The Gromov-Hausdorff distance in noncommutative geometry

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Noncommutative geometry: metric and spectral aspects
September, 28st 2022.

A geometry for the space of quantum spaces

Founding Allegory of Noncommutative Geometry

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- Motivated by mathematical physics.
- The focus is on various generalizations of the *Gromov-Hausdorff* distance.

Spectral Triples

Spectral triples have emerged as the preferred method to encode geometric information about quantum spaces.

Definition (Connes, 85)

A *spectral triple* $(\mathfrak{A}, \mathscr{H}, \not{D})$ is given by:

- a Hilbert space $\mathscr{H}$,
- a self-adjoint operator $\not{D}$ defined on a sense subspace $\mathrm{dom}(\not{D})$ of $\mathscr{H}$, with compact resolvent,
- a unital C*-algebra $\mathfrak{A}$, *-represented on $\mathscr{H}$,

such that

$$\mathfrak{A}_{\not{D}} = \{a \in \mathfrak{A} : a\,\mathrm{dom}(\not{D}) \subseteq \mathrm{dom}(\not{D}) \text{ and } [D, a] \text{ is bounded}\}$$

is a dense *-subalgebra of $\mathfrak{A}$.

The fundamental example

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Let M be a *compact spin Riemannian manifold*. Let $\mathbb{D}$ be the (closure of) Dirac operator of M, which acts on the Hilbert space $\mathscr{H}$ of square integrable sections of the spinor bundle of M. Then *$(C(M), \mathscr{H}, \mathbb{D})$ is a spectral triple.*

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Theorem (Connes, 89)

If M is moreover connected, then for all $x, y \in M$, the *distance from x to y* is:

$$\sup\left\{|f(x) - f(y)| : \left|\left|\left|[D\!\!\!/, f]\right|\right|\right|_{\mathscr{H}} \leqslant 1\right\}.$$

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The proof hinges on the fact: $[\not{D}, f]$ is the Clifford multiplication by $\operatorname{grad} f$, so $\left|\!\left|\!\left|[\not{D}, f]\right|\!\right|\!\right|$ is the *Lipschitz constant* of f.

The Monge-Kantorovich metric

Let (X, m) be a compact metric space. The *Lipschitz seminorm* L induced by m is:

$$\mathsf{L}(f) = \sup\left\{ \frac{|f(x) - f(y)|}{\mathsf{m}(x,y)} : x, y \in X, x \neq y \right\}$$

for all $f \in \mathfrak{sa}(C(X)) = C(X, \mathbb{R})$ (allowing ∞).

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The *Monge-Kantorovich metric* on $\mathscr{S}(C(X))$ is given for all Borel-regular probability measures μ, ν by:

$$\mathsf{mk}_{\mathsf{L}}(\mu,\nu) = \sup\left\{ \left| \int_X f\, d\mu - \int_X f\, d\nu \right| : f \in \mathfrak{sa}(C(X)), \mathsf{L}(f) \leqslant 1 \right\}.$$

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The Gelfand map $x \in (X, \mathsf{m}) \mapsto \delta_x \in (\mathscr{S}(C(X)), \mathsf{mk}_{\mathsf{L}})$ is an isometry.

Compact Quantum Metric Spaces

Definition (Connes, 89; Rieffel, 98; L., 13)

$(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space* when:

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5. $\mathsf{L}\left(\frac{ab+ba}{2}\right) \vee \mathsf{L}\left(\frac{ab-ba}{2i}\right) \leqslant F\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \mathsf{L}(a)\|b\|_{\mathfrak{A}}\right) + K\mathsf{L}(a)\mathsf{L}(b)$;

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We call L an *L-seminorm*.

Examples of Compact Quantum Metric Spaces

Example: Rieffel, 98

If α is an *action of a compact metric group* G with *continuous length function* ℓ, acting on a unital C*-algebra $\mathfrak{A}$, and if

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup\left\{ \frac{\|\alpha^g(a) - a\|_{\mathfrak{A}}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

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Example: Aguilar, L., 15

Let $\mathfrak{A} = \mathrm{cl}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)$ be *AF*, with $\mathfrak{A}_0 = \mathbb{C}$, $\mathfrak{A}_n$ finite dimensional for all $n \in \mathbb{N}$, with a faithful tracial state τ. Then there exists a *conditional expectation* $\mathbb{E}_n : \mathfrak{A} \to \mathfrak{A}_n$ for all $n \in \mathbb{N}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup_{n\in\mathbb{N}} \dim(\mathfrak{A}_n) \|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}.$$

Then $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space*.

Metric Spectral Triples

Definition (L., 18)

A spectral triple $(\mathfrak{A}, \mathscr{H}, \mathbb{D})$ is *metric* when $\mathsf{mk}_{\mathbb{D}}$ is a metric on the state space $\mathscr{S}(\mathfrak{A})$, which induces the *weak* topology*.

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In other words, $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a metric spectral triple, if and only if, setting:

$$\mathrm{dom}(\mathsf{L}_{\not{D}}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : a \cdot \mathrm{dom}(\not{D}) \subseteq \mathrm{dom}(\not{D}), [\not{D}, a] \text{ bounded}\}$$

and, for all $a \in \mathrm{dom}(\mathsf{L}_{\not{D}})$,

$$\mathsf{L}_{\not{D}}(a) = |||[\not{D}, a]|||_{\mathscr{H}},$$

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Examples of Metric Spectral Triples

Example: Connes' recipe for groups (89)

If G is a discrete group, acting on $\ell^2(G)$ via left regular representation, and if l is some length function over G, then setting:

$$\forall \xi \in \ell^2(G) \quad \mathbb{D}\xi : g \in G \mapsto l(g)\xi(g)$$

defines a spectral triple $(C^*_{\mathrm{red}}(G), \ell^2(G), \mathbb{D})$.

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Example: Aguilar, Kaad, 18

Dabrowski and Sitarz's spectral triple on the *Podles sphere* is metric.

Spectral triples from compact Lie group actions

Example: R., 98, 21

Let G be a compact Lie group. Let $\langle \cdot, \cdot \rangle_G$ be some inner product on the Lie algebra $\mathfrak{g}$ of G — giving a left invariant Riemannian metric over G.

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Let α be a strongly continuous action of G on a unital C*-algebra $\mathfrak{A}$ such that $\left\{a \in \mathfrak{A} : \forall g \in G \quad \alpha^g(a) = a\right\} = \mathbb{C}1_{\mathfrak{A}}$.

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For each $X \in \mathfrak{g}$, let α_X be the derivation induced by α — $\alpha_X(a) = \lim_{t\rightarrow 0} \frac{\alpha^{\exp(tX)}a - a}{t}$ for a is some dense *-subalgebra $\mathfrak{A}_\infty$ common to all these derivations.

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Define $d : a \in \mathfrak{A}_\infty \mapsto (X \in \mathfrak{g} \mapsto \alpha^X) \in \mathfrak{A}_\infty \otimes \mathfrak{g}'$.
Let $\mathfrak{C}$ be the Clifford algebra of $\mathfrak{g}$ for $\langle\cdot,\cdot\rangle_G$.

$$\not{D} : \mathfrak{A}_\infty \otimes H \xrightarrow{d\otimes 1} \mathfrak{A}_\infty \otimes \mathfrak{g}' \otimes H \to \mathfrak{A}_\infty \otimes \mathfrak{C} \otimes H \xrightarrow{1\otimes c} \mathfrak{A}_\infty \otimes H.$$

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The Gromov-Hausdorff Distance

Te *Hausdorff distance* Haus_d between two closed subsets A_1 and A_2 of a compact metric space (X, d) is defined by

$$\mathsf{Haus}_d(A_1, A_2) = \max_{\{j,k\}=\{1,2\}} \sup_{x \in A_j} \inf_{y \in A_k} d(x, y).$$

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Definition (Hausdorff, 1903; Edwards, 75; Gromov, 81)

The *Gromov-Hausdorff distance* between two compact metric spaces (X,m_X) and (Y,m_Y) is:

$$\inf\left\{ \mathsf{Haus}_{\mathsf{m}_Z}(\iota_X(X),\iota_Y(Y)) \middle| \begin{array}{l} (Z,\mathsf{m}_Z) \text{ compact metric space,} \\ \iota_X : X \hookrightarrow Z \text{ isometry,} \\ \iota_Y : Y \hookrightarrow Z \text{ isometry.} \end{array} \right\},$$

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The *Gromov-Hausdorff distance* is a *complete metric*, up to isometry, on the class of compact metric spaces.

Quantum Isometries

A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

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Definition (Rieffel (99), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \rightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and

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A *full quantum isometry* π is a *-isomorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) = \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

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Theorem (Rieffel, 99)

If $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a quantum isometry, then $\pi^* : \varphi \in \mathscr{S}(\mathfrak{B}) \mapsto \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is an isometry from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ into $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

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A *Lipschitz morphism* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$.

Definition (Rieffel (99), L. (13))

A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and

$$\forall b \in \mathrm{dom}(\mathsf{L}_{\mathfrak{B}}) \qquad \boxed{\mathsf{L}_{\mathfrak{B}}(b) = \inf\{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.}$$

A *full quantum isometry* π is a *-isomorphism such that $\pi(\mathrm{dom}(\mathsf{L}_{\mathfrak{A}})) = \mathrm{dom}(\mathsf{L}_{\mathfrak{B}})$ and $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

Theorem (L., 18)

If $(\mathfrak{A}_1, \mathscr{H}_1, \not{D}_1)$ and $(\mathfrak{A}_2, \mathscr{H}_2, \not{D}_2)$ are two *unitarily equivalent metric spectral triples*, then $(\mathfrak{A}_1, \mathsf{L}_{\not{D}_1})$ and $(\mathfrak{A}_2, \mathsf{L}_{\not{D}_2})$ are *fully quantum isometric*.

The Dual Gromov-Hausdorff Propinquity

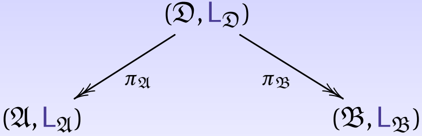

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

The Dual Gromov-Hausdorff Propinquity

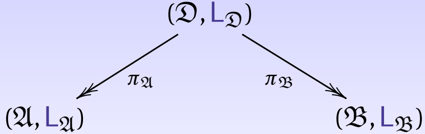

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Definition (The extent of a tunnel, L. 13,14)

The *extent* $\chi(\tau)$ of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{A}}^{*}(\mathscr{S}(\mathfrak{A}))\right), \mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_{\mathfrak{B}}^{*}(\mathscr{S}(\mathfrak{B}))\right)\right\},$$

where

$$\pi_{\mathfrak{A}}^{*} : \varphi \in \mathscr{S}(\mathfrak{A}) \mapsto \varphi \in \mathscr{S}(\mathfrak{D})$$

and similarly for $\pi_{\mathfrak{B}}^{*}$.

The Dual Gromov-Hausdorff Propinquity

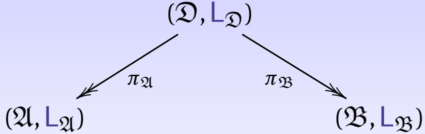

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Definition *(The Dual Propinquity, L. 13, 14)*

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The *dual propinquity* $\Lambda^{*}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ is given by:

$$\inf\left\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right\}.$$

The Dual Gromov-Hausdorff Propinquity

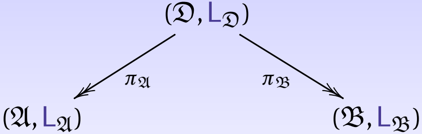

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

Theorem (L., 13)

The *dual propinquity* Λ^*, defined for any two quantum compact metric spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ by $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ by:

$$\inf\{\chi(\tau) : \tau \text{ any tunnel from } (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \text{ to } (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\}$$

is a *complete metric* up to *full quantum isometry*: $\Lambda((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0$ iff there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

The Dual Gromov-Hausdorff Propinquity

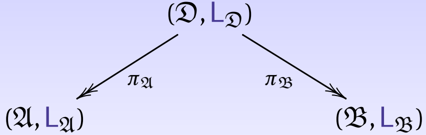

Figure: $\pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}$ are quantum isometries

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is a *complete metric* up to *full quantum isometry*. Moreover Λ^* induces the topology of the *Gromov-Hausdorff distance* on compact metric spaces.

Examples: Quantum and Fuzzy Tori

Example:

Let ℓ be a *continuous length function* on $\mathbb{T}^d$. For any $G \subseteq \mathbb{T}^d$ a closed subgroup and σ a multiplier of $\widehat{G}$, for any $a \in C^*(\widehat{G}, \sigma)$, set:

$$\mathsf{L}_{G,\sigma}(a) = \sup\left\{ \frac{\|\alpha^g(a) - a\|_{C^*(\widehat{G},\sigma)}}{\ell(g)} : g \in G \setminus \{1\} \right\}$$

where α is the dual action of G on $C^*(\widehat{G}, \sigma)$.

Rieffel showed in 1998 that $\left(C^*(\widehat{G}, \sigma), \mathsf{L}_{G,\sigma}\right)$ is a Leibniz quantum compact metric space.

Examples: Quantum and Fuzzy Tori

Example: L., 13

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where α is the dual action of G on $C^*(\widehat{G},\sigma)$.
If $(G_n)_{n\in\mathbb{N}}$ is a *sequence of closed subgroups* of $\mathbb{T}^d$ *converging* to $\mathbb{T}^d$ for the *Hausdorff distance* Haus_ℓ, and if $(\sigma_n)_{n\in\mathbb{N}}$ is a sequence of multipliers of $\mathbb{Z}^d$ converging pointwise to some σ, with $\sigma_n(g) = 1$ if g is the coset of 0 for $\widehat{G_n}$, then:

$$\lim_{n\to\infty} \Lambda^*((C^*(\widehat{G_n},\sigma_n), \mathsf{L}_{\widehat{G_n},\sigma_n}), (C^*(\mathbb{Z}^d,\sigma), \mathsf{L}_{\mathbb{Z}^d,\sigma})) = 0.$$

AF Algebras

Example: Aguilar, L. 15

Let $\mathfrak{A} = \mathrm{cl}(\bigcup_{n\in\mathbb{N}} \mathfrak{A}_n)$ be *AF*, with $\mathfrak{A}_0 = \mathbb{C}$, $\mathfrak{A}_n$ finite dimensional for all $n \in \mathbb{N}$, with a faithful tracial state τ. Then there exists a *conditional expectation* $\mathbb{E}_n : \mathfrak{A} \to \mathfrak{A}_n$ for all $n \in \mathbb{N}$. Set:

$$\forall a \in \mathfrak{A} \quad \mathsf{L}(a) = \sup_{n\in\mathbb{N}} \dim(\mathfrak{A}_n)\|a - \mathbb{E}_n(a)\|_{\mathfrak{A}}.$$

Then $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space*.

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Then $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space*.

We then have

$$\lim_{n\to\infty} \Lambda^*((\mathfrak{A}, \mathsf{L}), (\mathfrak{A}_n, \mathsf{L})) = 0.$$

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Then $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space*.

If $\mathfrak{A}_x = \mathrm{cl}\bigcup_{n\in\mathbb{N}} M_{\prod_{j=0}^{n} x_j}(\mathbb{C})$ is the UFH algebra for $x \in \mathbb{N}^{\mathbb{N}}$, with $\mathbb{N}^{\mathbb{N}}$ the Baire space, then

$$x \in \mathbb{N}^{\mathbb{N}} \longmapsto (\mathfrak{A}_x, \mathsf{L}_x)$$

is *continuous* from the Baire space to the space of quantum compact metric spaces with the *propinquity* (in fact, it is 2-Lipschitz).

AF Algebras

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Then $(\mathfrak{A}, \mathsf{L})$ is a *quantum compact metric space*.

For any $\theta \in \mathbb{R} \setminus \mathbb{Q}$, *Effros and Shen* constructed an inductive limit of finite dimensional C*-algebras with limit an AF algebra $\mathfrak{A}(\theta)$; using our construction, we obtain a *continuous map* $\theta \in \mathbb{R} \setminus \mathbb{Q} \mapsto \mathfrak{A}(\theta)$ from the Baire space to the space of quantum compact metric spaces with the *propinquity*.

1. *Metric Spectral Triples*
2. *The Gromov-Hausdorff Propinquity*
3. *Convergence of Metrical C*-correspondences*

Metrical C-correspondences*

If $(\mathfrak{A}, \mathscr{H}, \not{D})$ is a *metric spectral triple*, then

$$\mathrm{qvb}(\mathfrak{A}, \mathscr{H}, \not{D}) := (\mathscr{H}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\not{D}}, \mathbb{C}, 0)$$

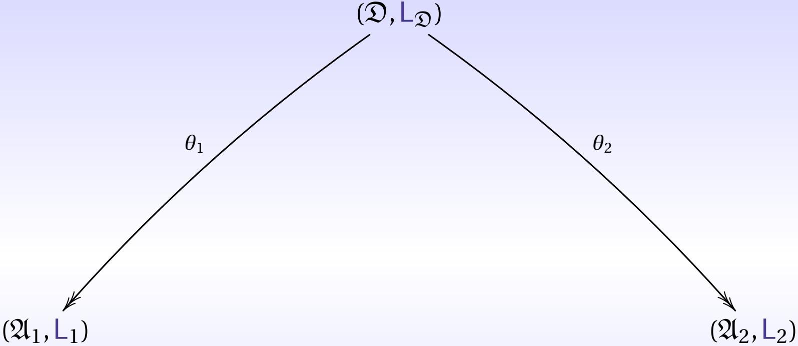
where $\mathsf{D} = \|\cdot\|_{\mathscr{H}} + \|\not{D}\cdot\|_{\mathscr{H}}$, is an example of the following structure.

Definition (L. (16,18,19))

A *metrical C*-correspondence* $\Omega = (\mathscr{M}, \mathsf{D}, \mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is given by:

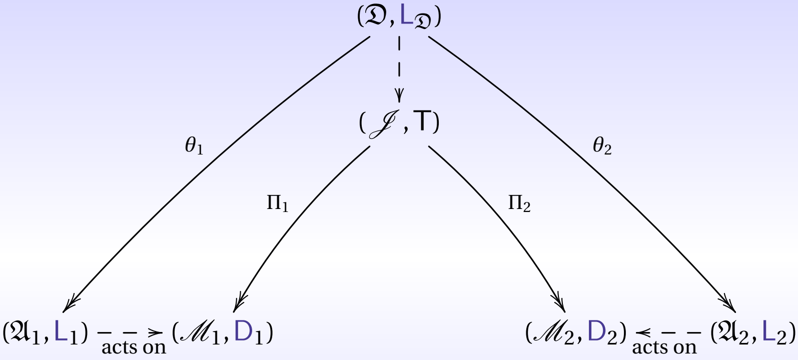
1. two *quantum compact metric spaces* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$,
2. an $\mathfrak{A}$-$\mathfrak{B}$ C*-correspondence $\mathscr{M}$, with $\mathfrak{B}$-valued inner product $\langle\cdot,\cdot\rangle_{\mathscr{M}}$,
3. D is a *norm* on a dense subspace of $\mathscr{M}$ such that:
   1. $\mathsf{D} \geqslant \|\cdot\|_{\mathscr{M}} := \sqrt{\langle\cdot,\cdot\rangle_{\mathscr{M}}}$
   2. $\{\omega \in \mathscr{M} : \mathsf{D}(\omega) \leqslant 1\}$ is compact in $(\mathscr{M}, \|\cdot\|_{\mathscr{M}})$,
   3. $\forall \eta, \omega \in \mathscr{M} \quad \max\{\mathsf{L}_{\mathfrak{B}}(\Re\langle\omega,\eta\rangle_{\mathscr{M}}), \mathsf{L}_{\mathfrak{B}}(\Im\langle\omega,\eta\rangle_{\mathscr{M}})\} \leqslant H\mathsf{D}(\omega)\mathsf{D}(\eta)$,
   4. $\forall \eta \in \mathscr{M} \quad \forall a \in \mathfrak{sa}(\mathfrak{A}) \quad \mathsf{D}(a\eta) \leqslant G(\|a\|_{\mathfrak{A}} + \mathsf{L}_{\mathfrak{A}}(b))\mathsf{D}(\eta)$.

Tunnels between Metrical C*-correspondeces



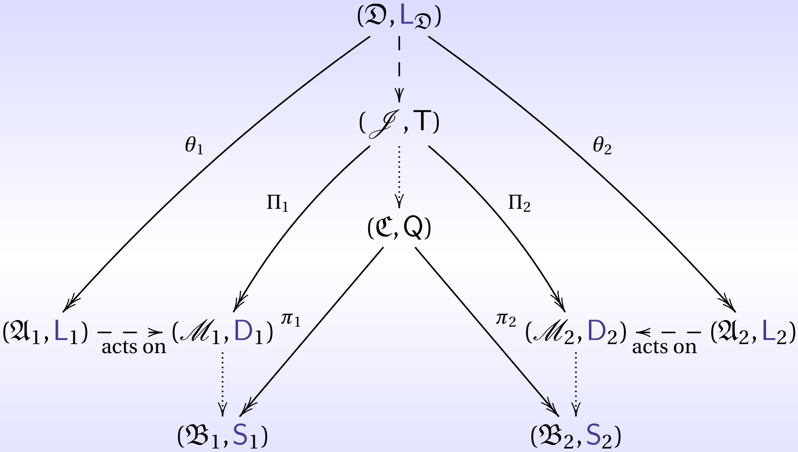
A tunnel: $\mathsf{L}_j(a) = \inf \mathsf{L}_{\mathfrak{D}}(\theta_j^{-1}(\{a\}))$.

Tunnels between Metrical C-correspondeces*



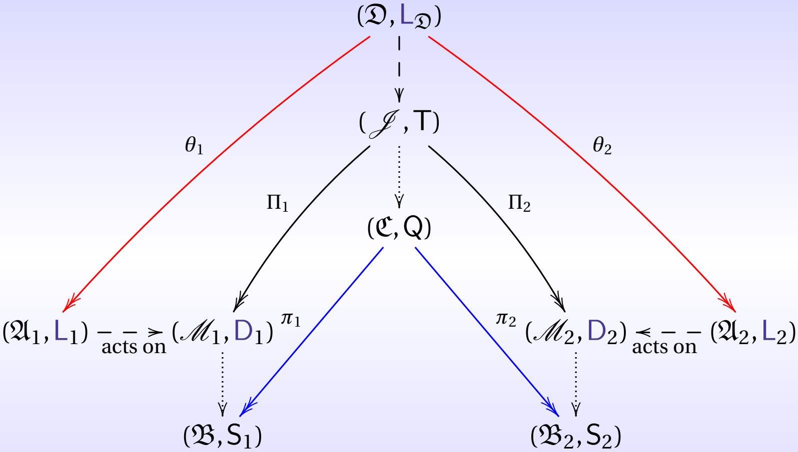
$\mathscr{J}$ is a $\mathfrak{D}$-module, $\mathsf{D}_j(\omega) = \inf \mathsf{T}(\Pi_j^{-1}(\{\omega\}))$, T D-norm

Tunnels between Metrical C-correspondeces*



$\mathscr{J}$ is a $\mathfrak{D}$-$\mathfrak{C}$-C*-corr; $(\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)$ tunnel.

Extent of Metrical Tunnels



$$\chi(\tau) = \max\left\{\chi\left((\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \theta_1, \theta_2)\right), \chi\left((\mathfrak{C}, \mathsf{Q}, \pi_1, \pi_2)\right)\right\}.$$

The metrical Propinquity

Definition (L. 16,18)

The *metrical propinquity* $\Lambda^{\mathsf{met}}(\mathbb{A}_1, \mathbb{A}_2)$ between two metrical C*-correspondences $\mathbb{A}_1$ and $\mathbb{A}_2$, is defined by

$$\inf\left\{\chi(\tau) : \tau \text{ is a tunnel from } \mathbb{A}_1 \text{ to } \mathbb{A}_2\right\}.$$

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A nontrivial example of convergence of modules for our metric is given by *Heisenberg modules over quantum tori*, where the D-norm arises from *Connes' connection*.

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When applied to spectral triples, the metrical propinquity provides metric information, but, maybe most importantly, it encodes the *convergence of the domains of the Dirac operators.*

The Heisenberg Modules (Connes, 81; Rieffel)

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$. Let $\mathscr{A}_\theta$ be the associated quantum torus.

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1. Start with a representation of $\left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R}^3 \right\}$ on $L^2(\mathbb{R})$:

$$\alpha_{\eth}^{x,y,t} \xi(s) = \exp(i\pi(t + 2xs))\xi(s + \eth y).$$

   Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.

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Promote it to $L^2(\mathbb{R}) \otimes \mathbb{C}^d$.
2. Let $W_1, W_2 \in U(d)$ with $W_1 W_2 = e^{2i\pi p/q} W_2 W_1$ and $W_1^n = W_2^n = 1$. We get a $\mathscr{A}_\theta = C^*(u_\theta, v_\theta)$-module with:
$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$

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$$(u_\theta^n v_\theta^m)\xi = W_1^n W_2^m \alpha_{\eth}^{n,m,0}\xi.$$
3. For Schwarz functions ξ, ω, set:
$$\langle \xi, \omega \rangle_{\mathscr{H}_\theta^{p,q,d}} = \sum_{n,m \in \mathbb{Z}} \langle u_\theta^n v_\theta^m \xi, \omega \rangle_{L^2(\mathbb{R},\mathbb{C}^d)} u_\theta^n v_\theta^m;$$
complete space of Schwarz functions to the *Heisenberg module* $\mathscr{H}_\theta^{p,q,d}$.

A Connection for The Heisenberg Modules

Fix $\theta \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, $d \in \mathbb{N} \setminus \{0\}$ such that $\eth = \theta - \frac{p}{q} \neq 0$. On the space $\mathscr{S}(\mathbb{C}^d)$ of $\mathbb{C}^d$-valued Schwarz functions (seen as a dense subspace of $\mathscr{H}_\theta^{p,q,d}$), we define

$$\forall \xi \in \mathscr{S}(\mathbb{C}^d) \quad \nabla_P \xi : s \in \mathbb{R} \mapsto \frac{d}{ds}\xi(s) \text{ and } \nabla_Q \xi : s \in \mathbb{R} \mapsto \frac{2i\pi s}{\eth}\xi(s).$$

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Connes and Rieffel define the *connection* ∇ on all $\xi \in \mathscr{S}(\mathbb{C}^d)$ by:

$$\nabla\xi : (x,y) \in \mathbb{R}^2 \mapsto \left(x\nabla_P + y\nabla_Q\right)\xi$$

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The *operator norm* $|||\nabla\xi|||_{\mathscr{H}_\theta^{p,q,d}}$ is given by:

$$\sup\left\{ \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}}\xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

D-norms for Heisenberg Modules

Theorem (L., 16)

Fix some norm $\|\cdot\|$ on $\mathbb{R}^2$. For all $\xi \in \mathscr{H}_\theta^{p,q,d}$, we set $\mathsf{D}_\theta^{p,q,d}(\xi)$ as:

$$\sup\left\{ \|\xi\|_{\mathscr{H}_\theta^{p,q,d}}, \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}} \xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}.$$

$\left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta\right)$ is a metrized quantum vector bundle.

Theorem (L., 17)

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$ and p, q, d fixed. If for all $\theta \in \mathbb{R}$, and $a \in \mathscr{A}_\theta$:

$$\mathsf{L}_\theta(a) = \sup\left\{ \frac{\left\| \beta_\theta^{\exp(ix),\exp(iy)} a - a \right\|_{\mathscr{A}_\theta}}{\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where β_θ is the dual action, and for all $\xi \in \mathscr{H}_\theta^{p,q,d}$ we set:

$$\mathsf{D}_\theta^{p,q,d}(\xi) = \sup\left\{ \frac{\left\| \alpha_{\eth}^{x,y,\frac{xy}{2}} \xi - \xi \right\|_{\mathscr{H}_\theta^{p,q,d}}}{2\pi|\eth|\|(x,y)\|} : (x,y) \in \mathbb{R}^2 \setminus \{0\} \right\}$$

where $\eth = \theta - p/q$, then:

$$\lim_{\vartheta \to \theta} \Lambda^{\mathsf{mod}}\left(\left(\mathscr{H}_\vartheta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\vartheta^{p,q,d}}, \mathsf{D}_\vartheta^{p,q,d}, \mathscr{A}_\vartheta, \mathsf{L}_\vartheta \right), \left(\mathscr{H}_\theta^{p,q,d}, \langle\cdot,\cdot\rangle_{\mathscr{H}_\theta^{p,q,d}}, \mathsf{D}_\theta^{p,q,d}, \mathscr{A}_\theta, \mathsf{L}_\theta \right) \right) = 0.$$

Thank you!

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