Dual of the dual for the distance in noncommutative geometry

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Introduction

In noncommutative geometry, Connes has defined a distance between states of an algebra $\mathcal{A}$, as a supremum.

For $\mathcal{A}$ commutative, this distance coincides with Kantorovich's dual formulation of the Wasserstein distance (or oder 1) between probability distributions in the theory of optimal transport.

The Wasserstein distance is an infimum, defined by minimising a cost.

- Is there some noncommutative cost associated with Connes distance ?
- Is there a dual formula of Connes distance as an infimum ?

Outline:

I. The metric aspect of noncommutative geometry

- spectral triple
- spectral distance
- optimal transport

II. Noncommutative cost

- State as probability measures
- Wasserstein distance on the space of states
- examples
- counterexample

III. Dual formula for the spectral distance

- Pull-back of a derivation
- The spectral distance as an infimum

I. The metric aspect of noncommutative geometry

Spectral triple

An involutive algebra $\mathcal{A}$, a faithful representation on $\mathcal{H}$, an operator D on $\mathcal{H}$ such that $[D, a]$ is bounded and $a[D - \lambda\mathbb{I}]^{-1}$ is compact for any $a \in \mathcal{A}$ and $\lambda \notin$ Sp D.

When a set of conditions is satisfied, then

Theorem Connes 1996-2008

$\mathcal{M}$ a compact Riemann manifold, then

$$(C^\infty(\mathcal{M}),\ \Omega^\bullet(\mathcal{M}),\ d + d^\dagger)$$

is a spectral triple.

When $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple with $\mathcal{A}$ unital commutative, then there exists a compact Riemannian manifold $\mathcal{M}$ such that $\mathcal{A} = C^\infty(\mathcal{M})$.

- The theorem extends to spin manifolds:

$$\Omega^\bullet(\mathcal{M}) \to L^2(\mathcal{M}, S), \qquad d + d^\dagger \to \partial\!\!\!/.$$

Spectral distance

Whatever $\mathcal{A}$, commutative or not, one defines on its space of states $\mathcal{S}(\mathcal{A})$, that is the set of normalized ($\mathbb{I} \to 1$), positive ($a^*a \to \mathbb{R}^+$) linear maps $\mathcal{A} \to \mathbb{C}$, the spectral distance (possibly infinite)

$$d_D(\varphi, \tilde{\varphi}) = \sup_{a \in \mathcal{A}} \{ |\varphi(a) - \tilde{\varphi}(a)| \, , \; \|[D, a]\| \leq 1 \}.$$

For the spectral triple of a compact riemannian manifold, this distance evaluated on the space of pure states

$$\mathcal{P}(C^\infty(\mathcal{M})) \simeq \mathcal{M}$$

gives back the geodesic distance on $\mathcal{M}$:

$$d_{d+d^\dagger}(\delta_x, \delta_y) = d_{\partial\!\!\!/}(\delta_x, \delta_y) = d_{\text{geo}}(x, y)$$

where $\delta_x : f \mapsto f(x)$ for $x \in \mathcal{M}$.

- Quid of the spectral distance between non-pure states ?

Optimal transport

$\mathcal{X}$ a locally compact Polish space, $c(x, y)$ a positive real function, the "cost". The minimal work W required to transport the probability measure μ_1 to μ_2 is

$$W(\mu_1, \mu_2) \doteq \inf_{\pi} \int_{\mathcal{X} \times \mathcal{X}} c(x, y) \, \mathrm{d}\pi$$

where the infimum is over all transportation plans, i.e. measures π on $\mathcal{X} \times \mathcal{X}$ with marginals μ_1, μ_2.

When the cost function c is a distance d, then

$$W(\mu_1, \mu_2) \doteq \inf_{\pi} \int_{\mathcal{X} \times \mathcal{X}} d(x, y) \, \mathrm{d}\pi$$

is a distance (possibly infinite) on the space of probability measures on $\mathcal{X}$, called the Wasserstein distance of order 1.

Proposition Rieffel 99, then D'Andrea, P.M. 2009

Let $\mathcal{X} = \mathcal{M}$ be a complete, connected, without boundary, Riemannian manifold. Then

$$W(\varphi, \tilde{\varphi}) = d_{d+d^\dagger}(\varphi, \tilde{\varphi}) \qquad \forall \varphi, \tilde{\varphi} \in \mathcal{S}(C_0(\mathcal{M}))$$

where W is the Wasserstein distance associated to the cost d_{geo}, while $d_{d+d^\dagger}$ is the spectral distance associated to

$$\left(C_c^\infty(\mathcal{M}), \Omega^\bullet(\mathcal{M}), d + d^\dagger\right).$$

i. Kantorovich duality:

$$W(\varphi, \tilde{\varphi}) = \sup_{\|f\|_{\mathrm{Lip}} \leq 1} \left(\int_{\mathcal{X}} f \mathrm{d}\mu - \int_{\mathcal{X}} f \mathrm{d}\tilde{\mu} \right)$$

with supremum on all real 1-Lipschitz $f \in C(\mathcal{X})$: $|f(x) - f(y)| \leq d_{\text{geo}}(x, y)$.

ii. For $f = f^*$, $\left\|[d + d^\dagger, f]^2\right\| = \|f\|_{\mathrm{Lip}}^2$.

iii. Any 1-Lip. f non-vanishing at infinity can be approximated by the 1-Lip.

$$f_n(x) \doteq f(x) e^{-d_{\text{geo}}(x_0, x)/n} \in C_0(\mathcal{M});$$

and any f_n is the uniform limit of a sequence of 1-Lip. functions in $C_c^\infty(\mathcal{M})$.

II. Noncommutative cost

In optimal transport, the cost function is retrieved as the Wasserstein distance between Dirac measures:

$$c(x, y) = W(\delta_x, \delta_y).$$

Commutative case:		**Noncommutative case:**
spectral distance $d_{d+d^\dagger}$	$\rightarrow$	spectral distance d_D
$\uparrow$		$\vert$
Kantorovich duality		Kantorovich duality ?
$\downarrow$		$\downarrow$
Wassertein distance W with cost $d_{\text{geo}}(x, y) = d_{d+d^\dagger}(\delta_x, \delta_y)$		Wasserstein distance W_D on the space of states, with cost d_D between pure states.

State as a probability measure

For any state φ of a separable, unital, C^*-algebra $\mathcal{A}$ there exists a non necessarily unique probability measure μ on the pure state space $\mathcal{P}(\mathcal{A})$ such that

$$\varphi(a) = \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\mu(\omega) \quad \text{where} \quad \hat{a}(\omega) \doteq \omega(a).$$

Genuine Wasserstein distance on the state space:

$$W(\varphi, \tilde{\varphi}) = \inf_{\pi} \int_{\mathcal{P}(\mathcal{A})\times\mathcal{P}(\mathcal{A})} d\pi \quad d_D(\omega, \tilde{\omega}),$$

where the infimum is on all the measure π on $\mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$ with marginals $\mu, \tilde{\mu}$.

- In case $\mu \neq \tilde{\mu}$ define the same state φ, the r.h.s. may be non-zero.

Take the minimum on all the measures μ, $\tilde{\mu}$ that yield the states $\varphi, \tilde{\varphi}$:

$$\min_{\mu,\tilde{\mu}} W(\mu, \tilde{\mu}).$$

- Not clear this satisfies the triangular inequality.

Wasserstein distance on the state space

Assume the optimal transport on $\mathcal{P}(\mathcal{A})$, with cost d_D, yields a distance W_D on $\mathcal{S}(\mathcal{A})$. Following Kantorovich, its dual formulation is

$$W_D(\varphi, \tilde{\varphi}) \doteq \sup_{a \in \mathrm{Lip}_D(\mathcal{A})} \left(\int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\mu(\omega) - \int_{\mathcal{P}(\mathcal{A})} \hat{a}(\omega)\, \mathrm{d}\tilde{\mu}(\omega) \right),$$

where

$$\mathrm{Lip}_D(\mathcal{A}) \doteq \{a \in \mathcal{A} \text{ such that } |\hat{a}(\omega_1) - \hat{a}(\omega_2)| \leq d_D(\omega_1, \omega_2)\ \forall \omega_1, \omega_2 \in \mathcal{P}(\mathcal{A})\}.$$

By construction W_D coincides d_D on $\mathcal{P}(\mathcal{A})$. Is this true on the whole of $\mathcal{S}(\mathcal{A})$?
One has

$$d_D(\varphi, \tilde{\varphi}) \leq W_D(\varphi, \tilde{\varphi}) \qquad \forall \varphi, \tilde{\varphi} \in \mathcal{S}(\mathcal{A}),$$

for

$$\|D, a\| \leq 1 \Longrightarrow \omega_1(a) - \omega_2(a) \leq d_D(\omega_1, \omega_2) \Longrightarrow a \in \mathrm{Lip}_D(\mathcal{A}).$$

If d_D were equal to W_D on the whole of $S(\mathcal{A})$, then computing the spectral distance would become a problem of optimal transport, and spectral triples would be "providers" of cost functions.

Proposition PM 2012

$W_D = d_D$ on any subspace of $S(\mathcal{A})$ obtained by the linear combinations of two fixed pure states ω_1, ω_2. Namely, denoting

$$\varphi_\lambda = \lambda\omega_1 + (1-\lambda)\omega_2 \quad \lambda \in [0,1]$$

then

$$d_D(\varphi_\lambda, \varphi_{\lambda'}) = W_D(\varphi_\lambda, \varphi_{\lambda'}) \quad \forall \lambda, \lambda' \in [0,1].$$

Examples

- **two-point space:**

$$\mathcal{A} = \mathbb{C}^2, \quad \mathcal{H} = \mathbb{C}^2, \quad D = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}$$

where $m \in \mathbb{C}$ and representation

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

This is a two-point space

$$\delta_1(z_1, z_2) \doteq z_1, \quad \delta_2(z_1, z_2) \doteq z_2$$

with distance

$$d_D(\delta_1, \delta_2) = \frac{1}{|m|}.$$

- For non pure states, $d_D = W_D$ since

$$\mathrm{Lip}_D(\mathbb{C}^2) = \left\{ a \in \mathbb{C}^2, \, |z_1 - z_2| \leq \frac{1}{|m|} \right\} = \left\{ a \in \mathbb{C}^2, \, \|[D, a]\| \leq 1 \right\}.$$

• **the sphere:** $\mathcal{A} = M_2(\mathbb{C})$.

Pure states: $\omega_\xi(a) = (\xi, a\,\xi) = \mathrm{Tr}(s_\xi\, a) \quad \forall a \in \mathcal{A}$

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \mathbb{C}P^1 \leftrightarrow \mathbf{x}_\xi = \begin{cases} x_\xi & = & \mathrm{Re}(\overline{\xi_1}\xi_2) \\ y_\xi & = & \mathrm{Im}(\overline{\xi_1}\xi_2) \\ z_\xi & = & |\xi_1|^2 - |\xi_2|^2 \end{cases} \in S^2.$$

Any non-pure state φ is probability density ϕ on S^2:

$$\varphi(a) = \int_{S^2} a(\mathbf{x}_\xi)\,\phi(\mathbf{x}_\xi)\,\mathrm{d}\mathbf{x}_\xi \quad \text{where} \quad a(\mathbf{x}_\xi) \doteq \omega_\xi(a).$$

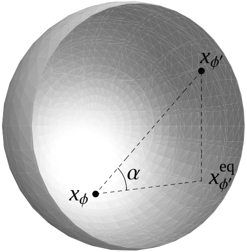
The density matrix s_φ such that $\varphi(a) = \mathrm{Tr}\,(s_\varphi\, a)$ actually depends only on the barycenter $\mathbf{x}_\phi = (x_\phi, y_\phi, z_\phi)$ of the probability density ϕ:

$$\mathbf{x}_\phi = (x_\phi, y_\phi, z_\phi) \quad \text{with} \quad x_\phi \doteq \int_{S^2} \phi(\mathbf{x}_\xi)\, x_\xi\, \mathrm{d}\mathbf{x}_\xi \ \text{ and similarly for } y_\phi, z_\phi.$$

$$\mathcal{S}(\mathcal{M}_2(\mathbb{C})) \ni \varphi \longrightarrow \mathbf{x}_\phi \in \mathcal{B}^2.$$

$\mathcal{H} = \mathbb{C}^2$, D any 2-by-2 matrix with distinct non-zero eigenvalues D_1, D_2,

$$d_D(\mathbf{x}_\phi, \mathbf{x}_{\phi'}) = \begin{cases} \frac{2}{|D_1 - D_2|}\, d_{Ec}(\mathbf{x}_\phi, \mathbf{x}_{\phi'}) & \text{if } z_\phi = z_{\phi'}, \\ +\infty & \text{if } z_\phi \neq z_{\phi'}. \end{cases}$$



$\mathcal{H} = M_2(\mathbb{C}) \otimes \mathbb{C}^2, D = -i\sqrt{\frac{2}{\theta}} \begin{pmatrix} 0 & [\Delta^*, \cdot] \\ -[\Delta, \cdot] & 0 \end{pmatrix}$ with $\Delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$d_D(\mathbf{x}_\phi, \mathbf{x}_{\phi'}) = \sqrt{\frac{\theta}{2}} \times \begin{cases} \cos\alpha \;\, d_{Ec}(\mathbf{x}_\phi, \mathbf{x}_{\phi'}) & \text{when } \alpha \leq \frac{\pi}{4}, \\ \frac{1}{2\sin\alpha}\, d_{Ec}(\mathbf{x}_\phi, \mathbf{x}_{\phi'}) & \text{when } \alpha \geq \frac{\pi}{4}. \end{cases}$$

- In both case $d_D = W_D$.

Counter-example

- **three point space:**

$$\mathcal{A} = \mathbb{C}^3, \qquad \mathcal{H} = \mathbb{C}^3, \qquad D = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ \alpha & \beta & 0 \end{pmatrix}$$

with $\alpha, \beta \in \mathbb{R}^+$ and representation

$$\pi(z_1, z_2, z_3) := \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix} \quad \forall\, (z_1, z_2, z_3) \in \mathbb{C}^3.$$

There are three pure states δ_i for $\mathcal{A}$,

$$\delta_i(z_1, z_2, z_3) = z_i \quad i = 1, 2, 3.$$

The space of states is the plain triangle with summit $\delta_1, \delta_2, \delta_3$.

W_D coincides with d_D on each edge of the triangle. But the two distances do not agree on the whole triangle.

Proposition

Rieffel 99, PM

Let φ, φ' be states in $\mathcal{S}(\mathbb{C}^3)$,

$$\varphi = \lambda_1\delta_1 + \lambda_2\delta_2 + (1-\lambda_1-\lambda_2)\delta_3, \qquad \varphi' = \lambda_1'\delta_1 + \lambda_2'\delta_2 + (1-\lambda_1'-\lambda_2')\delta_3$$

where $\lambda_i, \lambda_i' \in \mathbb{R}^+$, $i = 1, 2$ are such that $\Lambda_1 := \lambda_1 - \lambda_1'$ and $\Lambda_2 := \lambda_2 - \lambda_2'$ have the same sign. Then

$$W_D(\varphi, \varphi') = \frac{|\Lambda_1|}{\alpha} + \frac{|\Lambda_2|}{\beta}$$

while

$$d_D(\varphi, \varphi') = \sqrt{\frac{\Lambda_1^2}{\alpha^2} + \frac{\Lambda^2}{\beta^2}}$$

- There does not seem to be a cost systematically associated with the spectral distance.

Pull-back of a derivation

The Dirac operator of spectral triple $(\mathcal{A}, \mathcal{H}, D)$ defines a derivation

$$\begin{aligned} \nabla : \quad & \mathcal{A} \to \Omega^1_D(A) = \left\{ \sum_i a_i[D, b_i], a_i, b_i \in \mathcal{A} \right\} \subset B(\mathcal{H}) \\ & a \to [D, a] \end{aligned}$$

For any $B \subset \mathcal{B}(\mathcal{H})$ containing $\mathrm{Im}(\nabla)$, consider its Banach dual B^*, i.e. the set of linear functionals $\Phi : B \to \mathbb{C}$ bounded for the norm

$$||B|| = \sup_{b \in B, b \neq 0} \frac{|\Phi(b)|}{||b||}.$$

The pull back of the derivation is

$$\begin{aligned} \nabla^* : \quad & B^* \to \mathcal{L}(\mathcal{A}, \mathbb{C}) \\ & \Phi \to \nabla^* \Phi \end{aligned}$$

where $\mathcal{L}(\mathcal{A}, \mathbb{C})$ the set of linear functionals $\mathcal{A} \to \mathbb{C}$, and

$$\nabla^* \Phi(a) = \Phi(\nabla a) \quad \forall \, \Phi \in B^*, \, a \in \mathcal{A}.$$

The spectral distance as an infimum

(inspired from Chen, Georgiou, Ning and Tannenbaum).

Proposition D'Andrea, P.M. 2021

For any two states $\varphi, \psi \in \mathcal{S}(A)$ at finite spectral distance from one another, define

$$W(\varphi, \psi) := \inf_{\Phi \in B^*} \left\{ \|\Phi\| \text{ such that } \nabla^* \Phi = \varphi - \psi \right\} .$$

Then this expression is well defined, the infimum actually is a minimum, it does not depend on the choice of B^*, and one has

$$W(\varphi, \psi) = d_D(\varphi, \psi) \quad \forall \varphi, \psi \in \mathcal{S}(A).$$

- Dual formula for the spectral distance, but it does not involve any cost !

- **Euclidean space:** $\mathcal{A} = C_c^\infty(\mathbb{R}^{2m}), \quad \mathcal{H} = L^2(\mathbb{R}^{2m}, S), \quad D = -i\gamma^\mu \partial_\mu.$

An element in the Banach dual of $\mathcal{B}(\mathcal{H})$ is given by Radon measures $w_1, \ldots, w_{2m}$:

$$\Phi(\,.\,) = i \sum_\alpha \int_{\mathbb{R}^n} \langle \gamma^\alpha, \,.\, \rangle_{\mathrm{HS}} \, dw_\alpha .$$

The side condition $\nabla^* \Phi = \varphi - \psi$ reads (with $d\mathbf{w} = (dw_1, \ldots, dw_{2m})$)

$$\int_{\mathbb{R}^n} \boldsymbol{\nabla} f \cdot d\mathbf{w} = \int_{\mathbb{R}^n} (f \, d\mu - f \, d\nu) \qquad \forall\, f \in C_c^\infty(\mathbb{R}^n).$$

If $dw_\alpha = \omega_\alpha d\mathbf{x}$, $d\mu = \mu(x) d\mathbf{x}$, $d\nu = \nu(x) d\mathbf{x}$ for C^1-functions ω_α, μ, ν, one gets

$$\int_{\mathbb{R}^n} f \left\{ \boldsymbol{\nabla} \cdot \mathbf{w} + \mu - \nu \right\} d^n x = 0 \qquad \forall\, f \in C_c^\infty(\mathbb{R}^n).$$

That is $-\boldsymbol{\nabla} \cdot \mathbf{w} = \mu - \nu$.Moreover $\|\Phi\| = \int_{\mathbb{R}^n} |\mathbf{w}(x)| d^n x$. Therefore

$$W(\varphi, \psi) = \min_{\mathbf{w}} \left\{ \int_{\mathbb{R}^n} |\mathbf{w}(x)| d^n x \;\text{ such that }\; \boldsymbol{\nabla} \cdot \mathbf{w} = \mu - \nu \right\} .$$

- This is Beckmann's formula for dynamical optimal transport on $\mathbb{R}^{2m}$, which is well known to be dual to the Wasserstein distance with cost d_{geo}.

Conclusion and outlook

The spectral distance is a generalisation to the noncommutative setting of Kantorovich formula.

In optimal transport (on the euclidean space), Kantorovich formula is dual to both the Wasserstein distance (minimising a cost) and Beckmann formula (dealing with problems of flux).

The spectral distance admits

- in some example, **a dual formulation that involves a cost**: Wasserstein distance W_D on the space of states;
- in any case, **a dual formulation a la Beckmann**.

This might be helpful for explicit computation (providing lower bounds), and inspiring to explore further the metric aspect of noncommutative geometry: rather than trying to see the Higgs field as a cost, one may try to understand in what dynamical problem of optimal transport it is involved.

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