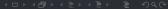
# Progress in spectral triples with twisted real structure

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## Background and definitions

## Definition (Real spectral triple)

A real spectral triple  $(\mathcal{A}, \mathcal{H}, D, J)$  is given by an action of the \*-algebra  $\mathcal{A}$  on the Hilbert space  $\mathcal{H}$ , with  $D = D^*$  an unbounded operator with  $(D - \lambda)^{-1} \in \mathcal{K}(\mathcal{H})$  and  $[D, a] \in \mathcal{B}(\mathcal{H})$  for all  $a \in \mathcal{A}$ , equipped with an antiunitary operator J such that  $J\mathcal{A}J^* \subset \mathcal{A}'$  and

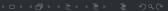
$$J^2 = \varepsilon 1_{\mathcal{H}}$$
 and  $DJ = \varepsilon' JD$ 

for  $\varepsilon, \varepsilon' \in \{-1, +1\}$ .

We also typically require the satisfaction of the first-order condition

$$[D, a]JbJ^* = JbJ^*[D, a]$$
 (1)

for all  $a, b \in \mathcal{A}$ .



## Definition (Spectral triple with twisted real structure)

A spectral triple with twisted real structure is a real spectral triple  $(\mathcal{A}, \mathcal{H}, D, J)$  equipped with a (bounded) operator  $\nu$  such that  $\bar{\nu}(\mathcal{A}) := \nu \mathcal{A} \nu^{-1} \simeq \mathcal{A}$ ,  $\nu J \nu = J$  and now

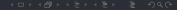
$$DJ\nu = \varepsilon'\nu JD$$

for  $\varepsilon' \in \{-1, +1\}$ .

The first-order condition is here replaced by the twisted first-order condition

$$[D, a]J\bar{\nu}^2(b)J^* = JbJ^*[D, a]$$
 (2)

for all  $a, b \in \mathcal{A}$ .



## Conformal transformation [Brzeziński, Ciccoli, Dąbrowski & Sitarz]

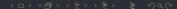
For a real spectral triple  $(\mathcal{A}, \mathcal{H}, D, J)$ , one can define a conformally transformed Dirac operator

$$D_k := JkJ^*DJkJ^*$$

for  $k, k^{-1} \in \mathcal{A}_+$ , such that  $D_k = D_k^*$  and

$$[D,a] \in \mathcal{B}(\mathcal{H}) \Longrightarrow [D_k,a] \in \mathcal{B}(\mathcal{H})$$

for all  $a \in \mathcal{A}$ , so that now  $(\mathcal{A}, \mathcal{H}, D_k, J, \nu)$  is a spectral triple with twisted real structure for the twist operator  $\nu = k^{-1}JkJ^*$ .



## Nibbling around the edges

Conformal transformtions

A twist operator  $\nu$  which is involutory  $\nu^2 = 1$  we refer to as being mild.

#### Conformal transformation (small generalisation)

For a spectral triple with mildly-twisted real structure  $(\mathcal{A}, \mathcal{H}, D, J, \chi)$  with

$$\chi^2 = 1$$
 and  $\chi \in \mathscr{A}'$ ,

one can define a conformally transformed Dirac operator as before so that now  $(\mathcal{A}, \mathcal{H}, D_k, J, \nu)$  is a spectral triple with twisted real structure for the twist operator  $\nu = \chi k^{-1} J k J^*$ .

## Nibbling around the edges

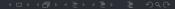
The even case

If a real spectral triple ( $\mathcal{A}$ ,  $\mathcal{H}$ , D, J) admits a  $\mathbb{Z}_2$  grading on  $\mathcal{H}$  implemented by the grading operator  $\gamma$ , it is called even. One requires that  $\gamma D = -D\gamma$  and

$$\gamma J = \varepsilon'' J \gamma \tag{3}$$

for  $\varepsilon'' \in \{-1, +1\}$ . It is not immediately obvious how  $\gamma$  should interact with  $\nu$  in the case of a twisted real structure. It was identifed in [1] that we should require  $\gamma \nu^2 = \nu^2 \gamma$ , but it was previously assumed that we should take (3) as-is (cf. [1, 2]). However, the fact that  $\nu JDJ^*\nu = \pm D$  suggests we should instead require

$$\gamma \nu J = \varepsilon'' \nu J \gamma. \tag{4}$$



## Bigger bites

Gauge transformations

It is clear that if fluctuations of Dirac operators are generated by 1-forms, then  $D = \varepsilon' \nu J D J^* \nu$  demands that fluctuations must be of the form

$$D_{\omega} = D + \omega + \varepsilon' \nu J \omega J^* \nu$$

for  $\Omega_D^1(\mathscr{A}) \ni \omega = \sum_j a_j [D, b_j]$  a 1-form. Note that  $v \in \mathscr{B}(\mathscr{H}) \implies v J \omega J^* v \in \mathscr{B}(\mathscr{H})$ . Interestingly,

$$\varepsilon' \nu Ja[D, b]J^* \nu = \bar{\nu}(JaJ^*)[D, \bar{\nu}^{-1}(JbJ^*)]_{\bar{\nu}^2}.$$

This suggests it should be possible to define Morita equivalence following the standard construction (right) and the construction for twisted spectral triples by [3] (left).

This is the case, with some minor modifications.

#### For reference...

For a unitary element  $u \in \mathcal{U}(\mathcal{A})$ , the gauge transformation of a gauge potential 1-form  $\omega \mapsto \omega^u$  and Dirac operator  $D \mapsto D^u$  are given by their standard definitions, i.e.,

$$\omega^{u} = u\omega u^{*} + u[D, u^{*}],$$

$$(D_{\omega})^{u} = D_{\omega^{u}}$$

$$= D + \omega^{u} + \varepsilon' \nu J \omega^{u} J^{*} \nu.$$

In the standard case, gauge transformations are implemented by the operator  $U := uJuJ^*$  such that  $D^u = UDU^*$ .

It is more natural to define the right  $\mathscr A$  -action on  $\mathscr H$  by

$$\psi \cdot a = \bar{\nu}^{-1} (Ja^* J^*) \psi$$

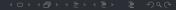
for  $a \in \mathcal{A}$  ,  $\psi \in \mathcal{H}$  . This means that we have

Ad(u)(
$$\psi$$
) =:  $\nabla \psi = u v^{-1} J u J^* v \psi$  and  
 $\widetilde{Ad}(u)(\psi) =: \widetilde{\nabla} \psi = u v J u J^* v^{-1} \psi$ .

These operators implement the gauge transformations by

$$D^{u} = \tilde{V}DV^{-1}$$
.

*Rmk*: Gauge transformations commute with conformal transformations.

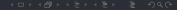


Note that the operators V and  $\tilde{V}$  are not unitary! However, the self-adjointness of Dirac operators

$$(D^{u})^{*} = (\tilde{V}DV^{-1})^{*} \stackrel{!}{=} D^{u} = \tilde{V}DV^{-1}$$

requires that  $\tilde{V}^* = V^{-1}$  and  $V^* = \tilde{V}^{-1}$ . This suggests that we demand that  $\nu = \nu^*$  (up to sign), which in turn implies that gauge potentials are self-adjoint,  $\omega = \omega^*$ , as in the standard case.

*Rmk*: Self-adjointness of the twist operator is also consistent with the typical requirement of twisted spectral triples that the algebra automorphism  $\rho$  satisfies  $\rho(a^*) = (\rho^{-1}(a))^*$  for all  $a \in \mathcal{A}$ .



## Ongoing work

In this context, the bilinear form  $\mathfrak{A}_D(\psi, \phi) := \langle J\psi, D\phi \rangle$  is *not* gauge covariant or antisymmetric any more and must be modified, similar to [4]. It appears that the correct modification is given by

$$\mathfrak{A}_{\mathsf{D}}^{\bar{\nu}}(\psi,\phi) := \langle \mathsf{J}\psi, \nu^{-1}\mathsf{D}\phi \rangle,$$

where  $\nu = \nu^*$  is required but  $\nu = \nu^{-1}$  is not. However, requiring that

$$\operatorname{Tr}\left(f\left(\frac{D^2}{\Lambda^2}\right)\right) = \operatorname{Tr}\left(f\left(\frac{(D^u)^2}{\Lambda^2}\right)\right)$$

does appear to require that  $\nu = \nu^{-1}$ .

## Bigger bites

Second-order condition

Following [5] and subsequent papers, and [6], we define the second-order condition for ordinary spectral triples to be

$$[[D, a], J[D, b]J^*] = 0$$

for all  $a, b \in \mathcal{A}$ . Just as spectral triples with mildly-twisted real structures follow the first-order condition like untwisted spectral triples, it may be possible to construct examples which follow the second-order condition.

## Hodge spectral triple on the torus [D'Andrea, M. & Dąbrowski]

Consider the spectral triple for the torus  $T^2$  with coordinates (x, y) given by

$$\left(\mathscr{A}=C^{\infty}\left(T^{2}\right),\mathscr{H}=M_{2}\left(L^{2}\left(T^{2}\right)\right),D=i\sigma_{1}\delta_{x}+i\sigma_{2}\delta_{y}\right).$$

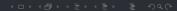
Under the vector space isomorphism  $\Sigma \colon \Omega_{\mathbb{C}}\left(T^{2}\right) \to M_{2}\left(C^{\infty}\left(T^{2}\right)\right)$  given by

$$\Sigma$$
:  $f_0 + f_1 dx + f_2 dy + f_3 dx \wedge dy \mapsto f_0 1 + i f_1 \sigma_1 + i f_2 \sigma_2 - i f_3 \sigma_3$ 

we have that

$$D = \Sigma \circ (d + d^*) \circ \Sigma^{-1}$$
 and  $\Sigma(df) = [D, f]$ .

This spectral triple can be equipped with a number of real structures, e.g.,  $P \circ C.C.$ ,  $\sigma_2 \circ C.C.$  but none satisfy the second-order condition.



## Hodge spectral triple on the torus (cont.)

However, the same spectral triple equipped with a mildly-twisted real structure can satisfy the second-order condition. The twisted real structures which do this are

$$\begin{split} J_1 \colon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto \begin{pmatrix} e^{i\theta}\bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & e^{-i\theta}\bar{\delta} \end{pmatrix}, \quad \nu_1 \colon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto \begin{pmatrix} e^{i\theta}\delta & \varepsilon'\beta \\ \varepsilon'\gamma & e^{-i\theta}\alpha \end{pmatrix}; \\ J_2 \colon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto \begin{pmatrix} \bar{\delta} & e^{i\phi}\bar{\beta} \\ e^{-i\phi}\bar{\gamma} & \bar{\alpha} \end{pmatrix}, \quad \nu_2 \colon \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &\mapsto \begin{pmatrix} \varepsilon'\alpha & e^{-i\phi}\gamma \\ e^{i\phi}\beta & \varepsilon'\delta \end{pmatrix}. \end{split}$$

Indeed, for  $J_{\Omega}$  the main anti-involution composed with complex conjugation on  $\Omega_{\mathbb{C}}(T^2)$ , we find that

$$\Sigma(J_{\Omega}\omega) = -J_1|_{\theta=\pi}\Sigma(\omega)$$

for any  $\omega \in \Omega_{\mathbb{C}}(T^2)$ .

#### Conclusions and outlook

The work conducted on this topic so far is still rather preliminary and there are many unanswered questions, big and small.

For example: Can the formalism be generalised to "semi-twisted" spectral triples, bridging the gap between twisted and untwisted spectral triples? Is there a twisted version of the second-order condition? What is its correct interpretation?

Next on the agenda: investigate whether the NCG Pati-Salam model admits a twisted real structure.

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