The Gromov-Hausdorff distance in noncommutative geometry

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Noncommutative geometry: metric and spectral aspects
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Matrix Models: Clock and Shift, Fuzzy Tori

$$C_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_n^{n-1} \end{pmatrix} \text{ and } S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

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with $r_n = \exp\left(\frac{2i\pi}{n}\right)$. Note that $C^*(C_n, S_n) = \mathfrak{B}(\mathbb{C}^n)$. Let $\mathbb{D}_n$ be defined on $\mathscr{H}_n = \mathfrak{B}(\mathbb{C}^n) \otimes \mathbb{C}^4$ be defined by:

$$\mathbb{D}_n = \frac{n}{2\pi}\left(\left[\frac{S_n + S_n^*}{2}, \cdot\right] \otimes \gamma_1 + \left[\frac{S_n - S_n^*}{2i}, \cdot\right] \otimes \gamma_2 + \left[\frac{C_n^* + C_n}{2}, \cdot\right] \otimes \gamma_3 + \left[\frac{C_n^* - C_n}{2i}, \cdot\right] \otimes \gamma_4\right),$$

with $\gamma_j\gamma_k + \gamma_k\gamma_j = \delta_j^k$ ($j, k \in \{1,2,3,4\}$). Then *$(C^*(C_n, S_n), \mathscr{H}_n, \mathbb{D}_n)$ is a spectral triple*.

A geometry for the space of quantum spaces

The Questions

1. How do we formalize convergence of spectral triples?
2. Can we construct new spectral triples as limits?

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1. *Connes' metric* on the state space of $\mathfrak{A}$,
2. An $\mathfrak{A}$-$\mathbb{C}$ C*-correspondence $\mathscr{H}$,
3. An action of $[0,\infty)$ on $\mathscr{H}$ by unitaries, given by $t \in [0,\infty) \longmapsto \exp(i t \not{D})$.

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Spectral Triples

Spectral triples have emerged as the preferred method to encode geometric information about quantum spaces.

Definition (Connes, 85)

A *spectral triple* $(\mathfrak{A}, \mathscr{H}, D\!\!\!/)$ is given by:

- a Hilbert space $\mathscr{H}$,
- a self-adjoint operator $D\!\!\!/$ defined on a sense subspace $\mathrm{dom}(D\!\!\!/)$ of $\mathscr{H}$, with compact resolvent,
- a unital C*-algebra $\mathfrak{A}$, *-represented on $\mathscr{H}$,

such that

$$\mathfrak{A}_{D\!\!\!/} = \{a \in \mathfrak{A} : a\,\mathrm{dom}(D\!\!\!/) \subseteq \mathrm{dom}(D\!\!\!/) \text{ and } [D, a] \text{ is bounded}\}$$

is a dense *-subalgebra of $\mathfrak{A}$.

Connes' distance

Let $(\mathfrak{A}, \mathscr{H}, D\!\!\!/)$ be a spectral triple. For any $a \in \mathfrak{A}_{\mathfrak{D}}$, we set

$$\mathsf{L}(a) = |||[D\!\!\!/, a]|||_{\mathscr{H}}.$$

We then set, for any $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$,

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\left\{\left|\varphi(a) - \psi(a)\right| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}.$$

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mk_{L} is the noncommutative analogue of the Monge-Kantorovich metric, called the *Connes metric* of $(\mathfrak{A}, \mathscr{H}, \not{D})$.

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Definition

A *spectral triple* $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is *metric* when its Connes' metric induces the weak* topology on the state space $\mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space* when:

1. $\mathfrak{A}$ is a *unital C*-algebra*,

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5. $\mathsf{L}\left(\frac{ab+ba}{2}\right) \vee \mathsf{L}\left(\frac{ab-ba}{2i}\right) \leqslant F\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}\right) + K\mathsf{L}(a)\mathsf{L}(b)$;

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6. $\{a \in \mathrm{dom}(\mathsf{L}) : \mathsf{L}(a) \leqslant 1\}$ is closed in $\mathfrak{A}$.

We call L an *L-seminorm*.

A characterization of compact quantum metric spaces

Theorem (Rieffel, 98)

Let $\mathfrak{A}$ be a unital C*-algebra and let L be a seminorm defined on a dense subspace of $\mathfrak{sa}(\mathfrak{A})$ with $\ker \mathsf{L} = \mathbb{R}1_{\mathfrak{A}}$. Define, for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$,

$$\mathsf{mk}_{\mathsf{L}}(\varphi,\psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1\right\}.$$

The metric mk_{L} induces the weak* topology on $\mathscr{S}(\mathfrak{A})$ if, and only if there exists a state $\mu \in \mathscr{S}(\mathfrak{A})$ such that $\left\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1, \mu(a) = 0\right\}$ is totally bounded.

Digression: Locally Compact quantum metrics

Theorem (L., 05)

Let $\mathfrak{A}$ be a separable C*-algebra, L a seminorm defined on a dense subspace of $\mathfrak{sa}(\mathfrak{A})$ with $\ker \mathsf{L} = \{0\}$ (if $\mathfrak{A}$ has no unit) or $\ker \mathsf{L} = \mathbb{R}1_{\mathfrak{A}}$. Define, for all $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$,

$$\mathsf{bL}_{\mathsf{L}}(\varphi,\psi) = \sup\left\{|\varphi(a) - \psi(a)| : a \in \mathfrak{sa}(\mathfrak{A}), \mathsf{L}(a) \leqslant 1, \|a\|_{\mathfrak{A}} \leqslant 1\right\}.$$

The metric bL_{L} induces the weak* topology on $\mathscr{S}(\mathfrak{A})$ if, and only if there exists a strictly positive $h \in \mathfrak{A}$ such that $\{hah \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) \leqslant 1, \|a\|_{\mathfrak{A}} \leqslant 1\}$ is totally bounded.

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What about the *Monge-Kantorovich metric*? It does not usually metrize the weak* topology on the state space, but it does on some special subsets, via Dobrushin (70). A noncommutative version of Dobrushin's work is found in *L., 13*.

Quantum Isometries

A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (99), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \quad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) = \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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Theorem (Rieffel, 99)

If $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a quantum isometry, then $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is an isometry from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ into $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

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Theorem (L., 18)

If $(\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2)$ are two *unitarily equivalent metric spectral triples*, then $(\mathfrak{A}_1, \mathsf{L}_{\not{D}_1})$ and $(\mathfrak{A}_2, \mathsf{L}_{\not{D}_2})$ are *fully quantum isometric*.

The Dual Gromov-Hausdorff Propinquity

$$\begin{array}{ccc} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) & \\ {}^{\pi_{\mathfrak{A}}}\swarrow & & \searrow^{\pi_{\mathfrak{B}}} \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{array}$$

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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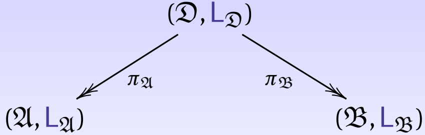

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Definition *(The extent of a tunnel, L. 13,14)*

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\},$$

where

$$\pi_{\mathfrak{A}}^{*} : \varphi \in \mathscr{S}(\mathfrak{A}) \longmapsto \varphi \circ \pi_{\mathfrak{A}} \in \mathscr{S}(\mathfrak{D})$$

and similarly for $\pi_{\mathfrak{B}}^{*}$.

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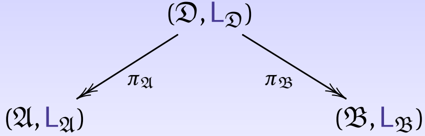

Figure: $\pi_\mathfrak{A}, \pi_\mathfrak{B}$ are quantum isometries

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The *dual propinquity* $\Lambda^*((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_\mathfrak{A}) \text{ to } (\mathfrak{B}, \mathsf{L}_\mathfrak{B})\right\}.$$

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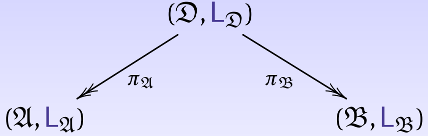

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Theorem (L., 13)

The *dual propinquity* Λ^*, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ by:

$$\inf\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\}$$

is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

The Dual Gromov-Hausdorff Propinquity

$$\begin{array}{ccc} & (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) & \\ \swarrow^{\pi_{\mathfrak{A}}} & & \searrow^{\pi_{\mathfrak{B}}} \\ (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) & & (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \end{array}$$

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

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Metrical C*-correspondences

If $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is a *metric spectral triple*, then

$$\mathrm{qvb}(\mathfrak{A}, \mathscr{H}, \mathbb{D}) := (\mathscr{H}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathbb{D}}, \mathbb{C}, 0)$$

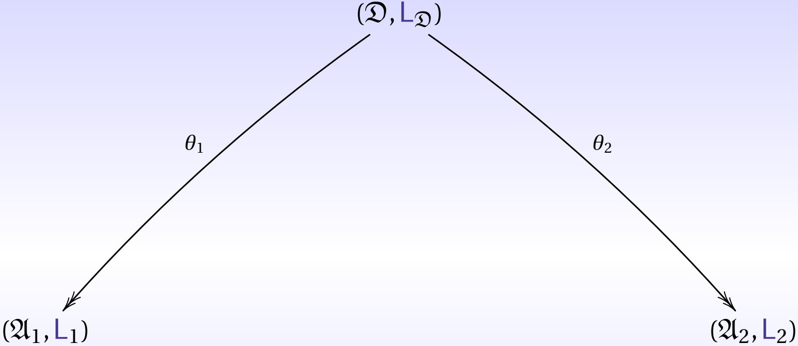
where $\mathsf{D} = \|\cdot\|_{\mathscr{H}} + \|\mathbb{D}\cdot\|_{\mathscr{H}}$, is an example of the following structure.

Definition (L. (16,18,19))

A *metrical C*-correspondence* $\Omega = (\mathscr{M}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is given by:

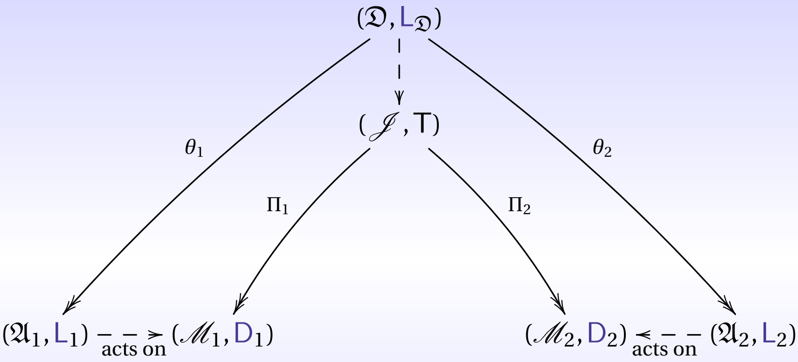
1. two *quantum compact metric spaces* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$,
2. an $\mathfrak{A}$-$\mathfrak{B}$ C*-correspondence $\mathscr{M}$, with $\mathfrak{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathscr{M}}$,
3. D is a *norm* on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
   3. $\forall \eta, \omega \in \mathscr{M} \quad \max\{\mathsf{L}_{\mathfrak{B}}(\Re\langle\omega,\eta\rangle_{\mathscr{M}}), \mathsf{L}_{\mathfrak{B}}(\Im\langle\omega,\eta\rangle_{\mathscr{M}})\} \leqslant H\mathsf{D}(\omega)\mathsf{D}(\eta)$,
   4. $\forall \eta \in \mathscr{M} \quad \forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{D}(a\eta) \leqslant G(\|a\|_{\mathfrak{A}} + \mathsf{L}_{\mathfrak{A}}(b))\mathsf{D}(\eta)$.

Tunnels between Metrical C-correspondeces*



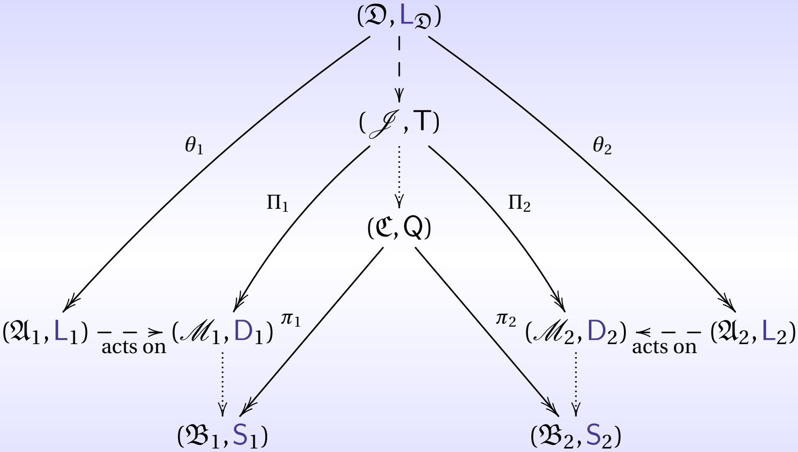
A tunnel: $\mathsf{L}_j(a) = \inf \mathsf{L}_{\mathfrak{D}}(\theta_j^{-1}(\{a\}))$.

Tunnels between Metrical C*-correspondeces



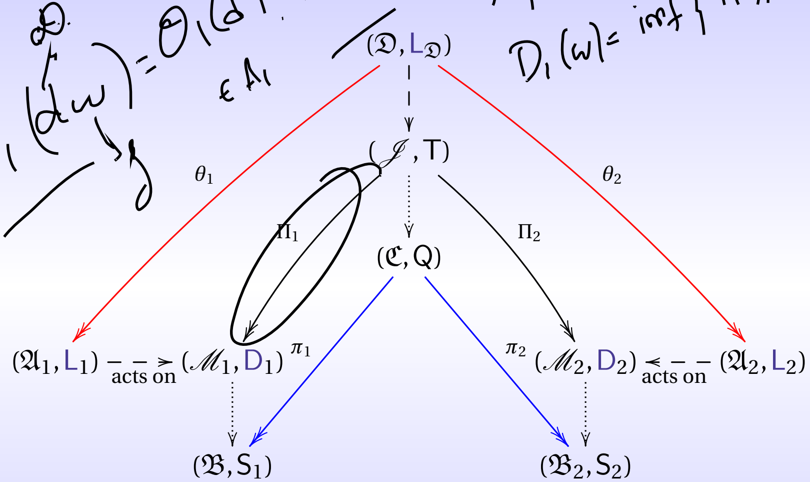
$\mathscr{J}$ is a $\mathfrak{D}$-module, $\mathsf{D}_j(\omega) = \inf \mathsf{T}(\Pi_j^{-1}(\{\omega\}))$, T D-norm

Tunnels between Metrical C*-correspondeces



$\mathscr{J}$ is a $\mathfrak{D}$-$\mathfrak{C}$-C*-corr; $(\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)$ tunnel.

Extent of Metrical Tunnels



$$\chi(\tau) = \max\left\{\chi\left((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2)\right), \chi\left((\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)\right)\right\}.$$

The metrical Propinquity

Definition (L. 16,18)

The *metrical propinquity* $\Lambda^{\mathsf{met}}(\mathbb{A}_1, \mathbb{A}_2)$ between two metrical C*-correspondences $\mathbb{A}_1$ and $\mathbb{A}_2$, is defined by

$$\inf \left\{ \chi(\tau) : \tau \text{ is a tunnel from } \mathbb{A}_1 \text{ to } \mathbb{A}_2 \right\}.$$

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A nontrivial example of convergence of modules for our metric is given by *Heisenberg modules over quantum tori*, where the D-norm arises from *Connes' connection*.

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When applied to spectral triples, the metrical propinquity provides metric information, but, maybe most importantly, it encodes the *convergence of the domains of the Dirac operators.*

GPS

Phase

The phase of a Dirac operator

What do we need to add to our construction of our metric on spectral triples so that distance 0 means equivalence by conjugation?

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The phase of a Dirac operator

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If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a spectral triple then:

$$t \in \mathbb{R} \mapsto U_t = \exp(i t \not{D})$$

is a strongly continuous action of $\mathbb{R}$ on $\mathscr{H}$ such that:

$$\forall t \in \mathbb{R}, \xi \in \mathscr{H} \quad \mathsf{D}(U_t \xi) = \|U_t \xi\|_{\mathscr{H}} + \|\not{D} U_t \xi\|_{\mathscr{H}} = \mathsf{D}(\xi).$$

Phase

The phase of a Dirac operator

What do we need to add to our construction of our metric on spectral triples so that distance 0 means equivalence by conjugation?

If $(\mathfrak{A}, \mathscr{H}, D\!\!\!/)$ is a spectral triple then:

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Next step

How do we incorporate convergence of group actions?

Covariant Reach of a Tunnel

Definition (L., 18)

Let $\tau = ((\mathscr{J}, \mathsf{T}, \ldots), (\Pi_1, \ldots), (\Pi_2, \ldots))$ be a metrical tunnel from $(\mathfrak{A}_1, \mathscr{H}_1, \mathsf{D}_1)$ to $(\mathfrak{A}_2, \mathscr{H}_2, \mathsf{D}_2)$. Let $\varepsilon > 0$. The *reach* $\rho^m(\tau|\varepsilon)$ of τ is

$$\underbrace{\sup_{\substack{\xi \in \mathscr{H}_j \\ \mathsf{D}_j(\xi) \leqslant 1}} \inf_{\substack{\eta \in \mathscr{H}_k \\ \mathsf{D}_k(\eta) \leqslant 1}}}_{\substack{\text{Hausdorff distance} \\ \forall\exists}} \overbrace{\sup_{0 \leqslant t \leqslant \frac{1}{\varepsilon}}}^{\substack{\text{orbital} \\ \text{uniform}}} \underbrace{\sup_{\substack{\omega \in \mathscr{J} \\ \mathsf{T}(\omega) \leqslant 1}} \left| \langle U_j^t \xi, \Pi_j(\omega) \rangle_{\mathscr{H}_j} - \langle U_k^t \eta, \Pi_k(\omega) \rangle_{\mathscr{H}_k} \right|}_{\text{distance}}$$

Covariant Reach of a Tunnel

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The *ε-magnitude* $\mu^m(\tau|\varepsilon)$ of τ is $\max\left\{\chi(\tau), \rho^m(\tau|\varepsilon)\right\}$.

Covariant Reach of a Tunnel

Definition (L., 18)

Let $\tau = ((\mathscr{J}, \mathsf{T}, \ldots), (\Pi_1, \ldots), (\Pi_2, \ldots))$ be a metrical tunnel from $(\mathfrak{A}_1, \mathscr{H}_1, \mathbb{D}_1)$ to $(\mathfrak{A}_2, \mathscr{H}_2, \mathbb{D}_2)$. Let $\varepsilon > 0$. The *reach* $\rho^m(\tau|\varepsilon)$ of τ is

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The *ε-magnitude* $\mu^m(\tau|\varepsilon)$ of τ is $\max\{\chi(\tau), \rho^m(\tau|\varepsilon)\}$.

This construction is a special case for our notion of convergence of actions of monoids over compact quantum metric spaces and their modules (which usually involves means to also approximate proper monoids).

The Spectral Propinquity

Definition (L., 18)

The *spectral propinquity* $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1, \mathscr{H}_1, \mathbb{D}_1), (\mathfrak{A}_2, \mathscr{H}_2, \mathbb{D}_2))$ between two metric spectral triples $(\mathfrak{A}_1, \mathscr{H}_1, \mathbb{D}_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, \mathbb{D}_2)$ is

$$\inf\left\{ \frac{\sqrt{2}}{2}\varepsilon, \varepsilon > 0 : \exists \tau \text{ tunnel from } (\mathfrak{A}_1, \mathscr{H}_1, \mathbb{D}_1) \right. \\ \left. \text{to } (\mathfrak{A}_2, \mathscr{H}_2, \mathbb{D}_2) \text{ such that } \mu^m(\tau|\varepsilon) \leqslant \varepsilon \right\}.$$

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Theorem (L., 18)

The *spectral propinquity* Λ^{spec} is a *metric* on the class of spectral triples, up to unitary equivalence, i.e. $\Lambda^{\mathrm{spec}}((\mathfrak{A}_1,\mathscr{H}_1,\mathbb{D}_1),(\mathfrak{A}_2,\mathscr{H}_2,\mathbb{D}_2)) = 0$ if, and only if there exists a *unitary* $U:\mathscr{H}_1\to\mathscr{H}_2$ such that $U\mathrm{dom}(\mathbb{D}_1)=\mathrm{dom}(\mathbb{D}_2)$,

$$U\mathbb{D}_1U^* = \mathbb{D}_2 \text{ and } \mathrm{Ad}_U \text{ *-isomorphism from } \mathfrak{A}_1 \text{ to } \mathfrak{A}_2.$$

GPS

Matrix Models: Clock and Shift, Fuzzy Tori

$$C_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & r_n^{n-1} \end{pmatrix} \text{ and } S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

with $r_n = \exp\left(\frac{2i\pi}{n}\right)$. Note that $C^*(C_n, S_n) = \mathfrak{B}(\mathbb{C}^n)$.

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with $r_n = \exp\left(\frac{2i\pi}{n}\right)$. Note that $C^*(C_n, S_n) = \mathfrak{B}(\mathbb{C}^n)$. Let $\mathbb{D}_n$ be defined on $\mathscr{H}_n = \mathfrak{B}(\mathbb{C}^n) \otimes \mathbb{C}^4$ be defined by:

$$\mathbb{D}_n = \frac{n}{2\pi}\left(\left[\frac{S_n + S_n^*}{2}, \cdot\right] \otimes \gamma_1 + \left[\frac{S_n - S_n^*}{2i}, \cdot\right] \otimes \gamma_2 + \left[\frac{C_n^* + C_n}{2}, \cdot\right] \otimes \gamma_3 + \left[\frac{C_n^* - C_n}{2i}, \cdot\right] \otimes \gamma_4 \right),$$

with $\gamma_j\gamma_k + \gamma_k\gamma_j = \delta_j^k$ $(j, k \in \{1,2,3,4\})$. Then *$(C^*(C_n, S_n), \mathscr{H}_n, \mathbb{D}_n)$ is a spectral triple*.

Convergence of Fuzzy tori

Theorem (L., 21)

The sequence $\left(C^*(C_n, S_n), \mathfrak{B}(\mathbb{C}^n) \otimes \mathbb{C}^4, \not{D}_n\right)_{n\in\mathbb{N}}$, where

$$\not{D}_n = \frac{n}{2\pi}\left(\left[\frac{S_n + S_n^*}{2}, \cdot \right] \otimes \gamma_1 + \left[\frac{S_n - S_n^*}{2i}, \cdot \right] \otimes \gamma_2 + \left[\frac{C_n^* + C_n}{2}, \cdot \right] \otimes \gamma_3 + \left[\frac{C_n^* - C_n}{2i}, \cdot \right] \otimes \gamma_4 \right),$$

converges, for the spectral propinquity, to the spectral triple $(C(T^2), L^2(\mathbb{T}^2) \otimes \mathbb{C}^4, \not{D}_{\mathbb{T}^2})$, where $C(\mathbb{T}^2) = \left\{ \left(e^{i\theta}, e^{i\psi}\right) : \theta, \psi \in [0, 2\pi] \right\}$, and on a dense subspace of $L^2(\mathbb{T}^2) \otimes \mathbb{C}^4$, we set

$$\not{D}_{\mathbb{T}^2} = \cos(\psi)\partial_\theta \otimes \gamma_1 + \sin(\psi)\partial_\theta \otimes \gamma_2 + \cos(\theta)\partial_\psi \otimes \gamma_3 + \sin(\theta)\partial_\psi \otimes \gamma_4.$$

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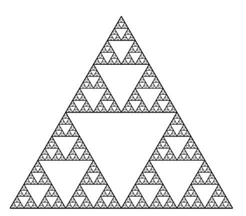
This result is extended to more general fuzzy/quantum tori (L., 21).

Piecewise C^1 Fractals Curves

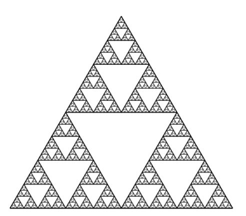


The Sierpińsky gasket

Let $\boxed{\mathscr{J} = \oplus_{n=1}^{\infty} \oplus_{j=1}^{3^n} L^2([-1,1])}$ and $\boxed{\partial(\xi) = \left(2^n \xi'_{n,j}\right)_{n\in\mathbb{N}, j\in\{1,\ldots,3^n\}}}$.

For $n \in \mathbb{N} \setminus \{0\}$, let $C_{n,1}, \ldots, C_{n,3^n}$ be affine functions from $[0,1]$, which parametrize every edge of the level n triangles in $\mathscr{S}\mathscr{G}_n$.

Piecewise C^1 Fractals Curves



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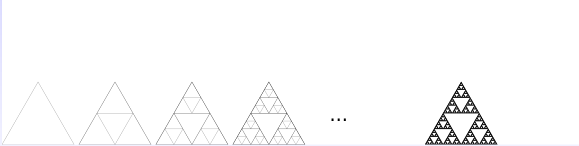
For $n \in \mathbb{N} \setminus \{0\}$, let $C_{n,1}, \ldots, C_{n,3^n}$ be affine functions from $[0,1]$, which parametrize every edge of the level n triangles in $\mathscr{S}\mathscr{G}_n$. For $f \in C(\mathscr{S}\mathscr{G}_\infty)$, we set $\boxed{f \cdot \xi_{n,j} := f \circ C_{n,j}(|\cdot|)\xi_{n,j}}$.

$\left(C(\mathscr{S}\mathscr{G}_\infty), \mathscr{J}, \partial\right)$ is a spectral triple, constructed by *Lapidus et al*.

Convergence of certain fractals



For each $n \in \mathbb{N}$, let ∂_n be the restriction of ∂ to $\mathscr{J}_N := \oplus_{n=1}^{N} \oplus_{j=1}^{3^N} L^2([-1,1])$, and let $C(\mathscr{S}\mathscr{G}_N)$ acts on $\mathscr{J}_N$.

Theorem (Landry, Lapidus, L., 20)

Let $(C(\mathscr{S}\mathscr{G}_\infty), \mathscr{J}, \partial)$ be the spectral triple over the Sierpiński gasket $\mathscr{S}\mathscr{G}_\infty$, and for each $n \in \mathbb{N}$, let $(\mathscr{S}\mathscr{G}_n, \mathscr{J}_n, \partial_n)$ be its "restriction" to the finite graph $\mathscr{S}\mathscr{G}_n$. Then

$$\lim_{n\to\infty} \Lambda^{\mathrm{spec}}((C(\mathscr{S}\mathscr{G}_\infty), \mathscr{J}, \partial), (C(\mathscr{S}\mathscr{G}_n), \mathscr{J}_n, \partial_n)) = 0.$$

Convergence of the Spectrum

Theorem (L., 21)

If $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples converging to $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$ for the spectral propinquity, then:

$$\mathrm{Sp}\,(\mathbb{D}_\infty) = \left\{\lambda \in \mathbb{R} : \forall_{\mathbb{N}}\, n \quad \exists \lambda_n \in \mathrm{Sp}\,(\mathbb{D}_n) \quad \lambda = \lim_{n\to\infty} \lambda_n \right\}.$$

Moreover, under reasonable assumptions, the *multiplicity of the eigenvalues converge as well.*

Convergence of multiplicities

Theorem (L., 21)

If $(\mathfrak{A}_n, \mathscr{H}_n, D_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples *converging, for the spectral propinquity,* to a metric spectral triple $(\mathfrak{A}_\infty, \mathscr{H}_\infty, D_\infty)$, and if $\lambda \in \mathrm{Sp}(D_\infty)$, such that:

1. there exists $\delta > 0$ and $N \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $n > N$, the intersection $\mathrm{Sp}(D_n) \cap (\lambda - \delta, \lambda + \delta)$ is a *singleton* $\{\lambda_n\}$,
2. if $(\mathrm{multiplicity}(\lambda_n | D_n))_{n\in\mathbb{N}}$ *converges*,

then

$$\lim_{n\to\infty} \mathrm{multiplicity}(\lambda_n | D_n) = \mathrm{multiplicity}(\lambda | D_\infty).$$

Convergence of bounded functional calculus

Theorem (L., 21)

If $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)_{n\in\mathbb{N}}$ is a sequence of metric spectral triples *converging* to $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$ for the spectral propinquity, if

$$\tau_n = \left[\left(\mathscr{J}_n, \mathsf{T}_n, \ldots\right), (\Pi_n, \ldots), (\Theta_n, \ldots)\right]$$

is a metrical tunnel from $(\mathfrak{A}_n, \mathscr{H}_n, \mathbb{D}_n)$ to $(\mathfrak{A}_\infty, \mathscr{H}_\infty, \mathbb{D}_\infty)$, let $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and such that $\mu^m(\tau_n|\varepsilon_n) \leqslant \varepsilon_n$, and if $f \in C_b(\mathbb{R})$, then,

$$\lim_{n\to\infty} \sup_{\substack{\xi\in\mathscr{H}_n \\ \mathsf{D}_j(\omega)\leqslant 1}} \inf_{\substack{\eta\in\mathscr{H}_\infty \\ \mathsf{D}_\infty(\eta)\leqslant 1}} \sup_{\substack{\omega\in\mathscr{J}_n \\ \mathsf{T}_n(\omega)\leqslant 1}} \left|\langle f(\mathbb{D}_n)\xi, \Pi_n(\omega)\rangle_{\mathscr{H}_n} - \langle f(\mathbb{D}_\infty)\eta, \Theta_n(\omega)\rangle_{\mathscr{H}_\infty}\right| = 0,$$

and similarly with $\mathscr{H}_n$ and $\mathscr{H}_\infty$ switched.

Thank you!

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- *The modular Gromov-Hausdorff propinquity*, F. Latrémolière, Dissertationes Math. **544** (2019), 70 pp. 46L89 (46L30 58B34)
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- *The Gromov-Hausdorff propinquity for metric spectral triples*, F. Latrémolière, *Adv. Math.* **404** (2022), paper no. 108393, 56 pp.