Progress in spectral triples with twisted real structure

Adam Magee

Scuola Internazionale Superiore di Studi Avanzati (SISSA)

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Outline

Background and definitions

Definition (Real spectral triple)

A real spectral triple $(\mathscr{A}, \mathscr{H}, D, J)$ is given by an action of the *-algebra $\mathscr{A}$ on the Hilbert space $\mathscr{H}$, with $D = D^*$ an unbounded operator with $(D - \lambda)^{-1} \in \mathscr{K}(\mathscr{H})$ and $[D, a] \in \mathscr{B}(\mathscr{H})$ for all $a \in \mathscr{A}$, equipped with an antiunitary operator J such that $J\mathscr{A}J^* \subset \mathscr{A}'$ and

$$J^2 = \varepsilon 1_{\mathscr{H}} \text{ and } DJ = \varepsilon' JD$$

for $\varepsilon, \varepsilon' \in \{-1, +1\}$.

We also typically require the satisfaction of the first-order condition

$$[D, a]JbJ^* = JbJ^*[D, a] \tag{1}$$

for all $a, b \in \mathscr{A}$.

Definition (Spectral triple with twisted real structure)

A spectral triple with twisted real structure is a real spectral triple $(\mathscr{A}, \mathscr{H}, D, J)$ equipped with a (bounded) operator ν such that $\bar{\nu}(\mathscr{A}) := \nu \mathscr{A} \nu^{-1} \simeq \mathscr{A}$, $\nu J \nu = J$ and now

$$DJ\nu = \varepsilon' \nu J D$$

for $\varepsilon' \in \{-1, +1\}$.

The first-order condition is here replaced by the twisted first-order condition

$$[D, a] J \bar{\nu}^2(b) J^* = J b J^* [D, a] \tag{2}$$

for all $a, b \in \mathscr{A}$.

Conformal transformation [Brzeziński, Ciccoli, Dąbrowski & Sitarz]

For a real spectral triple $(\mathscr{A}, \mathscr{H}, D, J)$, one can define a conformally transformed Dirac operator

$$D_k := JkJ^* DJkJ^*$$

for $k, k^{-1} \in \mathscr{A}_+$, such that $D_k = D_k^*$ and

$$[D, a] \in \mathscr{B}(\mathscr{H}) \implies [D_k, a] \in \mathscr{B}(\mathscr{H})$$

for all $a \in \mathscr{A}$, so that now $(\mathscr{A}, \mathscr{H}, D_k, J, \nu)$ is a spectral triple with twisted real structure for the twist operator $\nu = k^{-1}JkJ^*$.

Nibbling around the edges

Conformal transformtions

A twist operator ν which is involutory $\nu^2 = 1$ we refer to as being mild.

Conformal transformation (small generalisation)

For a spectral triple with mildly-twisted real structure $(\mathscr{A}, \mathscr{H}, D, J, \chi)$ with

$$\chi^2 = 1 \text{ and } \chi \in \mathscr{A}',$$

one can define a conformally transformed Dirac operator as before so that now $(\mathscr{A}, \mathscr{H}, D_k, J, \nu)$ is a spectral triple with twisted real structure for the twist operator $\nu = \chi k^{-1} J k J^*$.

Nibbling around the edges

The even case

If a real spectral triple $(\mathscr{A}, \mathscr{H}, D, J)$ admits a $\mathbb{Z}_2$ grading on $\mathscr{H}$ implemented by the grading operator γ, it is called even. One requires that $\gamma D = -D\gamma$ and

$$\gamma J = \varepsilon'' J\gamma \tag{3}$$

for $\varepsilon'' \in \{-1, +1\}$. It is not immediately obvious how γ should interact with ν in the case of a twisted real structure. It was identifed in [1] that we should require $\gamma\nu^2 = \nu^2\gamma$, but it was previously assumed that we should take (3) as-is (cf. [1, 2]). However, the fact that $\nu JDJ^*\nu = \pm D$ suggests we should instead require

$$\gamma\nu J = \varepsilon''\nu J\gamma. \tag{4}$$

Bigger bites

Gauge transformations

It is clear that if fluctuations of Dirac operators are generated by 1-forms, then $D = \varepsilon' \nu J D J^* \nu$ demands that fluctuations must be of the form

$$D_\omega = D + \omega + \varepsilon' \nu J \omega J^* \nu$$

for $\Omega^1_D(\mathscr{A}) \ni \omega = \sum_j a_j [D, b_j]$ a 1-form. Note that $\nu \in \mathscr{B}(\mathscr{H}) \implies \nu J \omega J^* \nu \in \mathscr{B}(\mathscr{H})$. Interestingly,

$$\varepsilon' \nu J a [D, b] J^* \nu = \bar{\nu}(J a J^*) [D, \bar{\nu}^{-1}(J b J^*)]_{\bar{\nu}^2}.$$

This suggests it should be possible to define Morita equivalence following the standard construction (right) and the construction for twisted spectral triples by [3] (left).

This is the case, with some minor modifications.

For reference...

For a unitary element $u \in \mathscr{U}(\mathscr{A})$, the gauge transformation of a gauge potential 1-form $\omega \longmapsto \omega^u$ and Dirac operator $D \longmapsto D^u$ are given by their standard definitions, *i.e.*,

$$\begin{aligned} \omega^u &= u\omega u^* + u[D, u^*], \\ (D_\omega)^u &= D_{\omega^u} \\ &= D + \omega^u + \varepsilon' \nu J \omega^u J^* \nu. \end{aligned}$$

In the standard case, gauge transformations are implemented by the operator $U := uJuJ^*$ such that $D^u = UDU^*$.

It is more natural to define the right $\mathscr{A}$-action on $\mathscr{H}$ by

$$\psi \cdot a = \bar{\nu}^{-1}(Ja^* J^*)\psi$$

for $a \in \mathscr{A}$, $\psi \in \mathscr{H}$. This means that we have

$$\mathrm{Ad}(u)(\psi) =: V\psi = u\nu^{-1}JuJ^* \nu\psi \text{ and}$$
$$\widetilde{\mathrm{Ad}}(u)(\psi) =: \tilde{V}\psi = u\nu JuJ^* \nu^{-1}\psi.$$

These operators implement the gauge transformations by

$$D^u = \tilde{V}DV^{-1}.$$

Rmk: Gauge transformations commute with conformal transformations.

Note that the operators V and $\tilde{V}$ are not unitary! However, the self-adjointness of Dirac operators

$$(D^u)^* = (\tilde{V} D V^{-1})^* \overset{!}{=} D^u = \tilde{V} D V^{-1}$$

requires that $\tilde{V}^* = V^{-1}$ and $V^* = \tilde{V}^{-1}$. This suggests that we demand that $\nu = \nu^*$ (up to sign), which in turn implies that gauge potentials are self-adjoint, $\omega = \omega^*$, as in the standard case.

Rmk: Self-adjointness of the twist operator is also consistent with the typical requirement of twisted spectral triples that the algebra automorphism ρ satisfies $\rho(a^*) = (\rho^{-1}(a))^*$ for all $a \in \mathscr{A}$.

Bigger bites

Spectral action

Ongoing work

In this context, the bilinear form $\mathfrak{A}_D(\psi, \phi) := \langle J\psi, D\phi \rangle$ is *not* gauge covariant or antisymmetric any more and must be modified, similar to [4]. It appears that the correct modification is given by

$$\mathfrak{A}_D^{\bar{\nu}}(\psi, \phi) := \langle J\psi, \nu^{-1} D\phi \rangle,$$

where $\nu = \nu^*$ is required but $\nu = \nu^{-1}$ is not. However, requiring that

$$\mathrm{Tr}\left(f\left(\frac{D^2}{\Lambda^2}\right)\right) = \mathrm{Tr}\left(f\left(\frac{(D^u)^2}{\Lambda^2}\right)\right)$$

does appear to require that $\nu = \nu^{-1}$.

Bigger bites

Second-order condition

Following [5] and subsequent papers, and [6], we define the second-order condition for ordinary spectral triples to be

$$[\,[D,a],\,J[D,b]J^*\,] = 0$$

for all $a, b \in \mathscr{A}$. Just as spectral triples with mildly-twisted real structures follow the first-order condition like untwisted spectral triples, it may be possible to construct examples which follow the second-order condition.

Hodge spectral triple on the torus [D'Andrea, M. & Dąbrowski]

Consider the spectral triple for the torus T^2 with coordinates (x, y) given by

$$\left(\mathscr{A} = C^\infty\left(T^2\right), \mathscr{H} = M_2\left(L^2\left(T^2\right)\right), D = i\sigma_1\partial_x + i\sigma_2\partial_y\right).$$

Under the vector space isomorphism $\Sigma\colon \Omega_{\mathbb{C}}\left(T^2\right) \to M_2\left(C^\infty\left(T^2\right)\right)$ given by

$$\Sigma\colon f_0 + f_1 dx + f_2 dy + f_3 dx \wedge dy \mapsto f_0 1 + if_1\sigma_1 + if_2\sigma_2 - if_3\sigma_3$$

we have that

$$D = \Sigma \circ (d + d^*) \circ \Sigma^{-1} \text{ and } \Sigma(df) = [D, f].$$

This spectral triple can be equipped with a number of real structures, *e.g.*, $P \circ \mathrm{c.\,c.}$, $\sigma_2 \circ \mathrm{c.\,c.}$ but none satisfy the second-order condition.

Hodge spectral triple on the torus (cont.)

However, the same spectral triple equipped with a mildly-twisted real structure can satisfy the second-order condition. The twisted real structures which do this are

$$J_1 : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta}\bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & e^{-i\theta}\bar{\delta} \end{pmatrix}, \qquad \nu_1 : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} e^{i\theta}\delta & \varepsilon'\beta \\ \varepsilon'\gamma & e^{-i\theta}\alpha \end{pmatrix};$$
$$J_2 : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \bar{\delta} & e^{i\phi}\bar{\beta} \\ e^{-i\phi}\bar{\gamma} & \bar{\alpha} \end{pmatrix}, \qquad \nu_2 : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \varepsilon'\alpha & e^{-i\phi}\gamma \\ e^{i\phi}\beta & \varepsilon'\delta \end{pmatrix}.$$

Indeed, for J_Ω the main anti-involution composed with complex conjugation on $\Omega_{\mathbb{C}}\left(T^2\right)$, we find that

$$\Sigma(J_\Omega\omega) = -J_1|_{\theta=\pi}\Sigma(\omega)$$

for any $\omega \in \Omega_{\mathbb{C}}\left(T^2\right)$.

Conclusions and outlook

The work conducted on this topic so far is still rather preliminary and there are many unanswered questions, big and small.

For example: Can the formalism be generalised to “semi-twisted” spectral triples, bridging the gap between twisted and untwisted spectral triples? Is there a twisted version of the second-order condition? What is its correct interpretation?

Next on the agenda: investigate whether the NCG Pati-Salam model admits a twisted real structure.

References

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