Tolerance Relations and Quantization

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Noncommutative Geometry: Metric and Spectral Aspects
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Talk based on:

- F. D'Andrea, G.Landi, F. Lizzi,
  Tolerance Relations and Quantization,
  Lett. Math. Phys. 112 (2022) 65; arXiv:2112.09698 [math.OA].

Further references:

- F. D'Andrea, F. Lizzi, P. Martinetti,
  Spectral geometry with a cut-off: topological and metric aspects,
  J. Geom. Phys. 82 (2014) 18–45; arXiv:1305.2605 [math-ph].
- A. Connes, W.D. van Suijlekom,
  Spectral Truncations in Noncommutative Geometry and Operator Systems,
  Commun. Math. Phys. 383 (2020), 2021–2067; arXiv:2004.14115 [math.QA].
- A. Connes, W.D. van Suijlekom,
  Tolerance relations and operator systems,
  arXiv:2111.02903 [math.OA].
- M. Gielen, W.D. van Suijlekom,
  Operator systems for tolerance relations on finite sets,
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  - Cut-off Λ $\rightsquigarrow$ minimal length
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- Finite-dim. case (tolerance relation on a finite set = finite simple graph)

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- Operator systems associated to tolerance relations on finite sets

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Tolerance relations

Definition.

Let S be a set. Then, $R \subset S \times S$ is a tolerance relation on S if it is

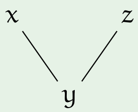
- reflexive: $(a, a) \in R \; \forall \, a \in S$
- symmetric: $(a, b) \in R \; \Rightarrow \; (b, a) \in R$

(Tolerance relation + transitivity = equivalence relation.)

Observation. R is a tolerance relation on S $\iff$ $(\underset{\substack{\uparrow \\ \text{vertices}}}{S}, \underset{\substack{\uparrow \\ \text{edges}}}{R \smallsetminus \Delta})$ is a simple graph.[†]

[†] Unweighted, undirected graph with no loops or multiple edges. Here $\Delta := \{(a, a) : a \in S\}$.

Example. A tolerance relation that is not an equivalence relation:



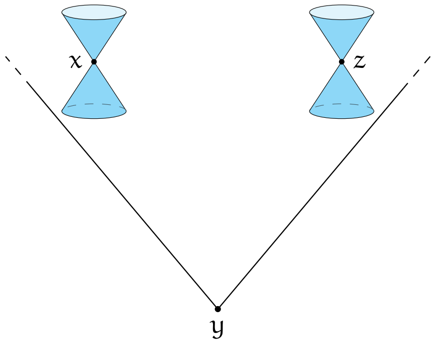
$S := \{x, y, z\}$

$R := \Delta \cup \{(x, y), (y, x), (y, z), (z, y)\}$

Real-life examples

① Causality in Special Relativity.

Here $S := \mathbb{R}^{1,3}$ and $a \sim b \overset{\text{def}}{\iff} \|a - b\| \geqslant 0$. Example:



In the picture:

$x \sim y$ and $y \sim z$ but $x \not\sim z$

Real-life examples

1. Causality in Special Relativity.
2. Equality in Wolfram Mathematica.

```
eps = 50*$MachineEpsilon;
x = 1;
y = x + eps;
z = y + eps;
{x == x, y == y, z == z, x == y, y == x, y == z, z == y, x == z, z == x}

(* Out:  {True, True, True, True, True, True, True, False, False}  *)
```

That is, $y := x + \varepsilon$, $z := y + \varepsilon$.

Since $\varepsilon \ll 1$ (finite storage capacity for numbers $=$ instrument with finite resolution):

$$x \sim y \qquad \text{and} \qquad y \sim z \qquad \text{but} \qquad x \not\sim z$$

Real-life examples

1. Causality in Special Relativity.
2. Equality in Wolfram Mathematica.
3. Proximity relations.

Let (S, d) be a metric space and $\varepsilon > 0$. Define $R \subset S \times S$ by:

$$(a, b) \in R \quad \Longleftrightarrow \quad d(a, b) < \varepsilon.$$

In some cases, proximity relations are transitive. E.g. if d is an ultrametric, i.e.

$$d(a, c) \leqslant \max\left\{d(a, b), d(b, c)\right\} \quad \forall\, a, b, c \in S.$$

Real-life examples

1. Causality in Special Relativity.
2. Equality in Wolfram Mathematica.
3. Proximity relations.
4. Covers.

Let S be a set and $\mathcal{U}$ a covering of S. A tolerance relation R on S is given by:

$$(x, y) \in R \quad \overset{\text{def}}{\Longleftrightarrow} \quad \left\{\exists\, A \in \mathcal{U} : x, y \in A\right\}.$$

- On a metric space with midpoint property (e.g. spectral distance):

  ϵ-proximity relation $\leftrightsquigarrow$ cover with open balls of radius $\varepsilon/2$.

- Compare with Sorkin and its approximations of topological spaces with posets.

Real-life examples

1. Causality in Special Relativity.
2. Equality in Wolfram Mathematica.
3. Proximity relations.
4. Covers.
5. Tolerance relations associated to actions (later).

The convolution product

In the following, $X :=$ topological space and $R :=$ tolerance relation on X.

We say that R is étale of $\exists$ topology on R such that the projection

$$R \to X, \qquad (x,y) \mapsto x,$$

is a local homeomorphism.
(If R is transitive, this means that the associated groupoid is étale.)

Lemma

If R is Hausdorff and étale, the space $C_c(R)$ of compactly-supported continuous complex functions on R is a $*$-algebra, with product and involution:

$$(f \star g)(x,z) := \sum_{y \in X : x \sim y, y \sim z} f(x,y)g(y,z)$$

$$f^*(x,z) := \overline{f(z,x)}$$

Example

Let both X and R discrete (hence étale).

For $(i,j) \in R$, E_{ij} is the function on R that is 1 at the site (i,j) and zero everywhere else.

In such a basis of $C_c(R)$:

$$E_{ij} \star E_{kl} = \begin{cases} \delta_{jk} E_{il} & \text{if } (i,l) \in R \\ 0 & \text{otherwise} \end{cases}$$

The algebra is unital if and only if X is a finite set (i.e. the diagonal Δ is compact), with unit:

$$1 = \sum_{i \in X} E_{ii} \,.$$

- R equivalence relation $\implies$ $(C_c(R), \star)$ associative $\implies$ universal C*-algebra.
- R not transitive $\implies$ $\star$ is not nec. associative $\implies$ no good notion of representation.
- The definition of convolution product makes sense over an arbitrary field $\mathbb{F}$.
- If $X = \{1, \dots, n\}$, we can identify $C_c(R)$ with a subset of $M_n(\mathbb{F})$ (as a vector space, but $\star$ is the matrix product iff R is transitive).

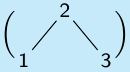
Finite-dimensional case

Notations. Here:

- $X := \{1, \dots, n\}$,
- R is a tolerance relation on X,
- the topology on X and R is discrete,
- Γ is the simple graph associated to R,
- $A(\Gamma) := \big(C_c(R) \subset M_n(\mathbb{F}), \star\big)$,
- $\Gamma = \bigcup \Gamma_i$ dec. into connected components.

Then:

- R is transitive $\iff$ each Γ_i is a complete graph.
- $A(\Gamma) = \bigoplus_i A(\Gamma_i)$, hence WLOG assume that Γ is connected.

Call $\mathfrak{A}_3 := A\Big($ 

$\Big)$. Then:

- $\mathfrak{A}_3$ is not power associative. $a := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \implies (a \star a) \star a \neq a \star (a \star a)$.
- TFAE:
  - (i) $A(\Gamma)$ is associative;
  - (ii) $A(\Gamma)$ is power associative;
  - (iii) $A(\Gamma)$ has no subalgebra isomorphic to $\mathfrak{A}_3$;
  - (iv) R is an equivalence relation.

Corollaries:

- $A(\Gamma)$ division algebra $\iff A(\Gamma) = \mathbb{F}$.
- Sedenions $\notin \{A(\Gamma)\}$.

Relations from actions

Act 1: group actions

- Given a group action $\alpha : G \times X \to X$, the relation "being on the same orbit":

$$R := \big\{ (x, \alpha_g x) : g \in G, x \in X \big\}$$

  is an equivalence relation.
- If G is discrete, X locally compact and Hausdorff and α continuous $\implies$ R is étale.
- If X compact, the completion of $(C_c(G), \star)$ is $C(X) \rtimes G$.
- If α is free and proper, $C(X) \rtimes G$ is Morita equivalent to $C(X/G)$.
- Examples: foliations, orbifolds, tilings of the plane, dynamical systems arising in number theory (Bost-Connes), ...
- Transitivity (and associativity) follows from the property:

$$\alpha_g \circ \alpha_h = \alpha_{gh} \quad \forall\, g, h \in G,$$

  and from associativity of the product in G.

  Generalizations: (1) α not a group action; (2) G not a group.

Relations from actions

Act 2: group quasi-actions

- Let A be a unital C*-algebra, G a group and

$$G \to \mathrm{Out}(A) := \mathrm{Aut}(A)/\mathrm{Inn}(A) \qquad (\star)$$

a group homomorphism. (E.g. A almost commutative $\Longrightarrow \mathrm{Out}(A) = \mathrm{Diff}(M)$.)

- Lift $(\star)$ to a map $\alpha : G \to \mathrm{Aut}(A)$ satisfying:

$$\alpha_g \circ \alpha_h = \mathrm{Ad}_{f(g,h)} \circ \alpha_{gh}$$

for a suitable $f : G \times G \to U(A)$. This was the motivation in [Bouwknegt, Hannabuss, Mathai, CMP 2006] for a theory of non-associative crossed products.

- A group quasi-action on a metric space (X, d) is a map $\alpha : G \times X \to X$ satisfying

$$d\big(\alpha_g\alpha_h(x), \alpha_{gh}(x)\big) \leqslant \varepsilon$$

for some fixed $\varepsilon > 0$.

Relations from actions

Act 3: magmas

- Here G is a set equipped with:
  - a binary operation $* : G \times G \to G$ (the "multiplication"),
  - a unary operation $(-)^{-1} : G \to G$ (the "right inversion"),
  - an element $1 \in G$ (the "right unit"),

  such that:

$$(g * h) * h^{-1} = g * 1 = g \qquad \forall\, g, h \in G.$$

  (Tentative name: magma with right inversion and unit.)

- A right action on a set X is a map $X \times G \to X$, $(x, g) \mapsto x \triangleleft g$, such that:

$$(x \triangleleft g) \triangleleft g^{-1} = x \triangleleft 1 = x \qquad \forall\, g \in G, x \in X.$$

  Given such an action, the image $R(G, X)$ of the canonical map:

$$X \times G \to X \times X, \qquad (x, g) \mapsto (x, x \triangleleft g),$$

  is a tolerance relation.

Relations from actions

Act 3: magmas

- An action is called $\dfrac{\text{free}}{\text{transitive}}$ $\Longleftrightarrow$ the canonical map is $\dfrac{\text{injective}}{\text{surjective}}$.

- The action of $G \curvearrowleft G$ by right mult. is free $\Longleftrightarrow$ $*$ is left-cancellative.

  If $g^{-1} * (g * h) = h\ \forall\, g, h \in G \quad \Longrightarrow \quad G \curvearrowleft G$ is both free and transitive.

- Example. $G := \mathbb{R} \smallsetminus \{0\}$, $x * y := x/y$. The action $G \curvearrowleft G$ is both free and transitive.

- Example. $G :=$ Moufang loop (e.g. unit octonions). $G \curvearrowleft G$ is free and transitive.

- Let G, X as before + topology; $\alpha, *, (-)^{-1}$ not nec. continuous.

  Then: $R(G, X)$ is étale $\Longleftrightarrow$ G is discrete.

Truncations

Let $B :=$ C*-algebra, $T : B \to B$ a linear map, $A := \mathrm{Im}(T)$, and $\star$ the product on A:

$$a \star b := T(ab) \qquad a, b \in A.$$

T is called idempotent if $T \circ T = T$ (which implies $B = A \oplus \ker T$ as vector spaces).

If T is idempotent and a completely positive contraction (CPC), then $\star$ is associative.

T is called a conditional expectation if one of the following equivalent conditions is satisfied:

(i) T is idempotent with $\|T\| = 1$,

(ii) T is positive, idempotent, and an A-bimodule map.

A conditional expectation is a CPC.

Example: $P \in B$ projection $\Longrightarrow T(x) := PxP\ \forall\, x \in B$ is a conditional expectation.

Example: $B = C(\mathbb{S}^1)$, $N \geqslant 1$, $T_1(f) := \sum_{|k| \leqslant N-1} e^{2\pi i k\theta} \widehat{f}(k)$ (Fourier partial sum)

$T_2 :=$ Cesàro sum

T_1 is idempotent but not positive; T_2 is positive but not idempotent.

States

- Every finite-dim. $A(\Gamma)$ is a truncation of a matrix algebra!

  If $X := \{1, \dots, n\}$, then $B := M_n(\mathbb{C})$ and $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is given by

$$T(x) := \sum_{(i,j)\in R} E_{ii} x E_{jj}$$

- $\mathcal{S}(A)$:= states in the sense of operator systems, i.e. restriction of states of $B = M_n(\mathbb{C})$.
- TFAE: (i) the map $\mathcal{S}(B) \to \mathcal{S}(A)$, $\varphi \mapsto \varphi|_A$, is injective (it is always surjective);
  (ii) the restriction of this map to pure states is injective;
  (iii) $A = B \ \big(= M_n(\mathbb{C})\big)$.
- A unit vector $v = (v_1, \dots, v_n) \in \mathbb{C}^n$ is called R-tolerant if the graph of the relation

$$R_v := \big\{(i,j) \in R : v_i v_j \neq 0\big\}$$

  is connected.

  (Analogous to “states with ε-connected support” in the case of a proximity relations, cf. [CvS21].)

- R-tolerant vector states of $M_n(\mathbb{C}) \overset{1:1}{\longleftrightarrow}$ pure states of A.

Positivity

Let $A = A(\Gamma) \subset M_n(\mathbb{C})$ and T as before. For $a \in A$, we write:

$$a \geqslant 0 \iff a \text{ is a positive semidefinite matrix}$$

$$a \succeq 0 \iff a = T(b) \text{ with } b \in M_n(\mathbb{C}) \text{ positive semidefinite}$$

Since T is idempotent

$$a \geqslant 0 \implies a \succeq 0.$$

Moreover, if $\exists\, b_k \in A$ s.t.

$$a = \sum\nolimits_k b_k \star b_k^* \tag{‡}$$

then $a = T\big(\sum_k b_k b_k^*\big) \succeq 0$.

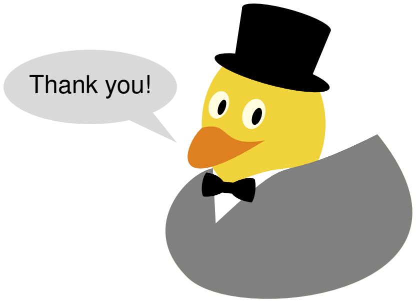
Proposition

- Each Γ_i has a dominant vertex $\iff$ every $a \succeq 0$ is of the form (‡).
- For $x, a \in M_n(\mathbb{C})$ let $\varphi_x(a) := \mathrm{Tr}(x^* a)$. Then, the map

$$(A, \succeq) \to (A^*, \geqslant), \qquad x \mapsto \varphi_x|_A,$$

  is an isomorphism of ordered vector spaces.

- States of $A \overset{1:1}{\longleftrightarrow}$ elements $\rho \in A$ s.t. $\rho \succeq 0$ & $\mathrm{Tr}(\rho) = 1$.



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