Lorentzian fermionic action by twisting Euclidean spectral triples

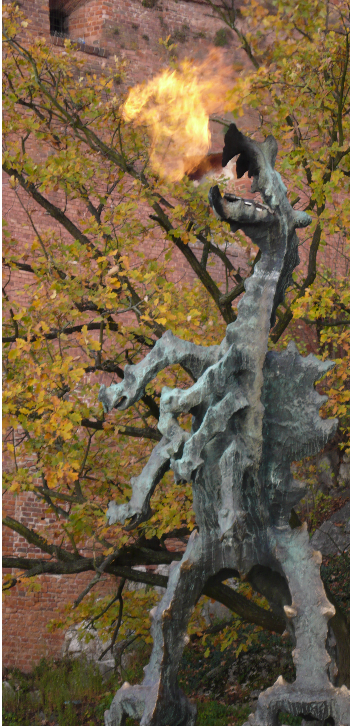
Pierre Martinetti

in collaboration with Devashish Singh

Università di Genova

NCG & Standard Model

Kraków, 8th November 2019



Besides a description of the Standard Model with (Euclidean) Einstein-Hilbert action, noncommutative geometry offers possibilities to go beyond SM:

- adding new-fermions (Stephan);
- get rid of the first-order condition (Chamseddine, Connes, Suijlekom), modifying the real structure (Brzezinski, Dabrowski, Sitarz), Clifford bundle (Dabrowski, D'Andrea, Sitarz.);
- non-associativity (Boyle, Farnsworth), Lorentzian structure (Besnard);
- using a twisted version of the original noncommutative geometry (Devastato, Lizzi, PM).

Besides a description of the Standard Model with (Euclidean) Einstein-Hilbert action, noncommutative geometry offers possibilities to go beyond SM:

- adding new-fermions (Stephan);
- get rid of the first-order condition (Chamseddine, Connes, Suijlekom), modifying the real structure (Brzezinski, Dabrowski, Sitarz), Clifford bundle (Dabrowski, D'Andrea, Sitarz.);
- non-associativity (Boyle, Farnsworth), Lorentzian structure (Besnard);
- using a twisted version of the original noncommutative geometry (Devastato, Lizzi, PM).

All but the first possibilities are minimal extension of the Standard Model: they produce an extra scalar field σ which stabilizes the electroweak-vacuum and permits to fit the Higgs mass, without adding new fermions.

Besides a description of the Standard Model with (Euclidean) Einstein-Hilbert action, noncommutative geometry offers possibilities to go beyond SM:

- adding new-fermions (Stephan);
- get rid of the first-order condition (Chamseddine, Connes, Suijlekom), modifying the real structure (Brzezinski, Dabrowski, Sitarz), Clifford bundle (Dabrowski, D'Andrea, Sitarz.);
- non-associativity (Boyle, Farnsworth), Lorentzian structure (Besnard);
- using a twisted version of the original noncommutative geometry (Devastato, Lizzi, PM).

All but the first possibilities are minimal extension of the Standard Model: they produce an extra scalar field σ which stabilizes the electroweak-vacuum and permits to fit the Higgs mass, without adding new fermions.

By using twisted noncommutative geometry, one gets in addition to σ another additional piece, which suprisingly turns out to be related to the transition from an Euclidean to a Lorentzian signature.

- Different from encoding the causal structure into some "Lorentzian NCG" (Barrett, Besnard, Eckstein , Franco, Dungen, Bochniak, Sitarz etc).

1. Standard Model in noncommutative geometry

2. Twisted spectral triple and Lorentz signature

3. Twisted fermionic action

Weyl equation

Dirac equation

1. Standard model in noncommutative geometry

Spectral triple: algebra $\mathcal{A}$ acting on a Hilbert $\mathcal{H}$ together with selfadjoint D s.t.

$$[D, a] \text{ is bounded} \quad \forall a \in \mathcal{A}.$$

Graded spectral triple: there exists $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{I}$, such that

$$\{\Gamma, D\} = 0, \quad [\Gamma, a] = 0 \quad \forall a \in \mathcal{A}.$$

Real spectral triple: there exists antilinear operator J such that

$$J^2 = \epsilon\mathbb{I}, \; JD = \epsilon' DJ, \; J\Gamma = \epsilon''\Gamma J$$

where $\epsilon, \epsilon', \epsilon'' = \pm 1$ define the *KO*-dimension $k \in [0, 7]$.

1. Standard model in noncommutative geometry

Spectral triple: algebra $\mathcal{A}$ acting on a Hilbert $\mathcal{H}$ together with selfadjoint D s.t.

$$[D, a] \text{ is bounded} \quad \forall a \in \mathcal{A}.$$

Graded spectral triple: there exists $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{I}$, such that

$$\{\Gamma, D\} = 0, \quad [\Gamma, a] = 0 \quad \forall a \in \mathcal{A}.$$

Real spectral triple: there exists antilinear operator J such that

$$J^2 = \epsilon \mathbb{I}, \; JD = \epsilon' DJ, \; J\Gamma = \epsilon'' \Gamma J$$

where $\epsilon, \epsilon', \epsilon'' = \pm 1$ define the *KO*-dimension $k \in [0, 7]$.

J implements a map $a \to a^\circ := Ja^* J^{-1}$ from $\mathcal{A}$ to the opposite algebra $\mathcal{A}^\circ$. This yields a right action of $\mathcal{A}$ on $\mathcal{H}$,

$$\psi a := a^\circ \psi,$$

which is asked to commute with the left action (order zero condition)

$$[a, Jb^* J^{-1}] = 0 \qquad \forall a, b \in \mathcal{A}.$$

As well, holds the first order condition

$$[[D, a], Jb^* J^{-1}] = 0 \qquad \forall a, b \in \mathcal{A}.$$

Fluctuation of the metric: substitution of D with the covariant Dirac operator

$$D_A = D + A + J\,A\,J^{-1}$$

where A is an element of the set of generalized 1-forms

$$\Omega^1_D(\mathcal{A}) := \left\{ a^i [D, b_i^\circ] \right\}.$$

Fluctuation of the metric: substitution of D with the covariant Dirac operator

$$D_A = D + A + J\,A\,J^{-1}$$

where A is an element of the set of generalized 1-forms

$$\Omega^1_D(\mathcal{A}) := \left\{a^i[D, b_i^\circ]\right\}.$$

Gauge transformation : conjugate action of the group $U(\mathcal{A})$ of unitaries of $\mathcal{A}$,

$$\mathrm{Ad}(u) : \psi \to u\psi u^* = u(u^*)^\circ\psi = uJuJ^{-1}\psi,$$

namely

$$\mathrm{Ad}(u)\, D_A\, \mathrm{Ad}(u)^{-1} = D_{A^u}$$

where

$$A^u := u[D, u^*] + uAu^*.$$

- Gauge transformations preserve the selfadjointness of D_A.

The spectral triple of the Standard Model:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_{32} + \gamma^5 \otimes D_F$$

where

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{32=2\times2\times8} = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^C,$$

$$D_F = \underbrace{\begin{pmatrix} 0_8 & M & 0_8 & 0_8 \\ M^\dagger & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & \bar{M} \\ 0_8 & 0_8 & M^T & 0_8 \end{pmatrix}}_{D_0} + \underbrace{\begin{pmatrix} 0_8 & 0_8 & M_R & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \\ M_R^\dagger & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \end{pmatrix}}_{D_R}.$$

- M contains the Yukawa couplings of the electron, the quarks up and down, and the (Dirac) mass of the electronic neutrino. M_R contains only one non-zero entry k_R (Majorana mass of the electronic neutrino).

The spectral triple of the Standard Model:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_F, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_{32} + \gamma^5 \otimes D_F$$

where

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_F = \mathbb{C}^{32=2\times 2\times 8} = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \mathcal{H}_R^c \oplus \mathcal{H}_L^C,$$

$$D_F = \underbrace{\begin{pmatrix} 0_8 & M & 0_8 & 0_8 \\ M^\dagger & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & \bar{M} \\ 0_8 & 0_8 & M^T & 0_8 \end{pmatrix}}_{D_0} + \underbrace{\begin{pmatrix} 0_8 & 0_8 & M_R & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \\ M_R^\dagger & 0_8 & 0_8 & 0_8 \\ 0_8 & 0_8 & 0_8 & 0_8 \end{pmatrix}}_{D_R}.$$

- M contains the Yukawa couplings of the electron, the quarks up and down, and the (Dirac) mass of the electronic neutrino. M_R contains only one non-zero entry k_R (Majorana mass of the electronic neutrino).

One also needs $\Gamma = \gamma^5 \otimes \gamma_F$ and $J = \mathcal{J} \otimes J_F$ with $\mathcal{J}$ the charge conjugation and

$$\gamma_F = \begin{pmatrix} \mathbb{I}_8 & & & \\ & -\mathbb{I}_8 & & \\ & & -\mathbb{I}_8 & \\ & & & \mathbb{I}_8 \end{pmatrix}, \quad J_F = \begin{pmatrix} 0_{16} & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0_{16} \end{pmatrix}.$$

Spectral action:

$$D_A = D + A + JAJ^{-1}$$

$$\left.\begin{array}{lcl} \mathcal{A} & = & C^\infty(\mathcal{M}) \otimes \mathcal{A}_F \\ \mathcal{H} & = & L_2(\mathcal{M}, S) \otimes \mathcal{H}_F \\ D & = & \partial\!\!\!/ \otimes \mathbb{I}_{32} + \gamma^5 \otimes D_F \end{array}\right\} \Longrightarrow A = \gamma^5 \otimes H - i \sum_\mu \gamma^\mu \otimes A_\mu.$$

- H: scalar field on $\mathcal{M}$ with value in $\mathcal{A}_F$ $\rightarrow$ Higgs.
- A_μ: 1-form field with value in $Lie(U(\mathcal{A}_F))$ $\rightarrow$ gauge field.

The asymptotic expansion $\Lambda \to \infty$ of the spectral action

$$\mathrm{Tr}\, f\left(\frac{D_A^2}{\Lambda^2}\right)$$

yields the bosonic SM Lagrangian coupled with Euclidean Einstein-Hilbert action.

The extra scalar field σ:

Because $m_H \leq 130\,\text{Gev}$, the quartic coupling of the Higgs field becomes negative at high energy, meaning the electroweak vacuum is meta-stable rather than stable. This instability can be cured by a new scalar field σ:

$$V(H,\sigma) = \frac{1}{4}(\lambda H^4 + \lambda_\sigma \sigma^4 + 2\lambda_{H\sigma} H^2 \sigma^2).$$

The extra scalar field σ:

Because $m_H \leq 130\,\text{Gev}$, the quartic coupling of the Higgs field becomes negative at high energy, meaning the electroweak vacuum is meta-stable rather than stable. This instability can be cured by a new scalar field σ:

$$V(H,\sigma) = \frac{1}{4}(\lambda H^4 + \lambda_\sigma \sigma^4 + 2\lambda_{H\sigma} H^2 \sigma^2).$$

In the SM spectral triple,

$$k_R \to k_R \sigma,$$

yields the required field, and alters the running of the parameters under the equations of the group of renormalization, so that to make the computation of m_H compatible with 125 Gev. Chamseddine, Connes 2012

Problem: σ cannot be obtained as a fluctuation of the metric, for

$$[\gamma^5 \otimes D_R, a] = 0 \qquad \forall a, b \in \mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F.$$

Problem: σ cannot be obtained as a fluctuation of the metric, for

$$[\gamma^5 \otimes D_R, a] = 0 \qquad \forall a, b \in \mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F.$$

But it can be obtained from the twisted fluctuation

$$[\gamma^5 \otimes D_R, a]_\rho \neq 0,$$

associated with the twisted spectral triple

$$(C^\infty(\mathcal{M}) \otimes \mathcal{A}_\mathsf{F}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_\mathsf{F}, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_{32} + \gamma^5 \otimes D_\mathsf{F}$$

where the automorphism ρ is the flip

$$\rho((f,g) \otimes m) = (g,f) \otimes m \qquad f,g \in C^\infty(\mathcal{M}), m \in \mathcal{A}_\mathsf{F}.$$

Devastato, Lizzi, P.M. 2014

Problem: σ cannot be obtained as a fluctuation of the metric, for

$$[\gamma^5 \otimes D_R, a] = 0 \qquad \forall a, b \in \mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathcal{A}_F.$$

But it can be obtained from the twisted fluctuation

$$[\gamma^5 \otimes D_R, a]_\rho \neq 0,$$

associated with the twisted spectral triple

$$(C^\infty(\mathcal{M}) \otimes \mathcal{A}_\mathrm{F}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S) \otimes \mathcal{H}_\mathrm{F}, \quad D = \partial\!\!\!/ \otimes \mathbb{I}_{32} + \gamma^5 \otimes D_\mathrm{F}$$

where the automorphism ρ is the flip

$$\rho((f, g) \otimes m) = (g, f) \otimes m \qquad f, g \in C^\infty(\mathcal{M}), m \in \mathcal{A}_\mathrm{F}.$$

Devastato, Lizzi, P.M. 2014

- Twisting the SM spectral triple is the correct mathematical way to implement the Grand Symmetry Fedele talked about.
- Many other ways to obtain σ. But as soon as one twists, one gets in addition a 1-form field coming from

$$[\partial\!\!\!/ \otimes \mathbb{I}_{32}, a]_\rho.$$

2. Twisted spectral triples and Lorentz signature

Given a triple $(\mathcal{A}, \mathcal{H}, D)$, instead of asking the commutators $[D, a]$ to be bounded, one asks the boundedness of the twisted commutators   Connes, Moscovici 2008

$$[D, a]_\rho := Da - \rho(a)D \qquad \text{for some fixed} \quad \rho \in \mathrm{Aut}(\mathcal{A}).$$

2. Twisted spectral triples and Lorentz signature

Given a triple $(\mathcal{A}, \mathcal{H}, D)$, instead of asking the commutators $[D, a]$ to be bounded, one asks the boundedness of the twisted commutators Connes, Moscovici 2008

$$[D, a]_\rho := Da - \rho(a)D \qquad \text{for some fixed} \quad \rho \in \text{Aut}(\mathcal{A}).$$

- Makes sense mathematically. Relevant to deal with type III algebras.
- Twisted spectral triples are compatible with the real structure.

Devastato, Landi, PM 2016/17

Twisted covariant Dirac operator:

$$D_{A_\rho} := D + A_\rho + J\, A_\rho\, J^{-1}$$

where A_ρ is an element of the set of twisted 1-forms

$$\Omega^1_D(\mathcal{A}, \rho) := \{a_i[D, b_i]_\rho, a_i, b_i \in \mathcal{A}\}.$$

Twisted gauge transformation: twisted adjoint action of $U = \mathrm{Ad}(u) = uJuJ^{-1}$:

$$\rho(U)\, D_{A_\rho}\, U^{-1} = D_{A^u_\rho},$$

where $\rho(U) = \rho(u)J\rho(u)J^{-1}$ with u a unitary of $\mathcal{A}$ and

$$A^u_\rho := \rho(u)[D, u^*]\rho + \rho(u)Au^*.$$

Landi, PM 2017

Twisted covariant Dirac operator:

$$D_{A_\rho} := D + A_\rho + J\,A_\rho\,J^{-1}$$

where A_ρ is an element of the set of twisted 1-forms

$$\Omega^1_D(\mathcal{A},\rho) := \{a_i[D,b_i]_\rho, a_i, b_i \in \mathcal{A}\}.$$

Twisted gauge transformation: twisted adjoint action of $U = \mathrm{Ad}(u) = uJuJ^{-1}$:

$$\rho(U)\,D_{A_\rho}\,U^{-1} = D_{A^u_\rho},$$

where $\rho(U) = \rho(u)J\rho(u)J^{-1}$ with u a unitary of $\mathcal{A}$ and

$$A^u_\rho := \rho(u)[D,u^*]\rho + \rho(u)Au^*.$$

Landi, PM 2017

- A gauge fluctuation $D_A \to UD_AU^{-1}$ preserved selfadjointness.
- A twisted fluctuation $D_{A_\rho} \to \rho(U)D_{A_\rho}U^{-1}$ has no reason to preserve it.

Twisted inner product: $\mathcal{H}$ an Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and ρ an automorphism of $\mathcal{B}(\mathcal{H})$.

Definition

A ρ-twisted inner product $\langle \cdot, \cdot \rangle_\rho$ is an inner product on $\mathcal{H}$ such that

$$\langle \Psi, \mathcal{O}\Phi \rangle_\rho = \langle \rho(\mathcal{O})^\dagger \Psi, \Phi \rangle_\rho \qquad \forall \mathcal{O} \in \mathcal{B}(\mathcal{H}),\ \Psi,\ \Phi \in \mathcal{H},$$

where † is the adjoint with respect to the initial inner product. We denote

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger.$$

the ρ-adjoint of $\mathcal{O}$.

- The ρ-twisted inner product is non-necessarily positive definite.

Twisted inner product: $\mathcal{H}$ an Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and ρ an automorphism of $\mathcal{B}(\mathcal{H})$.

Definition

A ρ-twisted inner product $\langle\cdot,\cdot\rangle_\rho$ is an inner product on $\mathcal{H}$ such that

$$\langle\Psi,\mathcal{O}\Phi\rangle_\rho = \langle\rho(\mathcal{O})^\dagger\Psi,\Phi\rangle_\rho \qquad \forall\mathcal{O}\in\mathcal{B}(\mathcal{H}),\ \Psi,\ \Phi\in\mathcal{H},$$

where † is the adjoint with respect to the initial inner product. We denote

$$\mathcal{O}^+ := \rho(\mathcal{O})^\dagger.$$

the ρ-adjoint of $\mathcal{O}$.

- The ρ-twisted inner product is non-necessarily positive definite.

If ρ an inner automorphism of $\mathcal{B}(\mathcal{H})$,

$$\rho(\mathcal{O}) = R\mathcal{O}R^\dagger \qquad \forall\mathcal{O}\in\mathcal{B}(\mathcal{H})$$

for a unitary operator R on $\mathcal{H}$, then a natural ρ-product is

$$\langle\Psi,\Phi\rangle_\rho = \langle\Psi,R\Phi\rangle.$$

Lorentzian inner product from twist: in the twisted spectral triple of the Standard Model, the flip ρ is an inner automorphism of $\mathcal{B}(L^2(\mathcal{M}, S))$, with

$$R = \gamma^0$$

the first Dirac matrix and $\mathcal{M}$ a Riemannian manifold.

- The ρ-twisted inner product is the Krein product for the space of spinors on a Lorentzian manifold.

Lorentzian inner product from twist: in the twisted spectral triple of the Standard Model, the flip ρ is an inner automorphism of $\mathcal{B}(L^2(\mathcal{M}, S))$, with

$$R = \gamma^0$$

the first Dirac matrix and $\mathcal{M}$ a Riemannian manifold.

- The ρ-twisted inner product is the Krein product for the space of spinors on a Lorentzian manifold.

- Furthermore, extending ρ to the whole of $\mathcal{B}(L^2(\mathcal{M}, S))$, one finds

  $$\rho(\gamma^0) = \gamma^0, \quad \rho(\gamma^j) = -\gamma^j \quad \text{for} \quad j = 1, 2, 3.$$

  The flip ρ is the square of the Wick rotation

  $$W(\gamma^0) = \gamma^0, \quad W(\gamma^j) = i\gamma^j.$$

  that is $\rho = W^2$.

Lorentzian inner product from twist: in the twisted spectral triple of the Standard Model, the flip ρ is an inner automorphism of $\mathcal{B}(L^2(\mathcal{M}, S))$, with

$$R = \gamma^0$$

the first Dirac matrix and $\mathcal{M}$ a Riemannian manifold.

- The ρ-twisted inner product is the Krein product for the space of spinors on a Lorentzian manifold.

- Furthermore, extending ρ to the whole of $\mathcal{B}(L^2(\mathcal{M}, S))$, one finds

  $$\rho(\gamma^0) = \gamma^0, \quad \rho(\gamma^j) = -\gamma^j \quad \text{for} \quad j = 1, 2, 3.$$

  The flip ρ is the square of the Wick rotation

  $$W(\gamma^0) = \gamma^0, \quad W(\gamma^j) = i\gamma^j.$$

  that is $\rho = W^2$.

- Krein-selfadjointness is preserved by twisted fluctuations. Twisted spectral triples may be a relevant tool to deal with Lorentzian spectral triple.

Devastato, Lizzi, Farnsworth, PM 2017

3. Twisted fermionic action

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with real structure J, the fermionic action for the covariant operator D_A is

$$S^f(D_A) = \mathfrak{A}_{D_A}(\tilde{\xi}, \tilde{\xi})$$

with $\tilde{\xi}$ the Grassman variables associated to $\xi \in \mathcal{H}^+ = \{\xi \in \mathcal{H}, \Gamma\xi = \xi\}$ and

$$\mathfrak{A}_{D_A}(\xi, \xi') = \langle J\xi, D_A\xi' \rangle.$$

- The presence of J is here to make the bilinear form $\mathfrak{A}_{D_A}$ antisymmetric.
- Restricting to $\mathcal{H}^+$ is required to solve the fermion doubling problem.
- Makes sense, for $\mathcal{H}^+$ are the elements of $\mathcal{H}$ with well defined chirality.

3. Twisted fermionic action

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with real structure J, the fermionic action for the covariant operator D_A is

$$S^f(D_A) = \mathfrak{A}_{D_A}(\tilde{\xi}, \tilde{\xi})$$

with $\tilde{\xi}$ the Grassman variables associated to $\xi \in \mathcal{H}^+ = \{\xi \in \mathcal{H}, \Gamma\xi = \xi\}$ and

$$\mathfrak{A}_{D_A}(\xi, \xi') = \langle J\xi, D_A\xi' \rangle.$$

- The presence of J is here to make the bilinear form $\mathfrak{A}_{D_A}$ antisymmetric.
- Restricting to $\mathcal{H}^+$ is required to solve the fermion doubling problem.
- Makes sense, for $\mathcal{H}^+$ are the elements of $\mathcal{H}$ with well defined chirality.

In the twisted case, with automorphism ρ implemented by a unitary R, one defines

$$S^f(D_{A_\rho}) = \mathfrak{T}_{D_{A_\rho}}(\tilde{\xi}, \tilde{\xi})$$

for $\xi \in \mathcal{H}_r := \{\xi \in \mathcal{H}, R\xi = \xi\}$ and

$$\mathfrak{T}_{D_{A_\rho}}(\xi, \xi') = \langle J\xi, RD_{A_\rho}\xi' \rangle.$$

- The matrix R guarantees the invariance under twisted gauge transformation.
- Restricting to $\mathcal{H}_r$ is to make the bilinear form $\mathfrak{T}_{D_{A_\rho}}$ antisymmetric.
- Does it make sense physically ?

Minimal twist of a manifold $\mathcal{M}$:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S), \quad D = \partial\!\!\!/; \quad \rho$$

where (with $\dim \mathcal{M} = 2m$)

$$\pi(f,g) = \begin{pmatrix} f\,\mathbb{I}_{2^{m-1}} & 0 \\ 0 & g\mathbb{I}_{2^{m-1}} \end{pmatrix}, \quad \rho(f,g) = (g,f) \qquad \forall (f,g) \in \mathcal{A}.$$

Minimal twist of a manifold $\mathcal{M}$:

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S), \quad D = \partial\!\!\!/; \quad \rho$$

where (with $\dim \mathcal{M} = 2m$)

$$\pi(f,g) = \begin{pmatrix} f\,\mathbb{I}_{2^{m-1}} & 0 \\ 0 & g\mathbb{I}_{2^{m-1}} \end{pmatrix}, \quad \rho(f,g) = (g,f) \qquad \forall (f,g) \in \mathcal{A}.$$

- This is the only possibility to twist a manifold keeping $\mathcal{H}$ and $\partial\!\!\!/$ untouched.
- The automorphism ρ is induced by $R = \gamma^0$.
- In dimension $0, 4$, there exist non-zero twisted selfadjoint fluctuations of $\partial\!\!\!/$:

$$\partial\!\!\!/_\rho = \partial\!\!\!/ - i\, f_\mu \gamma^\mu \gamma^5 \quad \text{with} \quad f_\mu \in C^\infty(\mathcal{M}, \mathbb{R}).$$

The twisted fermionic action, in dimension 4 is

$$S^f(\partial_\rho) = 2\int_{\mathcal{M}} d\mu\, \bar{\tilde{\zeta}}^\dagger \sigma_2 \left(if_0 \mathbb{I}_2 - \sum_{j=1}^{3} \sigma_j \partial_j\right) \tilde{\zeta} \quad \text{where} \quad \xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \in \mathcal{H}_r.$$

- The ∂_0 derivative is substituted with the component f_0 of the fluctuation.

The twisted fermionic action, in dimension 4 is

$$S^f(\partial_\rho) = 2\int_{\mathcal{M}} d\mu\, \bar{\tilde{\zeta}}^\dagger \sigma_2 \left(if_0 \mathbb{I}_2 - \sum_{j=1}^{3} \sigma_j \partial_j \right) \tilde{\zeta} \quad \text{where} \quad \xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \in \mathcal{H}_r.$$

- The ∂_0 derivative is substituted with the component f_0 of the fluctuation.

It reminds the Weyl lagrangian

$$i\psi_l^\dagger\, \tilde{\sigma}_M^\mu\, \partial_\mu \psi_l \quad \text{where} \quad \tilde{\sigma}_M^\mu := \left\{ \mathbb{I}_2, -\Sigma_{j=1}^3 \sigma_j \right\}.$$

The twisted fermionic action, in dimension 4 is

$$S^f(\partial_\rho) = 2\int_{\mathcal{M}} d\mu\, \bar{\tilde{\zeta}}^\dagger \sigma_2 \,(if_0\mathbb{I}_2 - \sum_{j=1}^{3} \sigma_j\partial_j)\,\tilde{\zeta} \quad \text{where} \quad \xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix} \in \mathcal{H}_r.$$

- The ∂_0 derivative is substituted with the component f_0 of the fluctuation.

It reminds the Weyl lagrangian

$$i\psi_l^\dagger\, \tilde{\sigma}_M^\mu\, \partial_\mu \psi_l \quad \text{where} \quad \tilde{\sigma}_M^\mu := \left\{\mathbb{I}_2, -\Sigma_{j=1}^3 \sigma_j\right\}.$$

It is tempting to identify $\tilde{\zeta}$ with ψ_l, then to assume

$$\partial_0\psi_l = if_0\tilde{\zeta},$$

that is

$$\tilde{\zeta}(x_0, x_j) = \psi_l(x_0, x_j) = e^{itf_0}\psi_l(x_j).$$

But then $\bar{\tilde{\zeta}}^\dagger\sigma^2 \neq i\psi_l^\dagger$.

Singh, PM, 2019

- Not enough degrees of freedom (already observed by v. Dungen and v. Suijlekom in their building of a spectral triple for electrodynamics).

Minimal twist of a doubled manifold:

$$\mathcal{A} = \left(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2\right) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2, \quad D = \eth \otimes \mathbb{I}_2$$

$$\pi(a = (f,g), a' = (f',g')) = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & g'\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & g\mathbb{I}_2 \end{pmatrix}, \quad \rho(a,a') = (a',a).$$

Minimal twist of a doubled manifold:

$$\mathcal{A} = \left(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2\right) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, \mathcal{S}) \otimes \mathbb{C}^2, \quad D = \eth \otimes \mathbb{I}_2$$

$$\pi(a = (f,g), a' = (f',g')) = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & g'\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & g\mathbb{I}_2 \end{pmatrix}, \quad \rho(a, a') = (a', a).$$

- Twisted fluctuation: $\partial_\rho \otimes \mathbb{I}_2 + g_\mu \gamma^\mu \otimes \gamma_F$ with g_μ a real field and γ_F the chirality of the two point space.
- Twist implemented by $R = \gamma^0 \otimes \mathbb{I}_2$.
- $\mathcal{H}_R$ spanned by $\{\xi \otimes e, \phi \otimes \bar{e}\}$ with $\xi = \begin{pmatrix} \zeta \\ \zeta \end{pmatrix}$, $\phi = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$, $\{e, \bar{e}\}$ basis of $\mathbb{C}^2$.

The fermionic action (with $g_\mu = 0$) is the integral of

$$\mathcal{L}^f_\rho := \bar{\tilde{\varphi}}^\dagger \sigma_2 \left(i f_0 - \sum_{j=1}^3 \sigma_j \partial_j \right) \tilde{\zeta}, \qquad f_0 \in C^\infty(\mathcal{M}, \mathbb{R}).$$

Yields the Weyl lagrangian identifying

$$\Psi_l := \tilde{\zeta}, \quad \Psi_l^\dagger := -i \bar{\tilde{\varphi}}^\dagger \sigma_2$$

and assuming

$$\partial_0 \Psi_l = i f_0 \Psi_l,$$

that is

$$\Psi_l(x_0, x_j) = \Psi_l(x_j) e^{i f_0 x_0}.$$

The fermionic action (with $g_\mu = 0$) is the integral of

$$\mathcal{L}^f_\rho := \bar{\tilde{\varphi}}^\dagger \sigma_2 \left(i f_0 - \sum_{j=1}^3 \sigma_j \partial_j \right) \tilde{\zeta}, \qquad f_0 \in C^\infty(\mathcal{M}, \mathbb{R}).$$

Yields the Weyl lagrangian identifying

$$\Psi_l := \tilde{\zeta}, \quad \Psi_l^\dagger := -i \bar{\tilde{\varphi}}^\dagger \sigma_2$$

and assuming

$$\partial_0 \Psi_l = i f_0 \Psi_l,$$

that is

$$\Psi_l(x_0, x_j) = \Psi_l(x_j) e^{i f_0 x_0}.$$

- The twisted fermionic action for a twisted doubled Euclidean manifold describes a plane wave solution of Weyl equation (in Lorentz signature), with x_0 as time coordinate.
- The zeroth-component of the real field f_μ parametrising the twisted-fluctuation gets interpreted as an energy.

Minimal twist of electrodynamics:

$$\mathcal{A}_{\text{ED}} = \left(C^\infty(\mathcal{M}) \otimes \mathbb{C}^2\right) \otimes\mathbb{C}^2, \; \mathcal{H} = L^2(\mathcal{M},\mathcal{S}) \otimes \mathbb{C}^4, \quad D = \eth \otimes \mathbb{I}_4 + \gamma^5 \otimes D_{\mathcal{F}}$$

$$D_{\mathcal{F}} = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix}, \; \pi(a,a') = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f\mathbb{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g'\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g'\mathbb{I}_2 \end{pmatrix}$$

v. Dungen, V. Suijlekom + PM, Singh

with $d \in \mathbb{C}$, $a = (f,g)$, $a' = (f',g')$.

Minimal twist of electrodynamics:

$$\mathcal{A}_{\text{ED}} = \left(C^\infty(\mathcal{M}) \otimes \ \mathbb{C}^2\right) \otimes\mathbb{C}^2,\ \mathcal{H} = L^2(\mathcal{M},\mathcal{S}) \otimes \mathbb{C}^4, \quad D = \eth \otimes \mathbb{I}_4 + \gamma^5 \otimes D_{\mathcal{F}}$$

$$D_{\mathcal{F}} = \begin{pmatrix} 0 & d & 0 & 0 \\ \bar{d} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{d} \\ 0 & 0 & d & 0 \end{pmatrix},\ \pi(a,a') = \begin{pmatrix} f\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & f'\mathbb{I}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f\mathbb{I}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g'\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & g\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & g'\mathbb{I}_2 \end{pmatrix}$$

v. Dungen, V. Suijlekom + PM, Singh

with $d \in \mathbb{C}$, $a = (f,g)$, $a' = (f',g')$.

- Twisted fluctuation: $D - if_\mu\gamma^\mu\gamma^5 \otimes \mathbb{I}' + g_\mu\gamma^\mu \otimes \mathbb{I}''$ with $\mathbb{I}' := \text{diag}(1,-1,1,-1)$, $\mathbb{I}'' := \text{diag}(1,1,-1,-1)$.
- Twist implemented by $R = \gamma^0 \otimes \mathbb{I}_4$.
- $\mathcal{H}_R$ spanned by $\{\Phi_1 \otimes e_l, \Phi_2 \otimes e_r, \Phi_3 \otimes \overline{e_l}, \Phi_4 \otimes \overline{e_r}\}$ where $\Phi_k := \begin{pmatrix} \varphi_k \\ \varphi_k \end{pmatrix}$ are Dirac spinors with Weyl components φ_k, and $\{e_l, e_r, \overline{e_l}, \overline{e_r}\}$ the basis of $\mathbb{C}^4$.

The lagrangian density corresponding to the fermionic action,

$$\mathcal{L}_\rho^f := \bar{\tilde{\varphi}}_1^\dagger \sigma_2 \left(i f_0 - \sum_j \sigma_j \mathfrak{D}_j \right) \tilde{\varphi}_3 - \bar{\tilde{\varphi}}_2^\dagger \sigma_2 \left(i f_0 + \sum_j \sigma_j \mathfrak{D}_j \right) \tilde{\varphi}_4 + \left(\bar{d}\, \bar{\tilde{\varphi}}_1^\dagger \sigma_2 \tilde{\varphi}_4 + d\, \bar{\tilde{\varphi}}_2^\dagger \sigma_2 \tilde{\varphi}_3 \right)$$

where $\mathfrak{D}_\mu := \partial_\mu - i g_\mu$, coincides with the Dirac lagrangian,

$$\mathcal{L}_M = i \Psi_l^\dagger \left(\mathfrak{D}_0 - \sigma_j \mathfrak{D}_j \right) \Psi_l + i \Psi_r^\dagger \left(\mathfrak{D}_0 + \sigma_j \partial_j \right) \Psi_r - m \left(\Psi_l^\dagger \Psi_r + \Psi_r^\dagger \Psi_l \right),$$

up to the identification $d = -im$,

$$\Psi = \begin{pmatrix} \Psi_l \\ \Psi_r \end{pmatrix} := \begin{pmatrix} \tilde{\varphi}_3 \\ \tilde{\varphi}_4 \end{pmatrix}, \qquad \Psi^\dagger = \begin{pmatrix} \Psi_l^\dagger, & \Psi_r^\dagger \end{pmatrix} := \begin{pmatrix} -i \bar{\tilde{\varphi}}_1^\dagger \sigma_2, & i \bar{\tilde{\varphi}}_2^\dagger \sigma_2 \end{pmatrix},$$

and assuming $\partial_0 \Psi = i f_0 \Psi$, that is $\Psi(x_0, x_j) = \Psi(x_j) e^{i f_0 x_0}$.

- The fermionic action for the twisted spectral triple of electrodynamics on a riemannian manifold describes a plane wave solution of the Dirac equation in lorentzian signature, and in the temporal (Weyl) gauge $\mathfrak{D}_0 = \partial_0$.

Conclusion

- The same process that turns the Dirac mass k_R of the neutrino into a scalar field generates a 1-form field, that - in the fermionic action - can be interpreted as the zero-th component of the energy-momentum in Lorentzian signature. Other components can be found by Lorentz transform ?
- Yet the manifold is still riemannian: not so important for the volume form, but what about the domain of integration ? (usually Wick rotation is considered locally).

Conclusion

- The same process that turns the Dirac mass k_R of the neutrino into a scalar field generates a 1-form field, that - in the fermionic action - can be interpreted as the zero-th component of the energy-momentum in Lorentzian signature. Other components can be found by Lorentz transform ?
- Yet the manifold is still riemannian: not so important for the volume form, but what about the domain of integration ? (usually Wick rotation is considered locally).
- In the standard model (with all masses but k_R to zero), ignoring the lorentzian interpretation, the spectral action for the supplementary fields is minimum when theses fields are zero, i.e. when there is no twist. Lorentzian (twisted) geometry as vacuum excitations around the (non-twisted) riemannian ?

Conclusion

- The same process that turns the Dirac mass k_R of the neutrino into a scalar field generates a 1-form field, that - in the fermionic action - can be interpreted as the zero-th component of the energy-momentum in Lorentzian signature. Other components can be found by Lorentz transform ?
- Yet the manifold is still riemannian: not so important for the volume form, but what about the domain of integration ? (usually Wick rotation is considered locally).
- In the standard model (with all masses but k_R to zero), ignoring the lorentzian interpretation, the spectral action for the supplementary fields is minimum when theses fields are zero, i.e. when there is no twist.
  Lorentzian (twisted) geometry as vacuum excitations around the (non-twisted) riemannian ?
- Maybe not so meaningful to look for a spectral characterization of lorentzian manifolds.
  Thermal time hypothesis: the flow of time is induced by the twist. Coherent with modular twist of Connes-Moscovici: $\rho = \sigma_{t=i}$. What σ for the flip ?

Conclusion

- The same process that turns the Dirac mass k_R of the neutrino into a scalar field generates a 1-form field, that - in the fermionic action - can be interpreted as the zero-th component of the energy-momentum in Lorentzian signature. Other components can be found by Lorentz transform ?
- Yet the manifold is still riemannian: not so important for the volume form, but what about the domain of integration ? (usually Wick rotation is considered locally).
- In the standard model (with all masses but k_R to zero), ignoring the lorentzian interpretation, the spectral action for the supplementary fields is minimum when theses fields are zero, i.e. when there is no twist.
  Lorentzian (twisted) geometry as vacuum excitations around the (non-twisted) riemannian ?
- Maybe not so meaningful to look for a spectral characterization of lorentzian manifolds.
  Thermal time hypothesis: the flow of time is induced by the twist. Coherent with modular twist of Connes-Moscovici: $\rho = \sigma_{t=i}$. What σ for the flip ?
- $i\, f^{\mu}\gamma^{\mu}\gamma^{5}$ a torsion term. Could help for spectral action (see Hanisch, Pfaffle, Stephan).

Lorentzian fermionic action
by twisting euclidean spectral triples,
with D. Singh, arXiv 1907.02485;

Lorentz signature and twisted spectral triples,
with A. Devastato, F. Lizzi and S. Farnsworth, JHEP (2018);

Gauge transformation for twisted spectral triples,
with G. Landi, Letters in Mathematical Physics (2018);

On twisting real spectral triples by algebra automorphism,
with G. Landi, Letters in Mathematical Physics (2016);

Twisted spectral triple for the standard model & spontaneous breaking of the grand symmetry,
with A. Devastato, Mathematical Physics, Analysis and Geometry (2017);

Grand symmetry, spectral action and the Higgs mass,
with A. Devastato, F. Lizzi, JHEP (2014).