$.

- Numerical explorations of the masses generated by the SSBM and constrained by $\hat{\phi}$-compatibility:
  - Fix the DOF in $B^i_{\mathcal{A},\alpha} = B^{i,\alpha'}_{\mathcal{A},\alpha} E^i_{\mathcal{A},\alpha'} + iB^{i,0}_{\mathcal{A},\alpha} \mathbb{1}_{n_i}$ ➞ fixes masses for the gauge fields $A^{i,\alpha}_{\mathcal{A},\mu}$
  - $\hat{\phi}$-compatibility ➞ transports these DOF into $B^j_{\mathcal{B},\beta} = B^{j,\beta'}_{\mathcal{B},\beta} E^j_{\mathcal{B},\beta'} + iB^{j,0}_{\mathcal{B},\beta} \mathbb{1}_{m_i}$ (inherited DOF).
  - SSBM on $\widehat{\mathcal{B}}$ ➞ fixes new DOF in $B^j_{\mathcal{B},\beta}$ with the constraints on the inherited DOF ➞ fixes masses for the $A^{j,\beta}_{\mathcal{B},\mu}$
  - How the masses of the $A^{j,\beta}_{\mathcal{B},\mu}$ are related to the masses of the $A^{i,\alpha}_{\mathcal{A},\mu}$ through the constraints imposed by ϕ?

Numerical explorations

- The space of configuration for the $B^i_{\mathcal{A},\alpha}$ is very large...
- Two special configurations for the minimum:

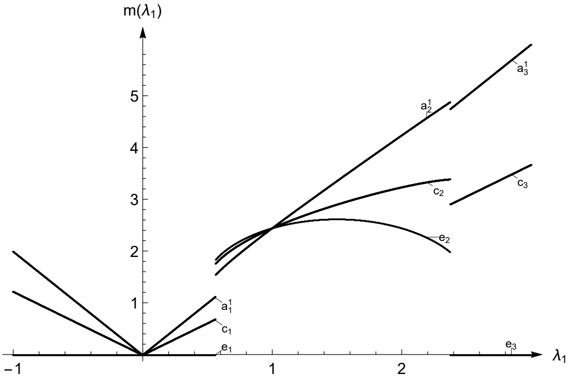
1. $B^i_{\mathcal{A},\alpha} = 0$, "null configuration" ➞ null masses for the $A^{i,\alpha}_{\mathcal{A},\mu}$.
2. $B^i_{\mathcal{A},\alpha} = E^i_{\mathcal{A},\alpha}$, "basis configuration" ➞ masses $\sqrt{2n_i}$ for the $A^{i,\alpha}_{\mathcal{A},\mu}$.

- Reduce the number of parameters to the $\lambda_i \in [-1, 3]$ with $B^i_{\mathcal{A},\alpha} = \lambda_i E^i_{\mathcal{A},\alpha}$ (= interpolation for $\lambda_i \in [0, 1]$).

  ➞ SSBM on $\widehat{\mathcal{B}}$ performed along the constraints induced by these configurations...

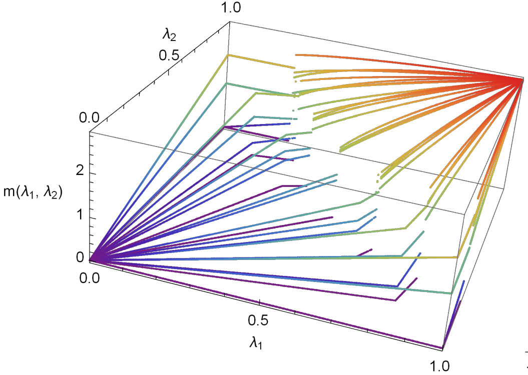
- Use MATHEMATICA.
- Numerical exploration for the cases: $M_2 \to M_3$, $M_2 \oplus M_2 \to M_4$, $M_2 \oplus M_2 \to M_5$, $M_2 \oplus M_3 \to M_5$.

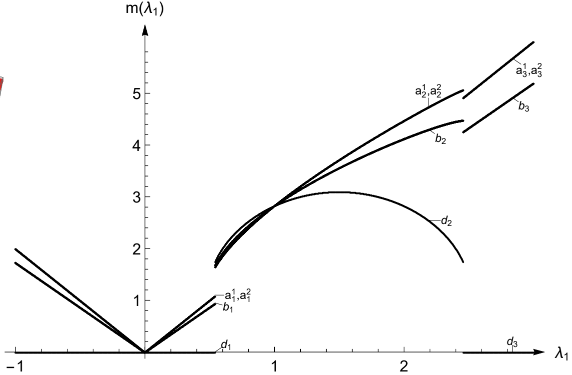
$M_2 \to M_3$



- Mass spectrum is not continuous.
- Several branches with degeneracies 3 (inherited DOF from M_2), 4, 1 (➡ $8 = 3^2 - 1$).
- Masses are preserved for inherited DOF for λ_1 close to 0 (numerically).

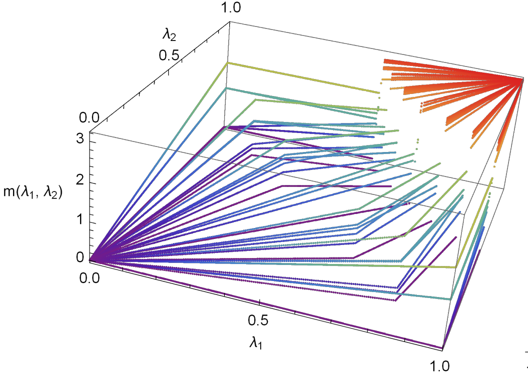
$M_2 \oplus M_2 \to M_4$

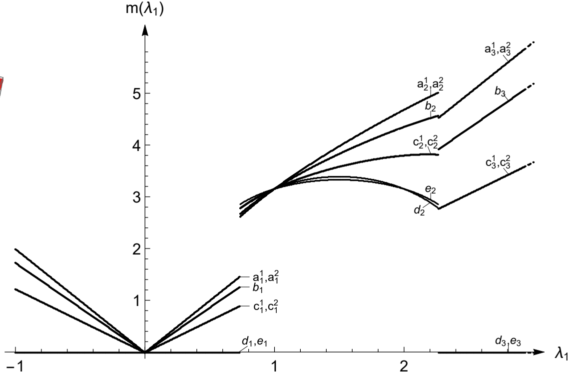




- On the left: square $(\lambda_1, \lambda_2) \in [-1, 3]^2$ (along selected paths). On the right: diagonal $\lambda_1 = \lambda_2$.
- Degeneracies 3 (inherited DOF from M_2), 3 (inherited DOF from M_2), 8, 1 (➡ $15 = 4^2 - 1$).
- Masses are preserved for inherited DOF for λ_1, λ_2 close to 0 (numerically).

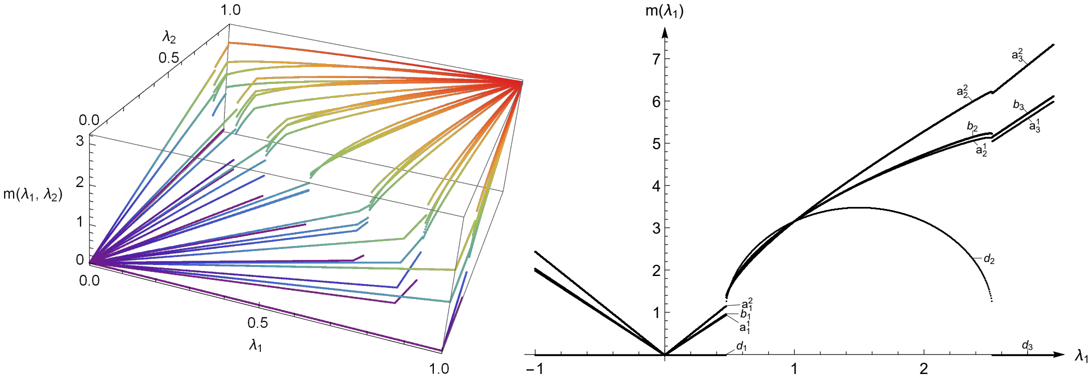
$M_2 \oplus M_2 \to M_5$





- Larger discontinuity and different position.
- Degeneracies 3 (inherited DOF from M_2), 3 (inherited DOF from M_2), 8, 4, 4, 1, 1 (➨ $24 = 5^2 - 1$).
- Masses are preserved for inherited DOF for λ_1, λ_2 close to 0 (numerically).

$M_2 \oplus M_3 \to M_5$



- Smaller discontinuity.
- Degeneracies 3 (inherited DOF from M_2), 8 (inherited DOF from M_3), 12, 1 (➡ $24 = 5^2 - 1$).
- Masses are preserved for inherited DOF for λ_1, λ_2 close to 0 (numerically).

Comments on the numerical explorations...

- Rich typology of mass spectra.
- We can “follow” the inherited DOF; their masses are preserved near the null configuration.
- Phenomenology is different for
  - the new DOF which commute with inherited DOF,
  - the new DOF which do not commute with inherited DOF.
- Interesting to study the “conflictual situations” $B^1_{\mathcal{A},\alpha} = 0$ and $B^2_{\mathcal{A},\alpha} = E^2_{\mathcal{A},\alpha}$.
- Position of the first discontinuity related to the ratio “number new of DOF”/“number of inherited DOF”.
- But this numerical study is based on strong simplifications: need more explorations...

Thank you for your attention