Causality in noncommutative deformation spacetimes

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Main goal

From the "usual" notion of Spectral Triple + Connes's spectral distance formula

$\rightarrow$ adaptation to:

- Spectral Triple corresponding to Lorentzian signature
- Spectral characterisation of Causality
- Spectral characterisation of distance in Lorentzian signature

$\rightarrow$ application to:

- Almost-commutative spaces (Kaluza-Klein like)
- "Quantum" deformation spaces
  - Moyal spacetime
  - Kappa-Minkowski spacetime

Lorentzian Spectral Triple

Which definition are we going to use?

- A Hilbert space $\mathcal{H}$
- A non-unital pre-C^*-algebra $\mathcal{A}$ with a representation on $\mathcal{H}$ as bounded operators
- A preferred unitization $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ which is a pre-C^*-algebra and such that $\mathcal{A}$ is an ideal of $\widetilde{\mathcal{A}}$
- An operator D densely defined on $\mathcal{H}$ such that
  - $a(1 + \langle D\rangle^2)^{-\frac{1}{2}}$ is compact $\forall a \in \mathcal{A}$, with $\langle D\rangle^2 = \frac{1}{2}(DD^* + D^*D)$
  - $[D, a]$ is bounded $\forall a \in \widetilde{\mathcal{A}}$
- A bounded operator $\mathcal{J}$ with $\mathcal{J}^2 = 1$, $\mathcal{J}^* = \mathcal{J}$, $[\mathcal{J}, a] = 0$,
  - $D^* = -\mathcal{J}D\mathcal{J}$
  - $\mathcal{J} = -N[D, \mathcal{T}]$ for $N \in \widetilde{\mathcal{A}}$, $N > 0$ and some (possibly unbounded) self-adjoint operator $\mathcal{T}$ such that $(1 + \mathcal{T}^2)^{-\frac{1}{2}} \in \widetilde{\mathcal{A}}$

The operator $\mathcal{J}$ is called the *fundamental symmetry*. Its role is to turn the positive definite inner product of the Hilbert space $\langle \cdot, \cdot \rangle$ into an indefinite inner product $(\cdot, \cdot) = \langle \cdot, \mathcal{J} \cdot \rangle$ (Krein space) on which the Dirac operator D is (skew)-selfadjoint.

For a general $\mathcal{J}$, the signature can correspond to a pseudo-Riemannian one. We restrict to Lorentzian signatures by requiring that $\mathcal{J} = -N[D, \mathcal{T}]$ which corresponds in the commutative case to $\mathcal{J} = iNc(d\mathcal{T}) = i\gamma^0$ with a global time function $\mathcal{T}$ and lapse function N.

Globally hyperbolic $\subseteq$ "$\mathcal{J} = -N[D, \mathcal{T}]$" $\subseteq$ "stably causal" Lorentzian manifolds

From Riemannian distance to causality

"Riemannian" distance formula:

$$d(p,q) = \sup_{f \in C(\mathcal{M})} \left\{ |f(q) - f(p)| \ : \ \|[D,f]\| \leq 1 \right\}$$

$\rightarrow$ based on a specific set of functions, defined using $[D, f]$

In Lorentzian geometry (at least stably causal spaces), the causality relation ($p \preceq q$ if and only if there is a future directed causal curve from p to q) can be completely recovered using the set (cone) of causal functions $\mathcal{C}$, which are the real-valued functions non-decreasing along every future directed causal curve :

$$\forall p, q \in \mathcal{M}, \quad p \preceq q \qquad \Longleftrightarrow \qquad \forall f \in \mathcal{C}, \ f(p) \leq f(q)$$

This set can be determined using spectral properties of the Dirac operator and the fundamental symmetry:

$$f \in \mathcal{C} \iff \forall \phi \in \mathcal{H}, \quad \langle \phi, \mathcal{J}[D,f]\phi \rangle \leq 0.$$

[Franco & Eckstein, Class. Quantum Grav. 2013]

How to derive a causal structure in NCG

1. Define the following subset among the (unitized) C^*-algebra *(causal cone)*:

$$\mathcal{C} = \left\{ a \in \widetilde{\mathcal{A}} \mid a = a^*, \forall \phi \in \mathcal{H} \left\langle \phi, \mathcal{J}[D, a]\phi \right\rangle \leq 0 \right\}$$

2. Check the condition (to guarantee that all states can be separated, Stone–Weierstrass):

$$\overline{\mathrm{span}_{\mathbb{C}}(\mathcal{C})} = \overline{\widetilde{\mathcal{A}}}$$

3. Define a causal relation (partial order) on $P(\widetilde{\mathcal{A}})$ (can be extended on $S(\widetilde{\mathcal{A}})$) by

$$\forall \chi, \xi \in P(\widetilde{\mathcal{A}}), \quad \chi \preceq \xi \qquad \Longleftrightarrow \qquad \forall a \in \mathcal{C},\ \chi(a) \leq \xi(a)$$

- get sufficient condition $\rightarrow$ transform the constraint on $\mathcal{C}$ between states (very difficult)
- get necessary condition $\rightarrow$ find a specific conter-exemple $a \in \mathcal{C}$ for each forbidden relation (even more difficult)

From causality to Lorentzian distance

- 1998, Parfionov & Zapatrin: first idea of dual formula (infimum instead of supremum)
- 2002, Moretti: using specific local causal functions (local "steep" functions) constrained by the gradient ∇f (+ first noncommutative attempt using Laplace-Beltrami-d'Alembert operator)
- 2010*, Franco: global formulation using global "steep" functions constrained by gradient ∇f (using non-smooth functions for the proof, and only for globally hyperbolic spacetimes)
- 2013, Franco & Eckstein: noncommutative formulation of the gradient ∇f condition (using smooth functions for the proof) and conjecture of the final formula
- 2014, Rennie & Whale: extension of the result $\star$ to non globally hyperbolic spacetimes
- 2017, Minguzzi: "smooth" proof of the result $\star$ leading to a complete proof
- 2018, Franco: final formulation for even and odd dimensions

Lorentzian distance formula

If $(\mathcal{M}, g)$ is a n-dimensional spin Lorentzian manifold which is either

- globally hyperbolic
- or stably causal such that the Lorentzian distance d is continuous and finite,

then for all $p, q \in \mathcal{M}$:

$$d(p,q) \;=\; \inf_{f \in C^1(\mathcal{M},\mathbb{R})} \{\, \max\{0, f(q) - f(p)\} \;:\; g(\nabla f, \nabla f) \leq -1 \,\}\,.$$

This formula can be generalized for even Lorentian spectral triples (with the chirality operator χ):

$$d(p,q) \;=\; \inf_{f \in \mathcal{A}, f = f^*} \{\, \max\{0, f(q) - f(p)\} \;:\; \forall \phi \in \mathcal{H}, \langle \phi, \mathcal{J}([D,f] + i\chi)\phi \rangle \leq 0 \,\}\,,$$

and odd Lorentian spectral triples:

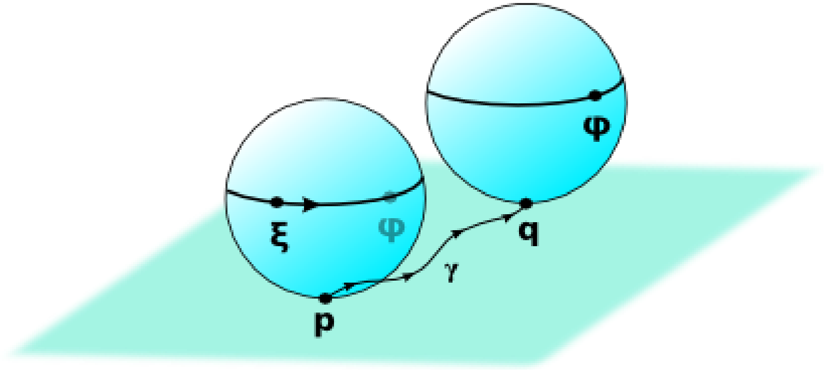
$$d(p,q) \;=\; \inf_{f \in \mathcal{A}, f = f^*} \{\, \max\{0, f(q) - f(p)\} \;:\; \forall \phi \in \mathcal{H}, \langle \phi, \mathcal{J}([D,f] \pm 1)\phi \rangle \leq 0 \,\}\,.$$

[Franco, J. Phys.: Conf. Ser. 2018 and references therein]

"Feasible" applications of causality (and metric): almost-commutative spacetimes

Set of pure states : "Kaluza-Klein" product between usual spacetime and discrete space.

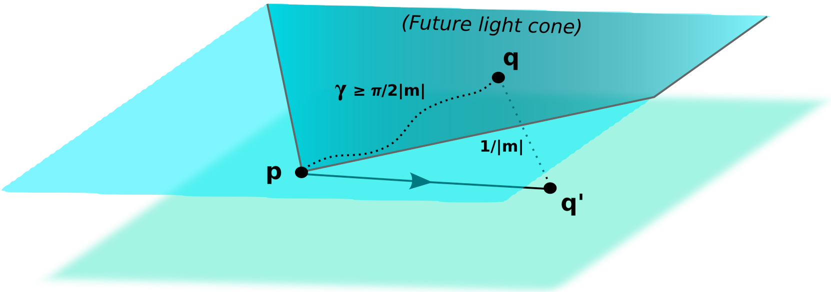
Example 1: $C^\infty(\mathcal{M}) \times M_2(\mathbb{C}) \to$ spacetime $\times$ S^2 sphere :



Complete causal structure : 2 pure states defines by (p, θ_ξ) and (q, θ_φ) on a same "parallel of latitude" are casually related if and only if:

$$\frac{|\theta_\varphi - \theta_\xi|}{l(\gamma)} \leq |d_1 - d_2| \quad \text{(difference of internal Dirac eigenvalues)}$$

Example 2: $C^\infty(\mathcal{M}) \times \mathbb{C} \oplus \mathbb{C} \rightarrow$ spacetime $\times$ two points :



Complete causal structure : Two points p and q' on separated sheets are causally related with $p \preceq q'$ if and only if they are causally related if considered on the same sheet and $l(\gamma) \geq \frac{\pi}{2|m|}$ (with $\frac{1}{|m|}$ corresponding to Connes' distance between the two sheets) [Franco & Eckstein: JGP 2015]

- First physical application: This limit corresponds exactly to the Zitterbewegung phenomera (fast oscillation of a free electron between two states) [Eckstein, Franco & Miller: PRD 2017]
- Second physical application: The Lorentzian distance formula reproduces the energy-momentum dispersion relation $E^2 = (pc)^2 + (mc^2)^2$ [Watcharangkool & Sakellariadou: PRD 2017]

"Less Feasible" application of causality (1) : 2-dim Moyal Minkowski spacetime

Moyal Lorentzian spectral triple $(\mathcal{A}, \widetilde{\mathcal{A}}, \mathcal{H}, D, \mathcal{J})$:

- $\mathcal{H} = L^2(\mathbb{R}^{1+1}) \otimes \mathbb{C}^2$ with the usual positive definite inner product
- $\mathcal{A}$ is the space of Schwartz functions $\mathcal{S}(\mathbb{R}^{1,1})$ with the "Weyl-Moyal" $\star$ product defined as

$$(f \star g)(x) := \frac{1}{(\pi\theta)^2} \int d^2y\, d^2z\; f(x+y) g(x+z) e^{-2i\, y^\mu\, \Theta^{-1}_{\mu\nu} z^\nu},$$

  with $\Theta_{\mu\nu} := \theta \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$, $\theta > 0$

- The preferred unitization $\widetilde{\mathcal{A}} = (\mathcal{B}, \star)$ is the unital algebra of smooth functions which are bounded together with all derivatives
- $D = -i\partial_\mu \otimes \gamma^\mu$ is the flat Dirac operator on $\mathbb{R}^{1,1}$ where $\gamma^0 = i\sigma^1$, $\gamma^1 = \sigma^2$ are the flat Dirac matrices
- $\mathcal{J} = i\gamma^0$ is the fundamental symmetry

Causal structure between coherent states:

Functions and states on Moyal can be easily described using as orthonormal basis the Wigner eigenfunctions of the two-dimensional harmonic oscillator $a = \sum_{mn} a_{mn} f_{mn}$ where

$$f_{mn} = \frac{1}{(\theta^{m+n} m! n!)^{1/2}} \bar{z}^{\star m} \star f_{00} \star z^{\star n} \text{ with } z = \frac{x_0 + i x_1}{\sqrt{2}}, f_{00} = 2e^{-\frac{x_0^2 + x_1^2}{\theta}}$$

The pure states are all the normalized vector states on the matrix representation:
$\omega_\psi(a) = 2\pi\theta \sum_{m,n} \psi_m^* a_{mn} \psi_n,\ 2\pi\theta \sum_m |\psi_m|^2 = 1$

The coherent states of $\mathcal{A}$ are the vector states defined, for any $\kappa \in \mathbb{C}$, by:

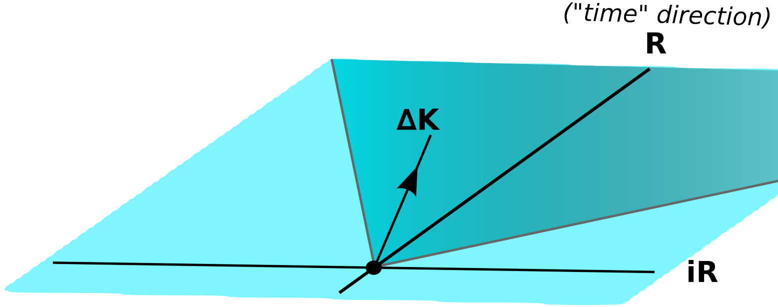
$$\varphi_m = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{|\kappa|^2}{2\theta}} \frac{\kappa^m}{\sqrt{m!\theta^m}}$$

The coherent states correspond to the possible translations under the complex scalar $\sqrt{2}\kappa$ of the ground state $|0>$, using $\kappa \in \mathbb{C} \cong \mathbb{R}^{1,1}$.

They are the states that minimize the uncertainty equally distributed in position and momentum. The classical limit of the coherent states ($\theta \to 0$) corresponds to the usual pure states on $\mathbb{R}^{1,1}$.

Let us suppose that two coherent states $\omega_\xi, \omega_\varphi$ correspond to the complex scalars $\kappa_1, \kappa_2 \in \mathbb{C}$. Those coherent states are causally related, with $\omega_\xi \preceq \omega_\varphi$, iff $\Delta\kappa = \kappa_2 - \kappa_1$ is inside the convex cone of $\mathbb{C}$ defined by $\lambda = \frac{1+i}{\sqrt{2}}$ and $\bar{\lambda} = \frac{1-i}{\sqrt{2}}$ (i.e. the argument of $\Delta\kappa$ is within the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$).

[Franco & Wallet, Contemp.Math. 2016]



This causal structure is similar to the one in Minkowski, except that we do not consider points but translations of Gaussian functions, so non-local states! In such a case, we can define a kind of "time" as translations under positive real scalars κ.

Causal structure between generalized coherent states:

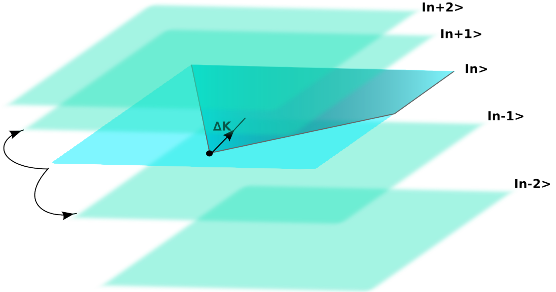
Can we have any causal relation between pure states of different energy level (the basic eigenstates of the harmonic oscillator) $|0>$, $|1>$, $|2>$, etc?

Currently we only have a sufficient condition: An eigenstate can "jump" from one energy state to another if there is at the same time a sufficient translation in the direction of the "time":

$$\Delta\kappa \in \mathbb{R} \text{ such that } \Delta\kappa \geq \frac{\pi}{2}\sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}} \quad \Longrightarrow \quad |n> \preceq \alpha_{\Delta\kappa}|n{+}1> \text{ and } |n{+}1> \preceq \alpha_{\Delta\kappa}|n>$$

where $\alpha_{\Delta\kappa}|n>$ is the translation of the eigenstate $|n>$ under $\Delta\kappa$ (using $\mathbb{C} \cong \mathbb{R}^{1,1}$).

This model represents waves packets under causal translations with a lower bound on time in order to change the energy level.



"Less Feasible" application of causality (2) : 2-dim κ-Minkowski spacetime

Too lazy to re-explain in detail κ-Minkowski (cf. Fedele Lizzi's talk…)

Exploration using a first choice of Dirac operator and states [Franco & Wallet, JPhysA, *to appear*] but other choices are possible, leading to similar results [Franco, Hersent, Maris & Wallet, *preprint soon*].

We don't currently have a necessary-sufficient condition for this space, only separated necessary or sufficient conditions.

Hibert space: $\mathbb{H} = \mathcal{H}_+ \oplus \mathcal{H}_0 \oplus \mathcal{H}_-, \qquad \mathcal{H}_{+,0,-} = (L^2(\mathbb{R}), ds) \otimes \mathbb{C}^2$
linked to the unitary irreducible representations $\pi_\pm$ and the trivial 1-dimensional one π_0.

Dirac operator: obtained from natural one-parameter groups of automorphisms of $C^*(\mathcal{G})$
(cf. Iochum, Masson, Sitarz, BCP 2012) :

$$\mathcal{D} = -\gamma^k \partial_k \otimes \mathbb{1}_3 = -i \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix} \otimes \mathbb{1}_3 \quad \text{with } \partial_\pm = \partial_0 \pm x_1 \partial_1, \qquad \mathcal{J} = i\gamma^0 \otimes \mathbb{1}_3.$$

Which pure states ?

An interesting set of pure states can be determined by a family of (cyclic) vector states:

$$\varphi_{\pm}^{\Phi}(f) = \langle \Phi, \pi_{\pm}(f)\Phi \rangle$$

for any $\Phi \in L^2(\mathbb{R})$ with $||\Phi|| = 1$ (this is only a subset of pure states but could give enough information due to Gel'fand–Raikov theorem).

Some explicit formulas:

- Representations: $(\pi_\nu(f)\,\phi)(s) = \frac{1}{2\pi}\int \mathrm{d}u\mathrm{d}v\; f(v, \nu e^{-s})\, e^{-iv(u-s)}\, \phi(u), \quad \nu = +, 0, -$
- States: $\varphi_{\pm}^{\Phi}(f) = \frac{1}{2\pi}\iiint \mathrm{d}s\mathrm{d}u\mathrm{d}v\; f(v, \pm e^{-s})\, e^{-iv(u-s)}\, \overline{\Phi}(s)\Phi(u)$
- "Rough" causality relation: $\varphi_{\pm}^{\Phi_1} \preceq \varphi_{\pm}^{\Phi_2} \iff$

  $$\iiint \mathrm{d}s\mathrm{d}u\mathrm{d}v\; f(v, \pm e^{-s})\, e^{-iv(u-s)}\, \Big[\overline{\Phi}_2(s)\Phi_2(u) - \overline{\Phi}_1(s)\Phi_1(u)\Big] \geq 0, \quad \forall f \in \mathcal{C}$$

  where $f \in \mathcal{C} \iff \bigoplus_{\nu=+,0,-} \begin{pmatrix} \pi_\nu(\partial_+ f) & 0 \\ 0 & \pi_\nu(\partial_- f) \end{pmatrix} \geq 0$

A sufficient causality condition: a "phase-momentum transport"

We get a sufficient condition for a causal evolution between two states $\varphi_+^{\Phi_1} \preceq \varphi_+^{\Phi_2}$ represented by the continuous evolution of $\Phi_t : \Phi_1 \rightsquigarrow \Phi_2$ if there exists $\forall t$ a function $\psi_t \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\alpha_t \in [-1, 1]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\Phi}_t(s)\Phi_t(u)\right) = i(u-s)\overline{\psi}_t(s)\psi_t(u) + \alpha_t\left(\overline{\psi}'_t(s)\psi_t(u) + \overline{\psi}_t(s)\psi'_t(u)\right)$$

Particular solution ($\psi_t = \Phi_t$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(u) - \alpha_t\Phi'_t(u) = iu\Phi_t(u).$$

For $\alpha_t = \alpha$ constant, this equation is a transport equation whose general solution is

$$\Phi_t(u) = \Phi_0(u+\alpha t)e^{itu}, \qquad \alpha \in [-1, 1].$$

This evolution represents a αt translation at the level of Φ_t simultaneously to a t translation at the level of $\mathcal{F}\Phi_t$, hence can be interpreted as a "phase-momentum transport".

A necessary causality condition: a "quantum causality constraint"

On the specific solution $\Phi_t(u) = \Phi_0(u + \alpha t)e^{itu}$, we have that $|\alpha| > 1$ is excluded.

On two generic states $\varphi_+^\Phi \preceq \varphi_+^\xi$, we can extract the following necessary constraint:

$$\frac{1}{2\pi}\int \mathrm{d}v\; v|\mathcal{F}\xi(v)|^2 - \frac{1}{2\pi}\int \mathrm{d}v\; v|\mathcal{F}\Phi(v)|^2 \geq \left|\int \mathrm{d}s\; s|\xi(s)|^2 - \int \mathrm{d}s\; s|\Phi(s)|^2\right|$$

We can interpret this as a constraint on the expectation value of some quantum operators (position/momentum) :

$$\langle\xi|P|\xi\rangle - \langle\Phi|P|\Phi\rangle \geq |\langle\xi|X|\xi\rangle - \langle\Phi|X|\Phi\rangle|$$

$$\iff \delta\langle P\rangle \geq |\delta\langle X\rangle|$$

$\sim$ quantum analogy to the classical speed of light limit $\delta t \geq |\delta x|$.

Which works for the future?

- Find "equivalent" sufficient/necessary conditions on Moyal/kappa
- Look at the Lorentzian distance more precisely
- If not feasible analytically, why not trying to compute those relations/metric using numerical methods (Monte-Carlo-like simulation to explore the compete causal structure) ?

Thank you for your attention!