# Spectral Metric and Einstein Functionals

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We define bilinear functionals of vector fields and differential forms, the densities of which yield the metric and Einstein tensors on even-dimensional Riemannian manifolds.
We generalise these concepts in non-commutative geometry and, in particular, we prove that for the conformally rescaled geometry of the noncommutative two-torus the Einstein functional vanishes.

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# Spectral Geometry:

Can one hear the shape of a drum?

An eminent spectral scheme that generates geometric objects on manifolds such as volume, scalar curvature, and other scalar combinations of curvature tensors and their derivatives is *prima facie* the small-time asymptotic expansion of the trace of heat kernel

$$\mathrm{Tr}\, e^{-t\Delta} = \sum_{n=0}^{\infty} t^{\frac{n-d}{2}} a_n.$$

Here the scalar laplacian $\Delta$ for a given Riemannian metric $g$ reads in local coordinates

$$\Delta = -\frac{1}{\sqrt{\det(g)}}\partial_a\big(\sqrt{\det(g)}g^{ab}\partial_b\big) \tag{1}$$

and $a_n = a_n(\Delta) = a_n(R, \partial R, \dots)$.

# Geometry from residues:

Using the Mellin transform the coefficients of this expansion can be transmuted into some values or residues of the zeta function of $\Delta$. In turn, some can be expressed using the Wodzicki residue $\mathcal{W}$

$$\mathcal{W}(P) := \int_M \left( \int_{|\xi|=1} tr\,\sigma_{-n}(P)(x,\xi)\ \mathcal{V}_\xi \right)\ d^n x. \qquad (2)$$

Here we focus on closed oriented $M$ of even dimension $n = 2m$. ♠
Then,

$$\mathcal{W}(\Delta^{-m}) = v_{n-1}\, vol(M),$$

where

$$v_{n-1} := vol(S^{n-1}) = \frac{2\pi^m}{\Gamma(m)},$$

is the volume of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

In the *localized* form (as a functional of $f \in C^\infty(M)$)

$$\mathsf{v}(f) := \mathcal{W}(f\Delta^{-m}) = v_{n-1} \int_M f\ vol_g.$$

↬ This is related to the asymptotic growth of eigenvalues of $\Delta$; clear e.g. from the Connes "trace thm." that here $\mathcal{W} = \mathrm{Tr}^+$. ↫

# Scalar curvature

A. Connes divulged in 90s a startling result, confirmed by Kastler and by Kalau-Walze:

$$\mathcal{W}(\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M R(g)\ vol_g, \tag{3}$$

which, up to a constant, is a Riemannian analogue of the Einstein-Hilbert action functional of general relativity in vaccum. Here $R = R(g)$ is the scalar curvature, that is the $g$-trace $R = g^{jk} R_{jk}$ of the Ricci tensor where $g^{jk} g_{k\ell} = \delta^j_\ell$.

A localised form of (3) is a functional on $C^\infty(M)$

$$\mathcal{R}(f) := \mathcal{W}(f\Delta^{-m+1}) = \frac{n-2}{12} v_{n-1} \int_M f R(g) vol_g. \tag{4}$$

$\looparrowright$ We have uncovered other two unforeseen functionals of a pair of vector fields $V$ and $W$ on $M$, viewed as derivations of $C^\infty(M)$:

# New functionals

**Def/Thm:**

*The functional*

$$\mathrm{g}^{\Delta}(\mathrm{V},\mathrm{W}) := \mathcal{W}\big(\mathrm{V}\mathrm{W}\Delta^{-\mathrm{m}-1}\big)$$

*is a bilinear, symmetric map, whose density is proportional to the metric $g$ evaluated on $V, W$*

$$\mathsf{g}^{\Delta}(V,W) = -\frac{v_{n-1}}{n}\int_M g(V,W)\, vol_g.$$

# New functionals

## Def/Thm: Metric functional

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## Def/Thm:

*The functional*

$$\mathrm{G}^{\Delta}(\mathrm{V},\mathrm{W}) := \mathcal{W}\big(\mathrm{V}\mathrm{W}\Delta^{-\mathrm{m}}\big)\,, \tag{5}$$

*is a bilinear, symmetric map, whose density is proportional to the Einstein tensor* $G := \mathit{Ric} - \frac{1}{2}R(g)g$ *evaluated on $V, W$*

$$\mathrm{G}^{\Delta}(V,W) = \frac{v_{n-1}}{6}\int_M G(V,W)\, vol_g.$$

# New functionals

## Def/Thm: Metric functional

*The functional*

$$\mathrm{g}^{\Delta}(\mathrm{V},\mathrm{W}) := \mathcal{W}\big(\mathrm{V}\mathrm{W}\Delta^{-\mathrm{m}-1}\big)$$

*is a bilinear, symmetric map, whose density is proportional to the metric* $g$ *evaluated on* $V, W$

$$\mathrm{g}^{\Delta}(V,W) = -\frac{v_{n-1}}{n}\int_M g(V,W)\, vol_g.$$

## Def/Thm: Einstein functional

*The functional*

$$\mathrm{G}^{\Delta}(V,W) := \mathcal{W}\big(VW\Delta^{-m}\big)\,, \tag{6}$$

*is a bilinear, symmetric map, whose density is proportional to the Einstein tensor* $G := Ric - \frac{1}{2}R(g)g$ *evaluated on* $V, W$

$$\mathrm{G}^{\Delta}(V,W) = \frac{v_{n-1}}{6}\int_M G(V,W)\, vol_g.$$

# Proof hint

**Algebra of symbols of pseudodifferential operators.**

Let $P$ and $Q$ be pseudodifferential operators with symbols:

$$\sigma(P)(x,\xi) = \sum_\alpha \sigma(P)_\alpha(x)\xi^\alpha, \qquad \sigma(Q)(x,\xi) = \sum_\beta \sigma(Q)_\beta(x)\xi^\beta. \tag{7}$$

The composition rule for the symbol of their product is

$$\sigma(PQ)(x,\xi) = \sum_\beta \frac{(-i)^{|\beta|}}{|\beta|!}\delta_\beta\sigma(P)(x,\xi)\partial_\beta\sigma(Q)(x,\xi), \tag{8}$$

where $\delta_\beta$ denotes the partial derivative wrt. $\xi^\beta$.

# Proof hint 2

**Taylor expansion in the normal coordinates:**

the metric

$$g_{ab} = \delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d + o(\mathbf{x^2}), \tag{9}$$

$$\sqrt{\det(g)} = 1 - \frac{1}{6} \mathrm{Ric}_{ab} x^a x^b + o(\mathbf{x^2}), \tag{10}$$

and the inverse metric

$$g^{ab} = \delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d + o(\mathbf{x^2}). \tag{11}$$

Here $R_{acbd}$ and $\mathrm{Ric}_{ab}$ are the components of the Riemann and Ricci tensor, respectively, at the point with $\mathbf{x} = 0$ and we use the notation $o(\mathbf{x^k})$ to denote the expansion up to the polynomial of order $k$ in the normal coordinates.

# Proof hint 3

Consequently, the symbols of $\Delta$ in normal coordinates are

$$\begin{aligned} \mathfrak{a}_2 =& \big(\delta_{ab} + \frac{1}{3} R_{acbd} x^c x^d\big) \xi_a \xi_b + o(\mathbf{x^2}), \\ \mathfrak{a}_1 =& \frac{2i}{3} \mathrm{Ric}_{ab} x^a \xi_b + o(\mathbf{x^2}). \end{aligned} \tag{12}$$

Next, the symbols of $\Delta^{-1}$ are given up to order respectively $\mathbf{x^2}, \mathbf{x}, \mathbf{1}$ by

$$\begin{aligned} \mathfrak{b}_2 &= ||\xi||^{-4} \big(\delta_{ab} - \frac{1}{3} R_{acbd} x^c x^d\big) \xi_a \xi_b + o(\mathbf{x^2}), \\ \mathfrak{b}_3 &= -\frac{2i}{3} \mathrm{Ric}_{ab} x^a \xi_b ||\xi||^{-4} + o(\mathbf{x}), \\ \mathfrak{b}_4 &= \frac{2}{3} \mathrm{Ric}_{ab} \xi_a \xi_b ||\xi||^{-6} + o(\mathbf{1}). \end{aligned} \tag{13}$$

# Proof hint 4

Thus the first three leading symbols of the operator $\Delta^{-k}$, $k > 0$

$$\sigma(\Delta^{-k}) = \mathfrak{c}_{2k} + \mathfrak{c}_{2k+1} + \mathfrak{c}_{2k+2} + \dots,$$

read

$$\begin{aligned} \mathfrak{c}_{2k} &= ||\xi||^{-2k-2} \left( \delta_{ab} - \frac{k}{3} R_{acbd} x^c x^d \right) \xi_a \xi_b + o(\mathbf{x^2}), \\ \mathfrak{c}_{2k+1} &= \frac{-2ki}{3||\xi||^{2k+2}} \mathrm{Ric}_{ab} x^b \xi_a + o(\mathbf{x}), \\ \mathfrak{c}_{2k+2} &= \frac{k(k+1)}{3||\xi||^{2k+4}} \mathrm{Ric}_{ab} \xi_a \xi_b + o(\mathbf{1}). \end{aligned} \tag{14}$$

Now, by combining all the above we show the statements. □

# Laplace-type operators

More generally, let

$$\Delta_{T,E} = -g^{ab}(\nabla_a \nabla_b - \Gamma^c_{ab}\nabla_c) + E$$

be a Laplace-type operator on a vector bundle $X$ of rank $r$, where $\nabla_a = \partial_a - T$ with $T \in End\ X$, and $E \in End\ X$.

By a lengthy computation:

**Thm**

*The functional*

$$\mathrm{g}^{\Delta_{T,E}}(V,W) := \mathcal{W}(\nabla_V \nabla_W \Delta_{T,E}^{-m-1})$$

*equals*

$$= r\,\mathrm{g}^{\Delta}(V,W).$$

*The functional*

$$\mathrm{G}^{\Delta_{T,E}}(V,W) := \mathcal{W}(\nabla_V \nabla_W \Delta_{T,E}^{-m})$$

*equals*

$$= \frac{v_{n-1}}{6} \int_M \Big( rG(V,W) + 3F(V,W) + 3\mathrm{Tr}E\, g(V,W) \Big) vol_g,$$

*where $F(V,W) = \mathrm{Tr}\ V^a W^b F_{ab}$ and $F_{ab}$ is the curvature of $\nabla_a$.*

# Spin Laplacian

A particularly interesting case is when $M$ is a $spin_c$ manifold $M$ and $X$ is a spinor bundle of rank $2^m$. The spin Laplace operator

$$\Delta^{(s)} := \nabla^{(s)*}\nabla^{(s)} = -\nabla^{(s)}_{e_i}\nabla^{(s)}_{e_i} + \nabla^{(s)}_{\nabla_{e_i}e_i}, \tag{15}$$

where $\nabla^{(s)}$ is the spin connection, biexpands in the order/normal coordinates as

$$\begin{aligned}\Delta^{(s)} &= -\partial_i\partial_i + \frac{1}{3}R_{ijk\ell}\, x^j x^k \partial_i\partial_\ell + o(\mathbf{x^2}) \\ &\quad + \frac{2}{3}R_{ij}\, x^i\partial_j + \frac{1}{4}R_{i\ell jk}\, x^\ell\gamma^j\gamma^k\partial_i + o(\mathbf{x}) \\ &\quad + o(\mathbf{1}),\end{aligned} \tag{16}$$

where $\gamma^j$ are the CAR gamma matrices.

Then by comparing the symbols it is easy to identify $\Delta^{(s)}$ as $\Delta_{T,E}$ for $T = \frac{1}{8}R_{abjk}\gamma^j\gamma^k x^a x^b$ & $E = 0$. Hence,

**Proposition**

$$\begin{aligned}\mathrm{g}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V\nabla^{(s)}_W(\Delta^{(s)})^{-n-2}\big) = 2^m\mathrm{g}^{\Delta}(V,W), \\ \mathrm{G}^{\Delta^{(s)}}(V,W) &:= \mathcal{W}\big(\nabla^{(s)}_V\nabla^{(s)}_W(\Delta^{(s)})^{-n}\big) = 2^m\mathrm{G}^{\Delta}(V,W)+0.\end{aligned} \tag{17}$$

# Dirac operator

Since we already *spin*, why not consider the Dirac operator (coupled do $U(1)$-gauge 1-form $A$):

$$D_A = i\gamma^j \nabla^{(s)}_{e_j} + A,$$

or actually its square $D_A^2$, which by the Lichnerowicz thm

$$D_A^2 = \Delta^{(s)} + \frac{1}{4}R + F,$$

where $F = \gamma^j\gamma^k F_{jk}$, and $F_{jk}$ is the curvature of $A$. Then

**Proposition**

*The spectral metric and Einstein functionals associated with the Dirac operator $D_A$ do not depend on $A$ and read, respectively,*

$$\mathrm{g}^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n-2}) = 2^m \mathrm{g}^{\Delta}(V,W),$$

$$\mathrm{G}^{D_A^2}(V,W) := \mathcal{W}(\nabla^{(s)}_V \nabla^{(s)}_W |D_A|^{-n})$$

$$= 2^m \Big( \mathrm{G}^{\Delta}(V,W) + \frac{v_{n-1}}{8} \int_M R(g)g(V,W) vol_g \Big).$$

# Spectral functionals on 1-forms

Spinors are fruitful also in order to define the "dual functionals" corresponding to contravariant tensors with "raised indices", which are $C^\infty(M)$-bilinear on 1-forms (co-vectors).
Can the contravariant metric and Einstein tensors be obtained from such functional using the spectral methods and $\mathcal{W}$?

We need to represent differential forms as differential operators.
A suitable way is to employ Clifford modules.
On a spin$_c$c manifold $M$ the Clifford representation of one-forms $v$ are 0-order differential operators $\hat{\nu}$, i.e. endomorphisms of the spinor bundle.
In fact they form a $C^\infty(M)$-bimodule generated by commutators of the Dirac operator with functions.
So the Dirac operator is <u>self-sufficient</u> for our purposes (and NCG-ready when assembled to a spectral triple of A. Connes).

# Metric and Einstein functionals on 1-forms

**Thm**

*The following spectral functionals of one-forms on a spin-c manifold $M$ of dimension $n$*

$$\begin{aligned} \mathrm{g}_D(v,w) &:= \mathcal{W}\big(\hat{v}\hat{w}D^{-n}\big), \\ \mathrm{G}_D(v,w) &:= \mathcal{W}\big(\hat{v}(D\hat{w}+\hat{w}D)D^{-n+1}\big) \\ &= \mathcal{W}\big((D\hat{v}+\hat{v}D)\hat{w}D^{-n+1}\big), \end{aligned} \tag{18}$$

*read*

$$\begin{aligned} \mathrm{g}_D(v,w) &= 2^m v_{n-1} \int_M g(v,w)\ vol_g, \\ \mathrm{G}_D(v,w) &= 2^m \frac{v_{n-1}}{6} \int_M G(v,w)\ vol_g, \end{aligned} \tag{19}$$

*where* $g(v,w) = g^{ab} v_a w_b$ *and* $G(v,w) = \big(Ric^{ab} - \frac{1}{2} R g^{ab}\big) v_a w_b$.

These perfectly (dually) match $\mathrm{g}^{\Delta}$ and $\mathrm{G}^{\Delta}$.

# Go quantum (= noncommutative)

Noncommutative tori are prominent examples of noncommutative manifolds. In particular, their smooth algebra $A = C^\infty(\mathbb{T}^n_\theta)$ possesses a faithful state $\tau$ invariant under external derivations $\delta_a$, $a = 1, \ldots, n$, interpreted as noncommutative vector fields.
It is then easy to identify a noncommutative counterpart of the flat-metric Laplace operator $\Delta = \sum_a \delta_a^2$,
and of Dirac operator $D = \sum_a \gamma^a \delta_a$ and 1-forms $\Omega_D(A)$.
Both of them generalise to the conformally rescaled geometry.

Considering for simplicity only the *strictly irrational* $\mathbb{T}^n_\theta$ ($\mathcal{Z}(A) = \mathbb{C}$) the trace extends to a factorized trace on the enlarged algebra $\hat{A} := A \otimes A^o$ given by $\tau(ab^o) = \tau(a)\tau(b^o)$,
where $A^o$ is a copy of $A$ in the commutant $A'$ of $A$. Such $\tau$ is still invariant under the extension of derivations to $\hat{A}$ (preserving $A^o$).
We use it to define the tracial state $\mathcal{W}$ on $\hat{A}$-valued symbols $\sigma(\xi)$ (where $\delta_a \mapsto \xi_a$ much the same as for $M$).

# Noncommutative 2-torus: vector fields

Let $h \in C^\infty(\mathbb{T}^2_\theta)$ be positive, invertible, with a bounded inverse. By a conformally rescaled Laplace operator for $\mathbb{T}^2_\theta$ we mean the selfadjoint operator on $H = L^2(\mathbb{T}^2_\theta, \tau)$

$$\Delta_h = h^{-1} \Delta\, h^{-1}.$$

This is motivated by the commutative case $\theta = 0$ where $\Delta_h$ is unitarily equivalent via $Ad_h$ to $h^{-2}\Delta$, which is a self-adjoint on $H_h = L^2(\mathbb{T}^2, \tau_h)$, where $\tau_h(a) = \tau(h^2 a)$. Accordingly, as vector fields we take (unitarily equivalent to derivations) selfadjoint operators

$$V_h = \sum_{a=1,2} V^a h \delta_a h^{-1}, \quad V^a \in A^o.$$

**Proposition**

*For the conformally rescaled Laplace operator for* $\mathbb{T}^2_\theta$

$$\mathrm{g}^{\Delta_h}(V_h, W_h) = \mathcal{W}\big(V_h W_h \Delta_h^{-2}\big) = \pi \tau(h^4) V^a W^a,$$

*whereas*

$$\mathrm{G}^{\Delta_h}(V_h, W_h) = \mathcal{W}\big(V_h W_h \Delta_h^{-1}\big) = 0.$$

# Noncommutative 2-torus: 1-forms

Motivated by another unitary equivalence in the classical case, as the conformal rescaling of $D$ we take $D_k = kDk$ on $H$ as [CM], however with $k > 0$ in $A^o \subset A'$, which assures that $(A, D_k, H)$ is a spectral triple and $\exists$ an $A$-bimodule $\Omega^1_{D_k}(A)$.
We propose the definition of metric and Einstein functionals for spectral triples of the same (classical) form as above

$$\mathrm{g}_D(v, w) = \mathcal{W}(vw|D|^{-n}),$$
$$\mathrm{G}_D(v, w) = \mathcal{W}(v\{D, w\}D|D|^{-n}),$$

where $v, w \in \Omega^1_{D_k}(A)$. On $\mathbb{T}^n_\theta$, $\Omega^1_{D_k}(A)$ is freely generated by $k^2\gamma^j$.

**Proposition**

*For the conformally rescaled spectral triple over $\mathbb{T}^2_\theta$ the metric functional for $v = k^2 V^a \sigma^a$ and $w = k^2 W^a \sigma^a$, $V^a, W^a \in A$, reads*

$$\mathrm{g}_{D_k}(v, w) = \tau(V^a W^a),$$

*whereas the spectral Einstein functional vanishes identically,*

$$\mathrm{G}_{D_k}(v, w) = 0.$$

# Noncommutative 4-torus: vector fields

We take a conformally rescaled Laplace operator on $H = L^2(T^4_\theta, \tau)$

as $\Delta_h = \sum_{a=1,2,3,4} \chi^{-1} \cdot \delta_a \cdot \chi \cdot \delta_a \cdot \chi^{-1}, \quad \text{where} \quad 0 < \chi = h^2 \in C^\infty(T^4_\theta)$.

**Proposition**

*The spectral metric & Einstein functionals on derivations for $\Delta_h$ on $T^4_\theta$ read*

$$\mathrm{g}^{\Delta_h}(V_h, W_h) = 2\pi^2 \tau(\chi^3 V^a W^a)$$

$$\begin{aligned}
\mathrm{G}^{\Delta_h}(V_h, W_h) = \frac{\pi^2}{12}\tau\Big( & -\chi V^a \delta_a\chi \chi^{-1} W^b \delta_b\chi - \chi W^a \delta_a\chi \chi^{-1} V^b \delta_b\chi \\
& +5 V^a \delta_a\chi \chi^{-1} W^b \delta_b\chi \chi + 5 W^a \delta_a\chi \chi^{-1} V^b \delta_b\chi \chi - V^a \delta_a\chi W^b \delta_b\chi \\
& - W^a \delta_a\chi V^b \delta_b\chi - 8 V^a W^b \delta_a\delta_b\chi \chi + 4\chi V^a W^b \delta_a\delta_b\chi \\
& + \big( -\delta_a\chi \chi^{-1} \delta_a\chi \chi - \chi \delta_a\chi \chi^{-1} \delta_a\chi - \delta_a\chi \delta_a\chi + 2\chi\Delta\chi + 2\Delta\chi\chi \big) V^b W^b \Big),
\end{aligned}$$

*where $V_h = \sum_{a=1,2} V^a \chi \delta_a \chi^{-1}, \quad V^a \in A^o$.*

# Noncommutative 4-torus: 1-forms

**Proposition**

*The metric and the Einstein functionals for the conformally rescaled spectral triple on the noncommutative 4-torus are, respectively,*

$$\mathrm{g}_{D_k}(v,w) = \tau\left(W^a V^a k^{-4}\right),$$

$$\mathrm{G}_{D_k}(v,w) = \tau\Big(V^a W^b \big(\frac{1}{3}k^{-4}(\delta_a k)k^2(\delta_b k) + \frac{2}{3}k^{-3}(\delta_a k)k^1(\delta_b k)$$
$$+k^{-2}(\delta_a k)(\delta_b k) + \frac{2}{3}k^{-1}(\delta_a k)k^{-1}(\delta_b k) - \frac{4}{3}k(\delta_a k)k^{-3}(\delta_b k)$$
$$-\frac{2}{3}k^2(\delta_a k)k^{-4}(\delta_b k) + \frac{2}{3}k^{-1}(\delta_a\delta_b k)$$
$$+\delta_{ab}\big(\frac{1}{3}k^{-1}(\delta_c k)k^{-1}(\delta_c k) + \frac{1}{3}k^2(\delta_c k)k^{-4}(\delta_c k)$$
$$+\frac{2}{3}k^1(\delta_c k)k^{-3}(\delta_c k) - \frac{2}{3}k^{-1}(\Delta k)\big)\Big),$$

*where $v = k^2 V^a \gamma^a, w = k^2 W^a \gamma^a$, and $V^a, W^a \in A$.*

# Outlook

Another situation where we can define the metric and Einstein functionals on differential forms concerns regular, finitely summable spectral triples (for simplicity with simple dimension spectrum). In this case there is a $\Psi$DO algebra and calculus of symbols as defined by Connes and Moscovici and there exists a tracial state. However, the methods of computation of our functionals are much involved and explicitly feasible only in some special cases.

Altogether, the spectral formulation of such geometric objects as the metric, curvature and other tensors should be beneficial for studying them globally both for the manifolds as well as for generalized geometries, like noncommutative geometry.

# Outlook 2

More concretely, it opens a possibility to study them on the analytic/operator level, and to compare with other settings (algebraic, differential) for quantum analogues of:

- Levi-Civita connection, torsion, ...
- metric spaces
- orbifolds and manifolds with singularities
- flat manifolds
- Einstein manifolds ($\leftrightarrow$ *Einstein spectral triples*) for which $\mathrm{G}_D$ is proportional to $\mathrm{g}_D$.

*Conjecture: For a 2-dimensional regular spectral triple* $\mathrm{G}_D = 0$.

Its validity would indicate some more robustness of noncommutative manifolds ($\leftrightarrow$ spectral triples).