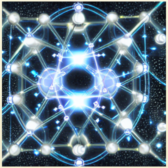
QUANTUM METRIC STRUCTURES ON q-DEFORMED SPACES



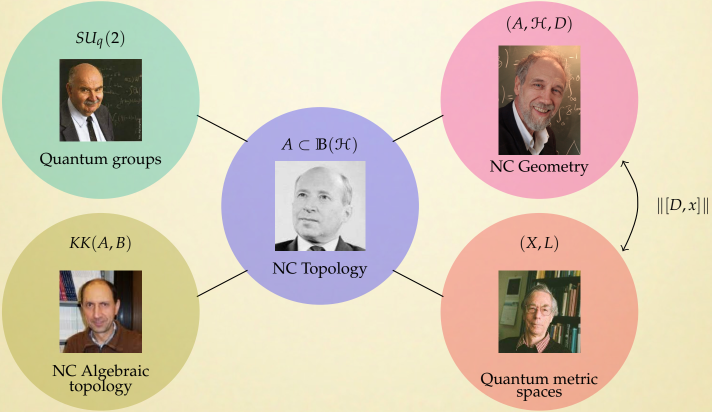
Noncommutative geometry: metric and spectral aspects

Jagiellonian University, Kraków

September 29, 2022

David Kyed

Based on joint works with Konrad Aguilar, Thomas Gotfredsen and Jens Kaad



QUESTION (RIEFFEL, 1990'S)

What is the non-commutative analogue of a compact metric space?

DEFINITION (RIEFFEL)

Let A be a unital C^*-algebra (or complete operator system) equipped with a seminorm $L\colon A \to [0,\infty]$ satisfying that $L(x^*) = L(x)$ for all $x \in A$. Then (A, L) is called a compact quantum metric space if

(i) $L(1) = 0$.

(ii) *The set* $\mathrm{Dom}(L) := \{a \in A \mid L(a) < \infty\}$ *is dense in* A.

(iii) $d_L(\mu, \nu) := \sup\{|\mu(a) - \nu(a)| : L(a) \leqslant 1\}$ *metrises the weak*-topology on* $\mathcal{S}(A)$.

In this case L is called a Lip-norm.

- If (X, d) is a compact metric space then $C(X)$ becomes a CQMS by setting $L_d(f) := \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)} : x \neq y\right\}$.

$\rightsquigarrow$ NCG examples

Examples from NCG

- A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ defines a seminorm

$$L_D(a) := \|[D, a]\| \qquad (a \in \mathcal{A})$$

- This sometimes – but not always – gives rise to a CQMS. Many examples were given in Frédéric's talk yesterday.
- Note that the domain of L_D matters: the most difficult one is $L_D^{\max}$ defined on

$$\mathrm{Lip}(A) := \{x \in A \mid [D, x] \text{ well-defined and bounded}\}$$

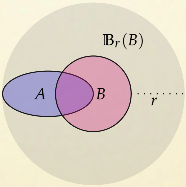
Here $A := \overline{\mathcal{A}} \subset \mathbb{B}(\mathcal{H})$.

$\rightsquigarrow$ next: classical GH-dist

Classical Gromov-Hausdorff distance

- Consider compact subsets A and B in a metric space.
- Then their *Hausdorff distance* is defined by

$$\mathrm{dist}_{\mathrm{H}}^{d}(A,B) := \inf\{r>0 \mid A \subset \mathbb{B}_r(B) \text{ and } B \subset \mathbb{B}_r(A)\}$$



- And for two compact metric spaces (X_1,d_1) and (X_2,d_2) their *Gromov-Hausdorff distance* is defined as

$$\mathrm{dist}_{\mathrm{GH}}(X_1,X_2) := \inf_{d}\left\{\mathrm{dist}_{\mathrm{H}}^{d}(X_1,X_2)\right\}$$

where the infimum runs over all metrics on $X_1 \sqcup X_2$ restricting to d_1 and d_2 respectively.

⇝ quantum-GH-distance

Quantum Gromov-Hausdorff distance

- If (A_1, L_1) and (A_2, L_2) are CQMS then a Lip-norm $L\colon A_1 \oplus A_2 \to [0,\infty]$ is called *admissible* if the induced quotient semi-norms on A_1 and A_2 agree with L_1 and L_2.
- The coordinate projections dualise to isometries

$$(\mathcal{S}(A_1), d_{L_1}) \hookrightarrow (\mathcal{S}(A_1 \oplus A_2), d_L) \hookleftarrow (\mathcal{S}(A_2), d_{L_2})$$

- And Rieffel then defines

$$\mathrm{dist}^{\mathrm{Q}}_{\mathrm{GH}}(A_1, A_2) := \inf \left\{ \mathrm{dist}^{d_L}_{\mathrm{H}}(\mathcal{S}(A_1), \mathcal{S}(A_2)) : L \text{ admissible} \right\}$$

- This is symmetric and satisfies the triangle inequality.
- But distance zero does not mean Lip-norm preserving $*$-isomorphism. This defect will be fixed in Frédéric's talks!
- $C\colon (X, d) \mapsto (C(X), L_d)$ is a contraction, but *not* an isometry.
- However, it is a homeomorphism onto its image.

$\rightsquigarrow$ continuity results

Convergence and continuity results

- Fuzzy spheres (i.e. matrix algebras) converge to the classical 2-sphere S^2 [Rieffel, 2004].
- – Non-commutative tori [Rieffel, 2004]
  – Spectral truncations [D'Andrea-Lizzi-Martinetti, 2014], [van Suijlekom, 2021]
  – Crossed products [Kaad-K, 2020]
  – Non-commutative solenoids [Latrémolière-Packer, 2017]
  – AF-algebras [Aguilar-Latrémolière, 2015]
  $\vdots$
- But q-deformations are conspicuous by their absence.

$\rightsquigarrow$ q-deformations

Quantum SU(2)

- The central object is Woronowicz' quantum $SU(2)$.
- This is the universal C^*-algebra $C(SU_q(2))$ generated by a and b subject to the relations arising by demanding that

$$u := \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix} \text{ be unitary.}$$

- This is a compact quantum group and the coordinate algebra

$$\mathcal{O}(SU_q(2)) := \mathrm{Alg}\{a, b, a^*, b^*\}$$

is a Hopf $*$-algebra: $(\mathcal{O}(SU_q(2)), \Delta, S, \epsilon)$
- Dually, one has the deformed enveloping Lie algebra $\mathcal{U}_q(\mathfrak{su}(2))$ with generators e, f, k, k^{-1} subject to the relations

$$kk^{-1} = 1 = k^{-1}k \qquad ek = qke \qquad kf = qfk$$

$$fe - ef = \frac{k^2 - k^{-2}}{q - q^{-1}} \qquad (q \neq 1)$$

$\rightsquigarrow$ the pairing

- There exists a non-degenerate pairing

$$\langle -, - \rangle : \mathcal{U}_q(\mathfrak{su}(2)) \times \mathcal{O}(SU_q(2)) \longrightarrow \mathbb{C}$$

- And thus a left- and right $\mathcal{U}_q(\mathfrak{su}(2))$-action on $\mathcal{O}(SU_q(2))$

$$\partial_\eta(a) := (1 \otimes \langle \eta, - \rangle)\Delta(a) \qquad \text{(left)}$$
$$\delta_\eta(a) := (\langle \eta, - \rangle \otimes 1)\Delta(a) \qquad \text{(right)}$$

for $\eta \in \mathcal{U}_q(\mathfrak{su}(2)), a \in \mathcal{O}(SU_q(2))$.
- The last ingredient needed is the circle action $S^1 \overset{\sigma}{\curvearrowright} C(SU_q(2))$ defined by rescaling a and b with $z \in S^1$.
- And its spectral subspaces

$$A_q^m := \{x \in C(SU_q(2)) \mid \forall z \in S^1 : \sigma_z(x) = z^m \cdot x\}$$
$$\mathcal{A}_q^m := \{x \in \mathcal{O}(SU_q(2)) \mid \forall z \in S^1 : \sigma_z(x) = z^m \cdot x\}$$

⇝ next: Podleś sphere

THE PODLEŚ SPHERE

- $C(S^2)$ is the fixed point algebra $C(SU(2))^{S^1}$ (Hopf fibration)
- The Podleś sphere is defined as $C(S_q^2) := C(SU_q(2))^{S^1} = A_q^0$
- It fits into a spectral triple $(\mathcal{O}(S_q^2), \mathcal{H}, D_q)$ where $\mathcal{O}(S_q^2) = \mathcal{O}(SU_q(2))^{S^1}$ and D_q is the closure of

$$\mathcal{D}_q = \begin{pmatrix} 0 & \partial_f \\ \partial_e & 0 \end{pmatrix} : \mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1} \longrightarrow \mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$$

  [Dąbrowski-Sitarz, 2003]
- Moreover, the commutator seminorm

$$L_q(x) := \|[D_q, x]\|, \qquad x \in \mathcal{O}(S_q^2).$$

  turns it into a QCMS [Aguilar-Kaad, 2018].
- They even proved it for $L_q^{\max}$.
- So the question in 2018 was: does S_q^2 converge to S^2 as q tends to 1?

Theorem (Aguilar-Kaad-K, 2021)

The family $(C(S_q^2), L_q^{\max})_{q\in]0,1]}$ *varies continuously in the quantum Gromov-Hausdorff distance. In particular,* $(C(S_q^2), L_q^{\max})$ *converges to* $(C(S^2), L_{d_{S^2}})$ *as* q *tends to* 1.

- The next natural step is to consider $SU_q(2)$ itself.
- Numerous Dirac operators have been proposed: [Masuda-Watanabe, 1994], [Bibikov-Kulish, 1997], [Chakraborty-Pal, 2002], [Dąbrowski-Landi-Sitarz-Suijlekom-Várilly, 2005], [Krähmer-Wagner, 2005], [Krähmer-Rennie-Senior, 2011], [Kaad-Senior, 2011], [Bhowmick-Voigt-Zacharias, 2015]……
- Most do not give spectral triples, but perhaps some of them could still provide a quantum "Riemannian metric"?
- Both the one suggested by Kaad-Senior and the one by Krähmer-Rennie-Senior seemed suitable.
- So we ended up considering a two parameter family $D_{t,q}$ where the t "interpolates" between these two candidates.

- Our substitute for the spinor bundle is $L^2(SU_q(2))^{\oplus 2}$.
- We then define a *horizontal Dirac* $\mathcal{D}_q^H := \begin{pmatrix} 0 & -q^{-1/2}\partial_{fk^{-1}} \\ -q^{1/2}\partial_{ek^{-1}} & 0 \end{pmatrix}$ with $\mathrm{Dom}(\mathcal{D}_q^H) = \mathcal{O}(SU_q(2))^{\oplus 2}$.
- And a *vertical Dirac* (with the same domain) by an additional parameter $t \in (0,1]$ and is given by

$$\mathcal{D}_t^V = \begin{pmatrix} t^{\frac{-n+1}{2}}\left[\frac{n-1}{2}\right]_t & 0 \\ 0 & -t^{\frac{-m-1}{2}}\left[\frac{m+1}{2}\right]_t \end{pmatrix}$$

  on $\mathcal{A}_q^n \oplus \mathcal{A}_q^m$.
- Here $[a]_t := \frac{a^t - a^{-t}}{t - t^{-1}}$ when $t \neq 1$ and $[a]_1 := a$.
- When $t = q \in (0,1)$, one has $\mathcal{D}_q^V = \frac{1}{q - q^{-1}} \begin{pmatrix} 1 - q\partial_{k^{-2}} & 0 \\ 0 & q^{-1}\partial_{k^{-2}} - 1 \end{pmatrix}$ and $\mathcal{D}_q^H + \mathcal{D}_q^V$ is the Dirac studied by Kaad-Senior.
- And $\mathcal{D}_q^H + \mathcal{D}_1^V$ is (almost) the Krähmer-Rennie-Senior Dirac.

⇝ properties

Properties of the Dirac operators

- Both $\mathcal{D}_q^H$ and $\mathcal{D}_t^V$ are essentially selfadjoint and we denote their closures by D_q^H and D_t^V.

Theorem (Kaad-K, 2022)

- *There exists a 1-parameter family of algebra automorphisms $(\sigma_r)_{r\in\mathbb{R}_+}$ of $\mathcal{O}(SU_q(2))$ such that the twisted commutators*

$$D_t^V\sigma_t(x) - \sigma_t^{-1}(x)D_t^V \quad \text{and} \quad D_q^H\sigma_q(x) - \sigma_q^{-1}(x)D_q^H,$$

extend to bounded operators $\partial_t^V(x)$ and $\partial_q^H(x)$.

- D_t^V *and* D_q^H *are $SU_q(2)$-equivariant.*
- *There exists an anti-unitary I with $I^2 = -1$ such that*

$$[\partial_t^V(x), IyI] = 0 = [\partial_q^H(x), IyI], \qquad x,y \in \mathcal{O}(SU_q(2))$$

- *When $t = q = 1$, one has $D_{S^3} = 2(D_1^H + D_1^V) + 1$.*

$\rightsquigarrow$ seminorm

- We may now define a seminorm

$$L_{t,q}(x) := \begin{cases} \|\partial_q^H(x) + \partial_t^V(x)\| & x \in \mathcal{O}(SU_q(2)) \\ \infty & x \in C(SU_q(2)) \setminus \mathcal{O}(SU_q(2)) \end{cases}$$

- When $t = q$, one has $\partial_q^H + \partial_q^V = \begin{pmatrix} \frac{\partial_k - \partial_{k^{-1}}}{q - q^{-1}} & -q^{-1/2}\partial_f \\ -q^{1/2}\partial_e & -\frac{\partial_k - \partial_{k^{-1}}}{q - q^{-1}} \end{pmatrix}$
- It too has a maximal version $L_{t,q}^{\max}$.
- The question now is: does $L_{t,q}^{\max}$ provide $SU_q(2)$ with a CQMS structure?
- If so, the same is automatically true for $L_{t,q}$.

⇝ next: CQMS

- The first clue is that $L_{t,q}^{\max}$ restricts to the commutator seminorm arising from the Dąbrowski-Sitarz spectral triple on $C(S_q^2)$.
- So by Aguilar-Kaad we know that $(C(S_q^2), L_{t,q}^{\max})$ is a CQMS.
- The *spectral bands* $B_q^M := \sum_{m=-M}^{M} A_q^m$ are operator systems and finitely generated projective (free) modules over $A_q^0 = C(S_q^2)$, and can therefore also be shown to be CQMS for the restriction of $L_{t,q}^{\max}$.
- With some additional care (and Schur multipliers) we were then able to bootstrap all the way up to $SU_q(2)$:

THEOREM (KAAD-K, 2022)

The pair $(C(SU_q(2)), L_{t,q}^{\max})$ is a CQMS.

$\rightsquigarrow$ continuity?

QUESTION

Is the map $(t,q) \mapsto (C(SU_q(2)), L^{\max}_{t,q})$ continuous with respect to the quantum Gromov-Hausdorff distance?

- For this one needs good *fuzzy approximations*.
- When studying S^2_q we built quantum fuzzy spheres.
- Using these, we now build fuzzy spectral M-bands.
- That is, for each $N \in \mathbb{N}$ we construct a finite dimensional sub-operator system

$$\mathrm{Fuzz}_N(B^M_q) \subseteq B^M_q.$$

- The hope then is that these vary continuously and approximate $SU_q(2)$ well enough when N and M tend to infinity.

⇝ continuity results

- To ease the arguments, we restrict to $t = q$ in the following.
- Firstly, the family $(\mathrm{Fuzz}_N(B_q^M), L_{q,q}^{\max})$ varies continuously for all N and M.
- This follows quite easily from finite-dimensionality and the fact that $C(SU_q(2))$ is a continuous field of C^*-algebras [Blanchard, 1996].
- To get that $\mathrm{Fuzz}_N(B_q^M)$ approximates $C(SU_q(2))$ we need:

Proposition (Kaad-K, 2022)

Let (X, L) be a CQMS and $Y \subseteq X$ a sub-CQMS. If there exists a unital contraction $\beta\colon X \to Y$ and $\varepsilon > 0$ s.t.

$$L(\beta(x)) \leqslant L(x) \quad \text{and} \quad \|\beta(x) - x\| \leqslant \varepsilon L(x),$$

then $\mathrm{dist}_Q(X, Y) \leqslant \varepsilon$.

- We thus need such $\beta_N^M\colon C(SU_q(2)) \to \mathrm{Fuzz}_N(B_q^M)$.

The Berezin transform

- We construct a suitable family of states χ_N^M and define
$$\beta_N^M(x) := (1 \otimes \chi_N^M)\Delta(x).$$
- They are chosen so that $\chi_N^M \underset{N,M}{\longrightarrow} \epsilon$ (weak*) from which it follows that $\beta_N^M \approx \mathrm{id}$ uniformly on the Lip unit ball.
- But why should these be Lip-contractions?
- Recall, $\partial_q^H + \partial_q^V = \begin{pmatrix} \frac{\partial_k - \partial_{k^{-1}}}{q-q^{-1}} & -q^{-1/2}\partial_f \\ -q^{1/2}\partial_e & -\frac{\partial_k - \partial_{k^{-1}}}{q-q^{-1}} \end{pmatrix} =: \partial_q$ on $\mathcal{O}(SU_q(2))$.
- Here is where the situation $t = q$ is quite special.

Lemma (Kaad-K, 2022)

It holds that $u\partial_q u^* = \begin{pmatrix} \frac{\delta_k - \delta_{k^{-1}}}{q-q^{-1}} & -q^{-1/2}\delta_f \\ -q^{1/2}\delta_e & -\frac{\delta_k - \delta_{k^{-1}}}{q-q^{-1}} \end{pmatrix} =: \delta_q$ *where* u *is the fundamental unitary corepresentation.*

- Clearly $L_{q,q}(x) = \|\delta_q(x)\|$ and β_N^M commutes with δ_q and is thus contractive for $L_{q,q}$ (more complicated for $L_{q,q}^{\max}$) $\rightsquigarrow$ max/min

- The first consequence of this is that for each q we have

$$\left(\mathrm{Fuzz}_N(B_q^M), L_{q,q}^{\max}\right) \underset{N,M\to\infty}{\longrightarrow} \left(C(SU_q(2), L_{q,q}^{\max}\right)$$

- Secondly, since $\mathrm{Fuzz}_N(B_q^M) \subseteq \mathcal{O}(SU_q(2))$ the two seminorms $L_{q,q}^{\max}$ and $L_{q,q}$ give the same CQMS structure:

Corollary (Kaad-K, 2022)

The seminorms $L_{q,q}^{\max}$ and $L_{q,q}$ define the same metric on the state space and

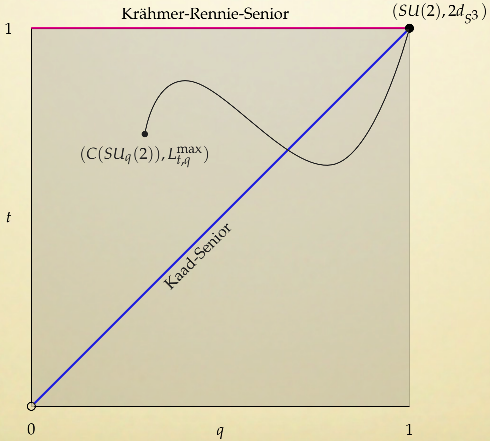
$$\mathrm{dist}_{GH}^{Q}\left((C(SU_q(2), L_{q,q}^{\max}); (C(SU_q(2), L_{q,q})\right) = 0$$

- The last hard analysis-problem to tackle is to show that the fuzzy approximation can be obtained *uniformly* around a given $q_0 \in (0, 1]$.
- We were able to deal with that as well, to obtain:

THEOREM (KAAD-K, 2022)

The family $(C(SU_q(2)), L^{\max}_{q,q})_{q\in(0,1]}$ *varies continuously in the quantum Gromov-Hausdorff distance*

- The main results hold true for general (t, q):



Thank You!

References:

1. *The quantum metric structure of quantum $SU(2)$*
   J. Kaad and D. Kyed
   Preprint, arXiv:2205.06043.
2. *The Podleś spheres converge to the sphere*
   K. Aguilar, J. Kaad and D. Kyed
   Communications in Mathematical Physics 392(3), (2022)
3. *Polynomial approximation of quantum Lipschitz functions*
   K. Aguilar, J. Kaad and D. Kyed
   Documenta Mathematica 27, (2022)
4. *Gromov-Hausdorff convergence of quantised intervals*
   T. Gotfredsen, J. Kaad and D. Kyed
   Journal of Mathematical Analysis and Applications 500(2), (2021)