Differential nested pairs of quantum homogeneous spaces

Based on a joint work with Réamonn Ó Buachalla

Noncommutative geometry: metric and spectral aspects, Krakow

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September 30, 2022

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- In this project we aim to include a suitable **differential structure** in this picture, generalizing Brzeziński and Majid's notion of a **quantum principle bundle**.

The classical picture

- Let G be a group, we have a fibration

$$M/N \to G/N \twoheadrightarrow G/M,$$

  for any two subgroups $N \subseteq M \subseteq G$.
- This fibration is principal if and only if N is a normal subgroup of M.

A right H-comodule algebra (P, Δ_R) is said to be a *H-Hopf–Galois extension of* $B := P^{\mathrm{co}(H)}$ if for $m_P : P \otimes_B P \to P$ the multiplication of P, a bijection is given by

$$\mathrm{can} := (m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R) : P \otimes_B P \to P \otimes H.$$

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Definition

A principal right H-comodule algebra is a right H-comodule algebra (P, Δ_R) such that P is a Hopf–Galois extension of $B := P^{\mathrm{co}(H)}$ and P is faithfully flat as a right and left B-module.

Let A be an Hopf algebra consider a left coideal subalgebra $B \subseteq A$ such that $B^+A = AB^+$.
We have a $\pi_B(A)$-coaction

$$\Delta_{R,\pi_B} := (\mathrm{id} \otimes \pi_B) \circ \Delta, \qquad \pi_B : A \to A/B^+A$$

We have $A^{\mathrm{co}(A/B^+A)} = B$.

Definition

If A is faithfully flat as a right B-module, we call B a **quantum homogeneous** A-**space**.

The right setting to extend the construction of a (principal) fibration by taking the quotient with respect to a (normal) subgroup is given by nested pairs $B \subseteq P \subseteq A$ of homogeneous quantum spaces.

The normalcy condition corresponds to the request that $\pi_B(P)$ is an Hopf algebra.

Definition

A **principal pair of quantum homogeneous spaces** (B, P) *is given by a pair of homogeneous spaces* $B \subseteq P$ *such that* $\pi_B(P)$ *is an Hopf algebra.*

- Of course $\pi_B(P)$ is not guaranteed to have a coalgebra structure (just like M might be not normal in N!).

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- However, even for the pair $B \subseteq P \subseteq A$ we end up within the domain of Brzeziński and Szymański's putative theory of *noncommutative fiber bundles*

Proposition (Generalised Hopf–Galois condition for nested pair of quantum homogeneous spaces)

The canonical map $\mathrm{can} : A \otimes_B A \to A \otimes \pi_B(A)$ *restricts to an isomorphism*

$$P \otimes_B P \to A \Box_{\pi_P} \pi_B(P),$$

in the category ${}^A_P\mathrm{Mod}$.

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- In order to get an honest bundle with homogeneous fibre we have to add some informations about the differential structures that are involved.

Let us recall how the principle case looks like.

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Definition (Brzeziński–Majid)

Let H be a Hopf algebra. A **quantum principal** *H-***bundle** *is a pair $(P, \Omega^1(P))$, consisting of a right H-comodule algebra (P, Δ_R) and a right-H-covariant calculus $\Omega^1(P)$, such that:*

1. *P is a Hopf–Galois extension of $B = P^{\mathrm{co}(H)}$.*
2. *If $N \subseteq \Omega^1_u(P)$ is the sub-bimodule of the universal calculus corresponding to $\Omega^1(P)$, we have $\mathrm{ver}(N) = P \otimes I$, for some Ad-sub-comodule right ideal*

$$I \subseteq H^+ := \ker(\varepsilon : H \to \mathbb{C}).$$

Where $\mathrm{ver} := \mathrm{can} \circ \mathrm{proj}_B$ and $\mathrm{Ad} : H \to H \otimes H$ is defined by $\mathrm{Ad}(h) := h_{(2)} \otimes S(h_{(1)})h_{(3)}$.

We have then the **short exact sequence**:

$$0 \longrightarrow P\Omega^1(B)P \xrightarrow{\iota} \Omega^1(P) \xrightarrow{\mathrm{ver}} P \otimes \Lambda^1(H) \longrightarrow 0,$$

- By abuse of notation ver denotes the map induced on $\Omega^1(P)$ by identifying $\Omega^1(P)$ as a quotient of $\Omega^1_u(P)$.

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- By abuse of notation ver denotes the map induced on $\Omega^1(P)$ by identifying $\Omega^1(P)$ as a quotient of $\Omega^1_u(P)$.
- A **principal connection** corresponds to a splitting of this sequence.
- This corresponds to the classical notion of a principal connection as an Horizontal complement to the Vertical component of the tangent space.

When we pass to homogeneous fibrations, things work well at the universal level

Proposition

For any nested pairs of quantum homogeneous spaces, an exact sequence in the category ${}^{A}_{P}\mathrm{Mod}^{\pi_B}$ *is given by*

$$0 \to P\Omega^1_u(B)P \xrightarrow{\iota} \Omega^1_u(P) \xrightarrow{\mathrm{ver}} A\Box_{\pi_P}\pi_B(P)^+ \to 0.$$

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Definition

A differential nested pair of quantum homogeneous spaces is a nested pair of quantum homogeneous spaces $B \subseteq P \subseteq A$ *together with a sub-object* $N_P \subseteq \Omega_u(P)$ *in the category* ${}^{A}_{P}\mathrm{Mod}^{\pi_B}_{P}$.

Definition

An **Ehresmann connection** *is a left P-module, right π_B-comodule, projection $\Pi : \Omega^1(P) \to \Omega^1(P)$ satisfying*

$$\ker(\Pi) = P\Omega^1(B)P.$$

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- Just like in the principal case, an Ehresmann connection corresponds to a splitting of the short sequence.
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- We say that an Ehresmann connection is **strong** if $(\mathrm{id} - \Pi)\big(\mathrm{d}P\big) \subseteq \Omega^1(B)P$
- We can now use these data to produce **bimodule connections** on homogeneous vector bundles over quantum homogeneous spaces.

Definition

For Ω^1 a fodc over an algebra B, and $\mathcal{F}$ a left B-module, a **left connection** *on $\mathcal{F}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ satisfying*

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla f, \qquad \textit{for all } b \in B, f \in \mathcal{F}.$$

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$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla f, \qquad \textit{for all } b \in B, f \in \mathcal{F}.$$

A left bimodule connection on $\mathcal{F}$ is a pair (∇, σ) where ∇ is a left connection and $\sigma : \mathcal{F} \otimes_B \Omega^1(B) \to \Omega^1(B) \otimes_B \mathcal{F}$ is a bimodule map satisfying

$$\sigma(f \otimes_B db) = \nabla(fb) - \nabla(f)b.$$

- We define a functor $\Psi : {}^{\pi_B}\mathrm{Mod} \to {}_P\mathrm{Mod}, V \mapsto P\Box_{\pi_B} V$,
- For any $\mathcal{F} := \Psi(V)$ we have a natural embedding

$$j : \Omega^1(B) \otimes_B \mathcal{F} \hookrightarrow \Omega^1(B) P \Box_{\pi_B} V,$$

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given by the multiplication map.
- A strong Ehresmann connection Π defines a connection ∇ on $\mathcal{F}$ by

$$\nabla : \mathcal{F} \to \Omega^1(B) \otimes_B \mathcal{F}, \qquad \sum_i p_i \otimes v_i \mapsto j^{-1}\big((\mathrm{id} - \Pi)(\mathrm{d}p_i) \otimes v_i\big).$$

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- Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, for $q \neq -1, 0, 1$ we have $U_q(\mathfrak{g})$, the **Drinfeld–Jimbo quantised enveloping algebra**.

Let's look at some examples from quantum flag manifolds.

- Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, for $q \neq -1, 0, 1$ we have $U_q(\mathfrak{g})$, the **Drinfeld–Jimbo quantised enveloping algebra**.
- We also have a correspondent $\mathcal{O}_q(G)$, the **quantum coordinate algebra of** G, where G is the compact, simply-connected, simple Lie group having $\mathfrak{g}$ as its complexified Lie algebra.
- There is an action $U_q(\mathfrak{g}) \otimes \mathcal{O}_q(G) \to \mathcal{O}_q(G)$.

- Take the Dynkin diagram of G and crossed an arbitrary number of nodes.

- We define the levi subalgebra of $U_q(\mathfrak{g})$

$$U_q(\mathfrak{l}_S) := \big\langle K_i, E_j, F_j \mid i = 1, \dots, l; j \in S \big\rangle.$$

- We call the **quantum flag manifold** associated to S coideal subalgebra of $U_q(\mathfrak{l}_S)$-invariants:

$$\mathcal{O}_q\big(G/L_S\big) := {}^{U_q(\mathfrak{l}_S)}\mathcal{O}_q(G).$$

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Theorem (Ó Buachalla, Somberg - Bhattacharjee, A.C., Díaz García- A.C., Mukhopadhyay)

Let $G = A_n, D_n, G_2$ and denote by $O_q(F_{\mathfrak{g}})$ the full quantum flag manifold of G. There exist exactly two right $\mathcal{O}_q(G)$-covariant complex structures $\Omega_q^{\pm}(F_{\mathfrak{g}})$ of classical dimension.

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- This calculi restrict to Heckenberger–Kolb calculi when we restrict to the irreducible cases!
- It holds that $\mathcal{O}_q\big(G/L_S\big) \subseteq O_q(F_{\mathfrak{g}})$. So we have a fibration where fibre, base and bundle are all quantum flag manifolds!

Example

For the Drinfeld–Jimbo quantum group $U_q(\mathfrak{sl}_{n+1})$ we consider the coloured Dynkin diagrams.

and

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This gives us the fibration

$$\mathcal{O}_q(\mathbb{CP}^n) \hookrightarrow \mathcal{O}_q(F_{SU_{n+1}}) \twoheadrightarrow \mathcal{O}_q(F_{SU_n}).$$

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- The zero map on $\Omega_q^{\pm}(G/L_S)$ is a strong connection.

- Morever when we we consider the restricted calculi $\Omega_q^{\pm}(G/L_S)$ we have a differential nested pair and hence a quantum bundle with homogeneous fibre!
- The zero map on $\Omega_q^{\pm}(G/L_S)$ is a strong connection.
- We can realize the connections on homogeneous vector bundles as associated to the zero map on $\Omega_q^{\pm}(G/L_S)$

Theorem (Work in progress)

Let $\mathcal{F}$ be a homogeneous vector bundle over $\mathcal{O}_q(G/L_S)$. There exist two unique right $\mathcal{O}_q(G)$-covariant connections $\nabla^{\pm} : \mathcal{F} \to \mathcal{F} \otimes_{\mathcal{O}_q(G/L_S)} \Omega_q^{\pm}(G/L_S)$. Moreover $\nabla^{\pm}$ are bimodule connections.

Thank you