Localization and Quantum Observers

in some Lie-algebra type noncommutative spacetimes

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I will discuss aspects of noncommutative spaces of the "simple kind", the one obtained by noncommuting coordinates.

Moyal plane, $\boxed{\kappa}$-Minowski and a couple of their variations are of this kind. But also most of quantum mechanics.

I will interpret the title of the conference, inasmuch the "spectral" part is concerned, by using a quantum mechanical, spectral, interpretations of these space.

In particular I will consider the observers as quantum objects.

You are familiar with our case study: Quantum Phase Space of a particle.

Phase space is a six-dimensional space spanned by (q^i, p_i). Quantization introduces the commutation relation $[q^i, p_j] = \mathrm{i}\hbar\delta^i_j$,

One can represent these as operators either on $L^2(q)$ or $L^2(p)$. In either case, three operators are multiplicative, and three are differential. $\widehat{q}$'s and $\widehat{p}$'s are unbounded selfadjoint operators with a dense domain. The spectrum is the real line (for each i).

Fourier transforms switch the two pictures. The choice is actually that of a complete set of observables, which are simultaneously diagonalised. Other choices are possible, like Energy, angular momentum. . .

p and q have no eigenvectors but improper eigenfunctions: distributions. Either plane waves or Dirac's δ.

$$\boxed{[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}}$$

This kind of noncommutative space, with $\boxed{\theta}$ a constant, often called Moyal space, has been used without any change (and often any analysis) to describe spacetime. You know the story, it came out in string theory and many people were rediscovering quantum mechanics.

Although some were doing something profound, DFR or Wess, to name a few. And also of interest is field theory on these spaces.

This NCG breaks Lorentz invariance, although it maintains translation invariance. It has two preferred directions, a vector and a pseudovector, which would characterize our universe.

This was not a problem for quantum mechanics we do not rotate coordinates into momenta.

We need quantum symmetries: quantum groups and Hopf algebras.

I am not going to introduce quantum groups and Hopf algebras to this audience. But I want to mention that one way to build noncommutative spaces and their symmetries in a coherent way is via a Drinfeld twist.

I like to consider the twist, a map from the tensor product of two algebras into itself, as deformation of the tensor product

$$\boxed{\mathcal{F} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}}$$

With a twist we can, with one stroke, deform the product, and the symmetries.

Another way to obtain deformed Hopf algebras is via the bicross product, which I like to see as a way to start from the dual space of the commutation relations of the coordinates, and the implementation of the Lorentz symmetry.

A non trivial commutation relations implies in general uncertainty with localization, due to Heisenberg principle. Hence it is impossible to talk of points, but we can use the generalised concept of state.

To describe symmetries I will take the passive point of view.

A Poincaré transformation relates two observers who are translate, rotated and boosted with respect to each other.

Since I am dealing with quantum spacetime an observer has a precise position in space and time, its measurements are events.

In a quantum spacetime, with quantum symmetries, also observers will be quantum.

I will start with κ-Minkowski. With $\lambda = \frac{1}{\kappa}$.

$$[x_0, x_i] = \mathrm{i}\lambda x_i$$

Note: Here and in the following I will often indicate only the nonzero commutators.

This oncommutative spacetime is the noncommutative operator algebra generated by these coordinate.

Note that x_0 acts as a dilation operator, and that it commutes with the angular polar coordinates, so that we have just:

$$[x_0, r] = \mathrm{i}\lambda r$$

There are two natural complete sets of commuting observables:

$$\{r, \theta, \varphi\}$$

$$\{x_0, \theta, \varphi\}$$

I ignore the fact that θ and φ are not good operators, but this will become relevant later.

For quantum mechanics the two sets are related by a Fourier transformation.

In this case a Mellin tranformation should be used.

A representation of the x^μ on $L^2(\mathbb{R}^3)$:

$$\widehat{x}^i\psi(x) = x^i\psi(x)$$

$$\widehat{x}^0\psi(x) = \mathrm{i}\lambda\left(\sum_i x^i\partial_{x^i} + \frac{3}{2}\right)\psi(x) = \mathrm{i}\lambda\left(r\partial_r + \frac{3}{2}\right)\psi(x).$$

Positions are multiplicative operators, time is dilation. The $3/2$ factor is necessary to make the operator symmetric. It is selfadjoint on all absolutely continuous functions.

The spectrum of the self-adjoint x_0 is the real line, with eigenvalues we call τ and the improper eigenfunctions are:

$$T_\tau = \frac{r^{-\frac{3}{2}-\mathrm{i}\tau}}{\lambda^{-\mathrm{i}\tau}} = r^{-\frac{3}{2}}\mathrm{e}^{-\mathrm{i}\tau\log\left(\frac{r}{\lambda}\right)}$$

The switch between the two complete set (r, θ, φ) or (τ, θ, φ), is a Mellin transform

$$\psi(r, \theta, \varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\tau \, r^{-\frac{3}{2}} \mathrm{e}^{-\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \widetilde{\psi}(\tau, \theta, \varphi)$$

$$\widetilde{\psi}(\tau, \theta, \varphi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{d}r \, r^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\tau \log\left(\frac{r}{\lambda}\right)} \psi(r, \theta, \varphi)$$

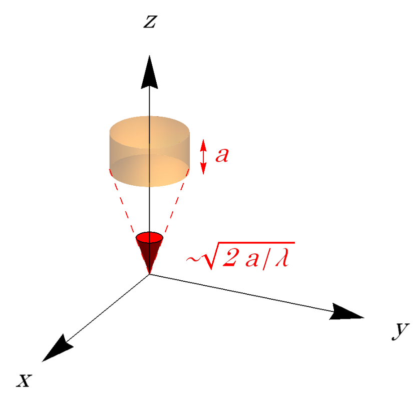
$|\psi|^2$ and $|\widetilde{\psi}|^2$ are the probability density to find the particle in position r or time τ respectively.

There is the uncertainty:

$$\Delta x^0 \Delta r \geq \frac{\lambda}{2} |\langle r \rangle|.$$

Consider the following localised in a small region.

$$\psi_{z_0,a}(r,\theta,\varphi) = \begin{cases} \sqrt{\frac{3\lambda}{2a\pi((a+z_0)^3 - z_0^3)}}, & z_0 \leq r \leq (z_0 + a) \text{ and } \cos\theta > 1 - \frac{a}{\lambda} \\ 0, & \text{otherwise} \end{cases}$$



In the limit $a \to 0$ the state is localised in z_0

The Mellin transform of this function, integrating out the angular variables, gives:

$$\int |\widetilde{\psi}_{z_0,a}|^2 \sin\theta \, d\theta = \left[\frac{a}{4\pi^2 z_0} - \frac{a^2}{8\lambda(\pi^2 z_0^2)} + \mathcal{O}(a^3) \right]$$

This tends to a constant which vanishes as $a \to 0$. Localising in space implies delocalising in time

The series expansion for a around 0, and z_0 around ∞, are the same. $|\widetilde{\psi}_{z_0}|^2 = \frac{\lambda}{4\pi^2 z_0} - \frac{a\lambda}{8\pi^2 z_0^2} + \frac{a^2\lambda(7-4\tau^2)}{192\pi^2 z_0^3} + \mathrm{O}\left(a^3\right)$

This means that a sharp localization of a particle far away from the origin implies that the particle cannot be localised in time. In accordance with the uncertainty for κ-Minkowski.

Implicitly in our discussion, when we were referring to states we were assuming the existence of an observer, located at the origin, measuring the localisation of states.

Since the observe can measure with absolute precision only states located at the origin, "here" and "now" make sense. States far away cannot be localised.

What about other observers? A different observer will be in general Poincaré transformed, i.e. translated, rotated and boosted. These operations are usually performed with an element of the Poincaré group. But now we have κ-Poincaré!

Require invariance under the transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$

But now the coordinate functions on the group are noncommutative, they are (in a particular basis, Zakrzewski)

$$[a^\mu, a^\nu] = \mathrm{i}\lambda \left(\delta^\mu{}_0\, a^\nu - \delta^\nu{}_0\, a^\mu\right) , \qquad [\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0$$

$$[\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\lambda \left[\left(\Lambda^\mu{}_\sigma \delta^\sigma{}_0 - \delta^\mu{}_0\right) \Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu \delta^0{}_\sigma - \delta^0{}_\nu\right) \eta^{\mu\rho} \right] .$$

In particular notice that translations are now noncommuting. With the same commutation relations of the coordinates.

We represented the κ-Minkowski algebra as operators. But in doing so we had implicitly chosen an observer.

In order to take into account the fact that there are different observers we enlarge the algebra (and consequently the space) to include the parameters of the new observers. We call then new set of states as $\mathcal{P}_\kappa$

Our (generalized) Hilbert space will now comprise not only functions on space-time (either functions of r or τ), but also functions of the a's and Λ's.

We can represent the κ-Poincaré group faithfully as

$$a^\rho = -\mathrm{i}\,\frac{\lambda}{2}\left[\left(\Lambda^\mu{}_\sigma\delta^\sigma{}_0 - \delta^\mu{}_0\right)\Lambda^\rho{}_\nu + \left(\Lambda^\sigma{}_\nu\delta^0{}_\sigma - \delta^0{}_\nu\right)\eta^{\mu\rho}\right]\Lambda^\nu{}_\alpha\frac{\partial}{\partial\omega^\mu{}_\alpha} + \mathrm{i}\frac{\lambda}{2}\left(\delta^\rho{}_0\, q^i\frac{\partial}{\partial q^i} + \delta^\mu{}_i\, q^i\right) + \frac{1}{2}\mathrm{h.c.}$$

Where ω are the parameters of the Lorentz transformation, and the Λ's are represented as multiplicative operators

We have therefore that, like spacetime, the space of observers is also non-commutative, and the noncommutativity is only present in the translation sector.

We now explore the space of observers, seen as states. First consider the observer located at the origin, which is reached via the identity transformation.

Define $|o\rangle_{\mathcal{P}}$ with the property:

$$ {}_{\mathcal{P}}\langle o|\, f(a,\Lambda)|o\rangle_{\mathcal{P}} = \varepsilon(f)\,, $$

with $f(a,\Lambda)$ a generic noncommutative function of translations and Lorentz transformation matrices, and ε the counit.

This state describes the Poincaré transformation between two coincident observers. The state is such that all combined uncertainties vanish. Coincident observers are therefore a well-defined concept in κ-Minkowski spacetime.

A change of observer transforms $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes 1$ and primed and unprimed coordinates correspond to different observers.

Identifying x with $\mathbb{1} \otimes x$ we generate an extended algebra $\mathcal{P} \otimes \mathcal{M}$ which extends κ-Minkowski by the κ-poincaré group algebra.

This algebra takes into account position states and observables

Remember that, just as we cannot sharply localize position states, neither we can sharply localize where the observer is.

Since Lorentz transformations commute among themselves, we can however say if two observers are just rotated with respect to each other

We can build the action of the position, translation and Lorentz transformations operator on generic functions of all those variables.

To simplify notations let us consider $1+1$ dimensions. In this case there are only two position coordinates, two translations coordinates and one Lorentz transformation parametrized by ξ

The relations are $\Lambda^0{}_0 = \Lambda^1{}_1 = \cosh\xi\,,\ \Lambda^0{}_1 = \Lambda^1{}_0 = \sinh\xi\,,$

$$[a^0, a^1] = \mathrm{i}\lambda\, a^1\,, \qquad [\xi, a^0] = -\mathrm{i}\lambda \sinh\xi\,, \qquad [\xi, a^1] = \mathrm{i}\lambda\,(1-\cosh\xi)\ .$$

And the action on $\mathcal{P}$ is

$$a^0 = \mathrm{i}\lambda q \frac{\partial}{\partial q} + \mathrm{i}\lambda \sinh\xi\, \frac{\partial}{\partial\xi}\,,\ \ a^1 = q + \mathrm{i}\lambda\,(\cosh\xi - 1)\,\frac{\partial}{\partial\xi}\,,$$

States (non entangled) will be objects of the kind $|g\rangle \otimes |f\rangle$

In particular $|g\rangle \otimes |o\rangle$ is a pure translation of the state at the origin.

The new observer measures coordinates with x'. The expectation values on (normalised) transformed state is

$$\langle x'^{\mu} \rangle = \langle g| \otimes \langle o|x'^{\mu}|g\rangle \otimes |o\rangle = \langle g|\Lambda^{\mu}{}_{\nu}|g\rangle\langle o|x^{\nu}|o\rangle + \langle g|a^{\mu}|g\rangle\langle o|o\rangle \,,$$

We get:

$$\langle x'^{\mu} \rangle = \langle g|a^{\mu}|g\rangle \,,$$

The expectation value of the transformed coordinates is completely defined by translations. This is natural, the different observers are comparing positions, not directions.

In general

$$\langle x'^{\mu_1} \dots x'^{\mu_n} \rangle = \langle g | a^{\mu_1} \dots a^{\mu_n} | g \rangle \langle o | o \rangle = \langle g | a^{\mu_1} \dots a^{\mu_n} | g \rangle \,.$$

Poincaré transforming the origin state $|o\rangle$ by a state with wavefunction $|g\rangle$ in the representation of the κ-Poincaré algebra, the resulting state will assign, to all polynomials in the transformed coordinates the same expectation value as what assigned by $|g\rangle$ to the corresponding polynomials in a^μ.

In other words, the state x'^μ is identical to the state of a^μ.

All uncertainty in the transformed coodinates $\Delta x'^\mu$ is introduced by the uncertainty in the state of the translation operator, Δa^μ.

It is also possible to see that the uncertainty of states increases with translation.

I can summarise saying that all observers can sharply localise states in their vicinity, and cannot localise states far away from them.

The apparent paradox of a state badly localisable by Alice, but which is well localised by Bob, is that Bob is badly localised by Alice, and of course viceversa.

All this is qualitatively perfectly compatible with the principle of relative locality (Amelino-Camelia, Kowalski-Glickman, Freidel, Smolin), which however starts in a quite different context: curved momentum space. In this analysis instead momentum does not appear explicitly, although it is present in the symmetry.

One of the tenets of Quantum Mechanics is that the observer is classical, usually macroscopic, and that therefore we “know” how to deal with them.

In quantum gravity this may not be the case. While it is true that the smallness of the Planckian constants suggests this, there may be amplifying effects, and conceptual aspects to deals with.

The group algebra approach, where the parameters of the Poincaré transformations do not commute is the key to understand the observer-dependent transformations

Transformations relating different frames belong to a noncommutative algebra. Hence localisability limitations.

Alternatively, the deformation can be seen as a deformation of the tensor product. This is evident in the case of a Drinfeld twist, and I give another example, based on a twist.

Angular Noncommutativity

First I will discuss the "time-like" case, which we call ϱ-Minkowski

$$[x^0, x^1] = -\mathrm{i}\varrho x^2 \;;\; [x^0, x^2] = \mathrm{i}\varrho x^1 \;;\; [x^0, x^3] = 0 \;;\; [x^1, x^2] = 0$$

This form of noncommutativity has a long history, Gutt, Lukierski, Woronowicz, Chaichian, Demichev, Presnajder, Tureanu and more recently Amelino-Camelia, Barcaroli, Loret, Bianco and Pensato.

A similar version can be built in which x^0 and x^3 are exchanged. I will mention it later.

Express the commutation relations in cylindrical coordinates (t, ρ, z, φ)

$$\text{“}[t, \varphi] = \mathrm{i}\varrho\text{”}\,; \;\; [t, z] = [t, \rho] = \text{“}[\rho, \varphi]\text{”} = [\rho, z] = 0$$

The inverted commas are there because φ is not a well defined self-adjoint operator.

The uncertainty is between time and the angular variable. Resist the temptation to write:

$$\Delta t \Delta \varphi \geq \frac{\varrho}{2}$$

In the $\{\rho, z, \varphi\}$ basis t is represented by the derivation operator $-\mathrm{i}\varrho\partial_\varphi$.

This operator has Discrete Spectrum!

A change of basis is given by the Fourier series. The eigenstates of momentum are $\mathrm{e}^{\mathrm{i}n\varphi}$, and they are completely delocalised in φ

On the other hand, a state completely localised in φ, given by a δ, which requires a superposition with equal weights of all eivenvalues of time.

$$\delta(\varphi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}n\varphi}$$

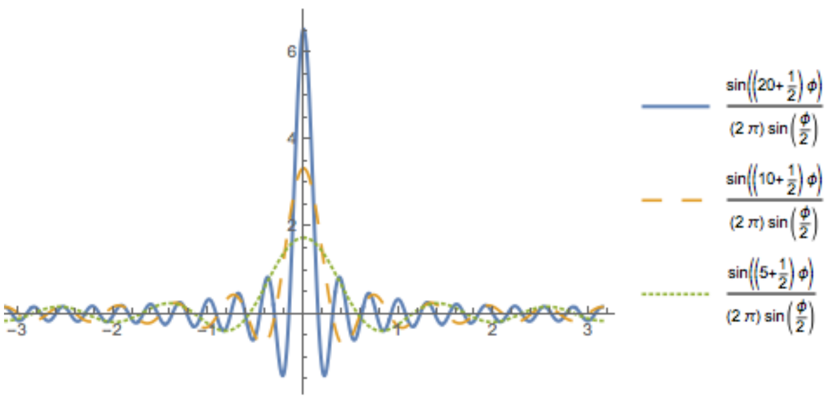
After a time measurement, which has given as result $n_0\varrho$, the system is in the eigenstate $\mathrm{e}^{\mathrm{i}n_0\varphi}$.

A slightly uncertain state uses a great number of Fourier modes to built a state peaked around some time, then the corresponding uncertainty is the angular variable is given by the fact that only a finite set of elements of the basis are available.

For ϱ Planckian of the quantum of time (also called a chronon), is $5.39\ 10^{-44}\, sec.$

The most accurate measurement of time is $\sim 10^{-19}\,sec.$ Heuristically the superposition of 10^{35} quanta of time is needed.

Approximate δ by the Dirichlet nucleus $\delta_N = \sum_{n=-N}^{N} e^{in\varphi} = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})\varphi}{\sin\frac{N}{2}\varphi}$



For $N = 5, 10, 15$.

For $N \sim 10^{35}$ the first zero of the nucleus is at $\varphi \sim 10^{-25}$. We may assume this to be the uncertainty in an angle determination. To translate this as an uncertainty in position we need ρ. For the radius observable universe ($10^{26}\,m$) the uncertainty is of the order of one metre.

Is this all pervading clicking a feature of our universe? Is time translation definitely lost? Putting time on a lattice may be disturbing.

Self-adjointness come to the rescue. Anybody who has studied the Aharonov-Bohm experiment knows that the momentum operator on a compact domain is a rich operator.

It is self-adjoint on periodic functions, but is also selfadjoint on functions periodic up to a phase. In this case the eigenfunctions are $\boxed{\mathrm{e}^{\mathrm{i}(n+\alpha)\varphi}}$.

The differences between states is unchanged, and the effect is a rigid shift. This however means that a different choices of selfadjointess domains. Time translations are undeformed, and two time translated observers will be in different, but equivalent domains.

In order to compare their results the two observers, again, have to compare representantions, and this is ruled by a coproduct.

Noticing that $[\partial_t, \partial\varphi] = 0$, the deformation can be built with a Drinfeld twist.

$$\mathcal{F}(x,y) = \exp\left\{-\frac{\mathrm{i}\varrho}{2}\left(\partial_{y^0}\left(x^2\partial_{x^1} - x^1\partial_{x^2}\right) - \partial_{x^0}\left(y^2\partial_{y^1} - y^1\partial_{y^2}\right)\right)\right\}$$

$$= \exp\left\{\frac{\mathrm{i}\varrho}{2}\left(\partial_{y^0}\partial_{\varphi_x} - \partial_{x^0}\partial_{\varphi_y}\right)\right\}$$

I will flash the coproduct

$$
\begin{aligned}
\Delta P_3 &= P_3 \otimes 1 + 1 \otimes P_3, \\
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta P_1 &= P_1 \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes P_1 + P_2 \otimes \sin\left(\frac{\varrho}{2} P_0\right) - \sin\left(\frac{\varrho}{2} P_0\right) \otimes P_2, \\
\Delta P_2 &= P_2 \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes P_2 - P_1 \otimes \sin\left(\frac{\varrho}{2} P_0\right) + \sin\left(\frac{\varrho}{2} P_0\right) \otimes P_1, \\
\Delta M_{01} &= M_{01} \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes M_{01} + M_{02} \otimes \sin\left(\frac{\varrho}{2} P_0\right) - \sin\left(\frac{\varrho}{2} P_0\right) \otimes M_{02} \\
&\quad - P_1 \otimes \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) + \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) \otimes P_1 \\
&\quad - P_2 \otimes \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) - \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) \otimes P_2, \\
\Delta M_{02} &= M_{02} \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes M_{02} - M_{01} \otimes \sin\left(\frac{\varrho}{2} P_0\right) + \sin\left(\frac{\varrho}{2} P_0\right) \otimes M_{01} \\
&\quad - P_2 \otimes \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) + \frac{\varrho}{2} M_{12} \cos\left(\frac{\varrho}{2} P_0\right) \otimes P_2 \\
&\quad + P_1 \otimes \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) + \frac{\varrho}{2} M_{12} \sin\left(\frac{\varrho}{2} P_0\right) \otimes P_1, \\
\Delta M_{03} &= M_{03} \otimes 1 + 1 \otimes M_{03} - \frac{\varrho}{2} P_3 \otimes M_{12} + \frac{\varrho}{2} M_{12} \otimes P_3, \\
\Delta M_{12} &= M_{12} \otimes 1 + 1 \otimes M_{12}, \\
\Delta M_{13} &= M_{13} \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes M_{13} + M_{23} \otimes \sin\left(\frac{\varrho}{2} P_0\right) - \sin\left(\frac{\varrho}{2} P_0\right) \otimes M_{23} \\
\Delta M_{23} &= M_{23} \otimes \cos\left(\frac{\varrho}{2} P_0\right) + \cos\left(\frac{\varrho}{2} P_0\right) \otimes M_{23} - M_{13} \otimes \sin\left(\frac{\varrho}{2} P_0\right) + \sin\left(\frac{\varrho}{2} P_0\right) \otimes M_{13}.
\end{aligned}
$$

Interesting in this context are the commutation relations of the Poincaré transformations.

Compare the two r matrices between κ and ϱ:

$$r_\kappa = \mathrm{i}\lambda M_{0\nu} \wedge P^\nu$$

$$r_\varrho = -i\varrho P_0 \wedge M_{12}$$

Note that $[M_{0\mu}, P_\mu] = 0$ while $[P_0, M_{12}] \neq 0$.

Unlike the κ-Minkowski case, which satisfies a modified Yang-Baxter equation, the ϱ satisfies the classical version.

For κ-Minkowski the commutation relations among the elements of the transformation are:

$$[\Lambda^{\mu}{}_{\alpha}, \Lambda^{\nu}{}_{\beta}] = 0.$$

$$[\Lambda^{\alpha}{}_{\beta}, a^{\rho}] = -\mathrm{i}\lambda((\Lambda^{\alpha}{}_{0} - \delta^{\alpha}{}_{0})\Lambda^{\rho}{}_{\beta} + (\Lambda_{0\beta} - g_{0\beta})g^{\alpha\rho}).$$

$$[a^{\mu}, a^{\nu}] = i\lambda(\delta^{\mu}{}_{0}a^{\nu} - \delta^{\nu}{}_{0}a^{\mu}),$$

We used them already for the localasability of the states. For example the fact that rotations are well definite is a consequence of these. The bicross-product nature of the algebra is important.

For $\boxed{\varrho}$-Minkowski things are simpler because of the twist:

$$\boxed{[\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma] = 0}$$

$$\boxed{[\Lambda^\mu{}_\nu, a^\rho] = \mathrm{i}\lambda \left[-\delta_0^\rho (\Lambda_{2\nu}\delta_1^\mu - \Lambda_{1\nu}\delta_2^\mu) + \Lambda^\rho{}_0 (\Lambda^\mu{}_1 \eta_{2\nu} - \Lambda^\mu{}_2 \eta_{1\nu}) \right]}$$

$$\boxed{[a^\mu, a^\nu] = \mathrm{i}\lambda \left[\delta_0^\mu (a_2\delta_1^\nu - a_1\delta_2^\nu) - \delta_0^\nu (a_2\delta_1^\mu - a_1\delta_2^\mu) \right] .}$$

Since the symmetry group is deformed according, we expect localization problems to arise also in observer transformation

Uncertainty relations:

$$\Delta a^\mu \Delta a^\nu \geq \frac{\varrho}{2}|\delta^\nu{}_0(\langle a_2\rangle\delta^\mu{}_1 - \langle a_1\rangle\delta^\mu{}_2) - \delta^\mu{}_0(\langle a_2\rangle\delta^\nu{}_1 - \langle a_1\rangle\delta^\nu{}_2)|,$$

$$\Delta\Lambda^\mu{}_\alpha \Delta\Lambda^\nu{}_\beta \geq 0$$

$$\Delta\Lambda^\mu{}_\nu \Delta a^\rho \geq \frac{\varrho}{2}|\delta^\varrho{}_0(\langle\Lambda_{2\nu}\rangle\delta^\mu{}_1 - \langle\Lambda_{1\nu}\rangle\delta^\mu{}_2) - \langle\Lambda^\varrho{}_0\Lambda^\mu{}_1\rangle g_{2\nu} + \langle\Lambda^\varrho{}_0\Lambda^\mu{}_2\rangle g_{1\nu}|$$

Case of pure Lorentz transformations, translational parameters are sharply localized in 0

$$\boxed{\delta^{\varrho}{}_{0}(\langle \Lambda_{2\nu}\rangle \delta^{\mu}{}_{1} - \langle \Lambda_{1\nu}\rangle \delta^{\mu}{}_{2}) - \langle \Lambda^{\varrho}{}_{0}\Lambda^{\mu}{}_{1}\rangle g_{2\nu} + \langle \Lambda^{\varrho}{}_{0}\Lambda^{\mu}{}_{2}\rangle g_{1\nu} = 0.}$$

The only admitted pure transformation is the identical one, and can be sharply localized. For $\boxed{\kappa}$-Poincaré a slightly different result was found by Mercati: just pure boosts are not admitted.

For the case of pure space translations, those along the 3-axis or the time axis exist without issues and can be sharply localized.

For pure translations along the 1- and 2-axes the result is different: if we consider, for example, the first case one would have $\boxed{\langle a^2 \rangle = 0}$, this is compatible with $\boxed{\Delta x^0 = 0}$, but this last condition imposes also that $\boxed{\langle a^1 \rangle = 0}$, the same being true switching 1with 2. This means that $\boxed{\varrho}$-Poincaré admits only pure time translations and pure space translations along the 3-axis.

For comparison, in the $\boxed{\kappa}$ case, it was found that the only possible pure translation is the temporal one.

One can also check that the identity transformation is OK in all case

If one considers what happens when we transform the states one reaches some conclusions (detail in paper with Vitale and Scala)

Define $L^2(SO(1,3) \times \mathbb{R}^3_q)$ as the space of states of an observer (i.e. the space of ϱ-Poincaré states) and $L^2(\mathbb{R}^3_x)$ as the space of observables (i.e. the space of states of ϱ-Minkowski spacetime); furthermore we assume that a generic state can be realized as a separable element $|\phi, \psi\rangle = |\phi\rangle \otimes |\psi\rangle$, a reasonable assumption since it reflects the fact that the relation between two inertial observers does not depend on the observed state.

Recall: we have at the same time a noncommutative spacetime on which observables are defined and a noncommutative observer state-space, in general a ϱ-Poincaré transformation between different observers could change localizability of states.

Identity transformation state is again OK

What the observer $\mathcal{O}'$ will measure after ϱ-Poincaré transforming the origin state? The starting state is

$$|\phi, o\rangle = |\phi\rangle \otimes |o\rangle,$$

$$\langle x'^{\mu}\rangle = \langle\phi| \otimes \langle o|(\Lambda^{\mu}{}_{\nu} \otimes x^{\nu} + a^{\mu} \otimes 1)|\phi\rangle \otimes |o\rangle = \langle\phi|\Lambda^{\mu}{}_{\nu}|\phi\rangle\langle o|x^{\nu}|o\rangle + \langle\phi|a^{\mu}|\phi\rangle.$$

Recalling that $\langle o|x^{\mu}|o\rangle = 0$, we have

$$\langle x'^{\mu}\rangle = \langle\phi|a^{\mu}|\phi\rangle.$$

This means that the two observers $\mathcal{O}$ and $\mathcal{O}'$ are comparing positions and not directions, so the expectation value is determined only by the mean value of translation operators.

It can be shown by an analogous computation that the result remains true also for a generic monomial in coordinates $x'^{\mu_1} \cdots x'^{\mu_n}$. In this case, the uncertainty of the transformed event coincides with that of the translation operator:

$$\Delta(x'^{\mu})^2 = \langle (x'^{\mu})^2 \rangle - \langle x'^{\mu} \rangle^2 = \langle (a^{\mu})^2 \rangle - \langle a^{\mu} \rangle^2 = \Delta(a^{\mu})^2.$$

Comparing with the κ-case the translational parameter can be localized, in both cases the uncertainty on the final state is zero. For ϱ-Poincaré this occurs when $\langle a^1 \rangle = \langle a^2 \rangle = 0$, namely, for pure translations along a^0, a^3 or even mixed translations in 03, while for κ-Poincaré this occurs only for pure temporal translations.

Another interesting case is that of a pure translation $x'^{\mu} = 1 \otimes x^{\mu} + a^{\mu} \otimes 1$ of a generic state. Therefore, one sees that acting with a pure translation leads in general to an increase in the state uncertainty. As for the comparison with the κcase, the same considerations apply as those relative to the origin

Bonus track: λ-Minkowski

Everything we did for ϱ-Minkowski can be repeated exchanging x^0 with x^3:

$$[x^3, x^1] = -\mathrm{i}\lambda x^2 \ ; \ [x^3, x^2] = \mathrm{i}\varrho x^1 \ ; \ [x^0, x^3] = 0 \ ; \ [x^1, x^2] = 0$$

We call this λ-Minkowski, and I will not repeat the previous analysis

It does however have an interesting application.

So far most of NCG happens is (quantum) spacetime, deformations which have a length scale, the phase space scale $\hbar$ is usually ignored with $\boxed{\lambda}$-Minkowski it possible to quantize configuration and momentum space in a sort of double quantization.

Both the usual phase space $\boxed{\hbar}$ quantization and spacetime quantuzation $\boxed{\varrho}$ come from twists

The quantum mechanics twist is

$$\boxed{\mathcal{F}_\hbar = \exp\left[-\frac{i\hbar}{2}\left(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_i}\right)\right]\}}$$

Now the change of variable in phase space:

$$\boxed{\tilde{x}^1 = x^1, \; \tilde{x}^2 = x^2, \; \tilde{x}^3 = x^3 + \frac{\lambda}{\hbar}(x^2 p_1 - x^1 p_2),}$$

$$\boxed{\tilde{p}_i = p_i}$$

It is not difficult to see that, using the λ-Minkowski together with the canonical commutation relations one obtains the right relations among the x.

Now we can construct a twist, which turns out to be abelian:

$$\mathcal{F} = \exp\left[-\frac{i\hbar}{2}\left(\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_i}\right) + \frac{i\lambda}{2}\frac{\partial}{\partial x^3} \wedge \left(\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) + \left(p_1\frac{\partial}{\partial p_2} - p_2\frac{\partial}{\partial p_1}\right)\right)\right] .$$

The twist nontrivial new commutations, necessary for phase-space coordinates to close a Lie algebra $[x^3, p_1] = i\lambda p_2, \qquad [x^3, p_2] = -i\lambda p_1 .$

Intriguingly, while the limit $\boxed{\hbar \to 0}$ is possible, and one obtains classical physics on $\boxed{\lambda}$-Minkowski, the one $\boxed{\lambda \to 0}$ does not lead to a well defined case. Showing a deep connection among the two scales.

With a twist is possible to study the symmetries of this quantum space, which turns out to be a deformation of Galieli.

Details in the paper

Final Remarks

The main message I want to convey is that quantum gravity will require Quantum Spacetime.

Quantum Spacetime in turn requires quantum observers.

This is of course true for quantum phase space as well. There we became (more or less) used to deal with the contradictions of the quantum/classical interaction. We learned how to deal with noncommuting observables for example

But a quantum spacetime will pose further challenges and other layers to our understanding.