Geometric notions from spectral action

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Main question

Given $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ what are the informations given by the spectral action

$\operatorname{Tr} f(|\mathcal{D}|)$?

or by its asymptotics?

Remark:

Heat kernel is sufficient

If $f(x) = \int_0^\infty d\phi(s)\, e^{-sx}$

$$\operatorname{Tr} f(|\mathcal{D}|) = \int_0^\infty d\phi(s)\ \operatorname{Tr} e^{-s\,|\mathcal{D}|}$$

Almost sufficient: counterexample with noncommutative torus

On the track of new perspectives

How to define Riemannian curvature on spectral triples?

- (i) Old ideas: Use of derivations as generalization of the tangent bundle

Chamseddine–Felder–Frölich, Dubois-Violette–Madore–Masson–Mourad, Landi, ...

Rosenberg, Bhowmick–Goswami–Joardar–Mukhopadhyay:

Levi-Civita connections on the module of 1-forms

Find differential calculi giving existence/uniqueness of L-C connection

Rennie: (to appear) analogue of Riemann for a spectral triple

- (ii) Everything follows from spectral action:

Connes–Tretkof + ...(quoted later)

$$a_2(\Delta_0)(x) = \frac{1}{(4\pi)^{d/2}}\, \frac{1}{6}\,\mathfrak{R} \qquad \text{for } \Delta_0 \text{ acting on } C^\infty(M)$$

$$a_2(\Delta_1)(x) = \frac{1}{(4\pi)^{d/2}}\, \operatorname{tr}\big(\tfrac{1}{6}\,\mathfrak{R} - \mathrm{Ric}_x\big) \qquad \text{for } \Delta_1 \text{ acting on } \Omega^1(M)$$

Ricci operator on T^*M defined by $\mathrm{Ric}_x(\xi)(X) := \mathrm{Ric}_x(\xi^\sharp, X)$

Floricel–Ghorbanpour–Khalkhali–Dong: Ricci functional for nc-torus

Plan

Sitarz: Metric, torsion and minimal operators on noncommutative tori (Warsaw, 2014)

I: The commutative case

II: Influence of the metric

The noncommutative torus

III: The scalar curvature + …

IV: The torsion

I: The commutative case

(M, g) Compact boundaryless smooth oriented Riemannian manifold
d $\dim(M)$

V Smooth hermitean vector bundle over M of fiber $\mathbb{C}^N$

P Differential operator of degree p

P strongly elliptic: its principal symbol $\sigma(x,\xi) \in M_N(\mathbb{C})$

$$\Re(\langle \sigma(x,\xi)\, v,\ v\rangle) \geq c\, \|\xi\|^p\, \|v\|^2$$
$$\text{for } (x,\xi) \in M \times T_x^* M,\ \xi \neq 0,\ v \in \mathbb{C}^N$$

P non necessarily (formally) selfadjoint

Consequence:
Spectrum(P) is discrete without accumulation point, contained in a symmetric angular sector with angle $< \pi$ around $\mathbb{R}^+$
$e^{-t\,P}$ is trace-class for $t > 0$

The Heat trace

The heat trace

$$\mathrm{Tr}_{L^2(C^\infty(V))}\ b\, e^{-t\,P}$$

$b \in C^\infty(\mathrm{End}(V))$
$t > 0$

Example

$$P = -g^{\mu\nu}\nabla_\mu\nabla_\nu, \quad b = \mathbb{1}$$

Heat equation

$$(\partial_t + P)\, s(x,t) = 0, \quad \lim_{t\to 0} s(x,t) = s_0(x)$$

$(e^{-t\,P} s_0)(x)$ is (unique) solution

Impossible to compute the heat trace if Spectrum(P) is unknown

Asymptotics of Heat trace

Known result:

The following (unique) asymptotics exists

$$\operatorname{Tr}\, b\, e^{-tP} \sim_{t\downarrow 0} t^{-d/p} \sum_{r=0}^{\infty} a_r(b,P)\,(t^{1/p})^r$$

and $a_{2r+1}(b,P) = 0 \quad$ for all r

Symbolic notation: the series does not converge in general, means only

$$\left| \operatorname{Tr}\, b\, e^{-tP} - t^{-d/p} \sum_{r=0}^{n} a_r(b,P)\, t^{r/p} \right| = \mathcal{O}_{t\to 0}(t^{(n-d)/p}) \qquad \text{for all n}$$

Asymptotics of Heat trace

Given a closed connected Riemannian manifold
With $P = -\mathrm{div} \circ \mathrm{grad}$, on can recover from asymptotics:
the dimension, volume, scalar curvature, etc

But two problems

1) There exists isospectral manifolds not isometric
('64 Milnor, '85 Sunada,...)

2) What is lost in the difference between the heat trace and its asymptotics?

Examples of physical phenomena exponentially small (i.e. asymptotics = 0)

What can be recovered from spectral action?

How to compute coefficients?

P as before

$e^{-t\,P}$: smoothing pseudodifferential operator (not polyhomogeneous !)

smooth kernel: $K(t;x,y) \in M_N(\mathbb{C})$

Localize the problem of asymptotics

- Via:

$$\operatorname{Tr} b\, e^{-t\,P} = \int_M \mathrm{dvol}_g(x)\ \operatorname{tr}\,[b(x)\,K(t;x,x)]$$

$$\operatorname{tr} = \text{trace of } M_N(\mathbb{C})$$

- Via the parametrix approach:

$$K(t;x,x) \sim_{t\to 0}\ t^{-d/p} \sum_{r\in\mathbb{N}} a_r(x)\,(t^{\,1/p})^r$$

This gives asymptotics of $\operatorname{Tr} b\, e^{-tP}$

The existing results

1) If the principal symbol of P is scalar

$$a_r := \int_M \mathrm{dvol}_g(x)\, a_r(x) \quad \text{are known for } r = 0, \cdots, 10$$

In particular,

$a_2(-g^{\mu\nu}\nabla_\mu\nabla_\nu) = c \int_M \mathrm{dvol}_g(x)\, \mathcal{R}(x)\,, \quad \mathcal{R}$: scalar curvature

2) If the principal symbol of P is not scalar:

only a few works

essentially P acting on differential forms

Gilkey, Branson, Fulling, Vassilevich, …

II: Influence of the metric

The noncommutative d-torus

$\Theta \in M_d(\mathbb{R})$ skew-symmetric, $C(\mathbb{T}^d_\Theta)$ is generated by d unitaries U_k

$$U_k U_\ell = e^{2i\pi\Theta_{k,\ell}} U_\ell U_k$$

$$\mathcal{A} := \{a = \sum_{k\in\mathbb{Z}^d} a(k)\, U_1^{k_1} \cdots U_d^{k_d} \mid a \in \mathcal{S}(\mathbb{Z}^d)\}$$

$\mathcal{H}$ = GNS-Hilbert space for the trace τ

Following Ha–Ponge (2019)

Riemannian metric:

- $\mathcal{K}$:= free left $\mathcal{A}$-module generated by the derivations δ_μ
  $X \in \mathcal{K}$ means $X = \sum_\mu X^\mu \delta_\mu$, $X^\mu \in \mathcal{A}$, $\quad \mathcal{K} \simeq \mathcal{A}^d$
- Metric on $\mathcal{K}$: $g \in GL_d(\mathcal{A})$), g and g^{-1} are positive, selfadjoint entries

Given the vector fields $X,\ Y \in \mathcal{K}$ then

$$g(X, Y) := \sum_{\mu,\nu} X^\mu\, g_{\mu\nu}\, (Y^\nu)^*$$

II: Influence of the metric

The Riemannian density and volume are

$$\nu_g := \sqrt{\det(g)} := \exp[\tfrac{1}{2}\,\mathrm{tr}(\log(g))], \qquad \mathrm{Vol}_g\,(\mathcal{A}) := \tau[\nu_g]$$

Laplace–Beltrami operator:

$$\Delta_g := \nu_g^{-1} \sum_{\mu,\nu=1}^{d} \delta_\mu\, \nu_g^{1/2}\, (g^{-1})_{\mu\nu}\, \nu_g^{1/2}\, \delta_\nu$$

Examples:

- Flat metric $g_{\mu\nu} = \delta_{\mu\nu}$, $\quad \nu_g = \mathbb{1}$ and $\Delta_g = \Delta := -\sum_\mu \delta_\mu^2$
- Conformal deformation: $g_{\mu\nu} = k^{-2}\,\delta_{\mu\nu}$ $\qquad$ where $0 < k \in \mathcal{A}$

$$\nu_g = k^{-d}$$

$$\Delta_g = k^2\,\Delta - \sum_{\mu=1}^{d} k^d\,\delta_\mu(k^{2-d})\,\delta_\mu$$

$$\simeq k\,\Delta\,k \qquad \text{if } d = 2$$

Δ_g: second order differential operator with "non-scalar principal symbol"

Naive approach for scalar curvature

Classical formula for scalar curvature

$$\begin{aligned}\mathfrak{R} = &+ g^{\mu\nu} g_{\rho\sigma}(\partial_\mu \partial_\nu g^{\rho\sigma}) - (\partial_\mu \partial_\nu g^{\mu\nu}) + g_{\rho\sigma}(\partial_\mu g^{\mu\nu})(\partial_\nu g^{\rho\sigma}) \\ &+ \tfrac{1}{2} g_{\rho\sigma}(\partial_\mu g^{\nu\rho})(\partial_\nu g^{\mu\sigma}) - \tfrac{1}{4} g^{\mu\nu} g_{\rho\sigma} g_{\alpha\beta}(\partial_\mu g^{\rho\sigma})(\partial_\nu g^{\alpha\beta}) \\ &- \tfrac{5}{4} g^{\mu\nu} g_{\rho\sigma} g_{\alpha\beta}(\partial_\mu g^{\rho\alpha})(\partial_\nu g^{\sigma\beta})\end{aligned}$$

Even for the nc-torus, the swap

$$g^{\mu\nu} \to (g^{-1})_{\mu\nu}\, k^2$$

does not work: no link with a_2

Back to commutative geometry: non-scalar case

Following I. Avramidi

Slightly different method avoiding the pseudodiff. theory

Local trivialisation: U open set around $x \in M$

$$(e^{-t\,P_{|U}}\mathcal{s})(x) = \int_U \mathrm{d}y\,|g|^{1/2}(y)\,\mathcal{K}_{loc}(t,x,y)\,\mathcal{s}(y)$$

The use of the Fourier transform $\widehat{\mathcal{s}}\,(\xi) := (2\pi)^{-d/2}\int_U \mathrm{d}y\,e^{-iy\cdot\xi}\,\mathcal{s}\,(y)$

$$\mathcal{K}_{loc}(t,x,y)\,\mathcal{s}\,(y) = (2\pi)^{-d}|g|^{-1/2}(y)\Big[\int_{\mathbb{R}^d}\mathrm{d}\xi\,e^{-iy\cdot\xi}(e^{-t\,P_{|U}}e^{ix\cdot\xi})\Big]\,\mathcal{s}\,(y)$$

Commutative non-scalar case

For $\xi \in \mathbb{R}^d$, define

$$\mathcal{P}(x,\partial,\xi) := e^{-ix\cdot\xi} P_{|U}(x,\partial) e^{ix\cdot\xi}, \qquad \text{on } C^\infty(V_{|U})$$

Then for $P_{|U}(x,\partial) = \sum_{|\alpha|\leq p} c_\alpha(x)\, \partial_x^\alpha$

$$\mathcal{P}(x,\partial,\xi) = P_{|U}(x,\partial + i\xi)$$
$$\mathcal{P}(x,\partial,0) = P_{|U}(x,\partial), \qquad \sigma^{P_{|U}}(x,\xi) = \mathcal{P}(x,\partial,\xi)\, \mathbb{1}_N$$

Viewed on constant section (Leibniz rule)

$$\boxed{\mathcal{K}_{loc}(t;x,x) = (2\pi)^{-d} |g|^{-1/2}(x) \int_{\mathbb{R}^d} \mathrm{d}\xi\; e^{-t\,\mathcal{P}(x,\partial,\xi)}}$$

Duhamel–Volterra series

Formally, the Duhamel formula

$$e^{-t\,(A+B)} = e^{-tA} - \int_0^t ds_1\, e^{(s_1-t)\,A}\, B\, e^{-s_1\,(A+B)}$$

gives by iteration the Volterra series

$$e^{-t\,(A+B)} = \sum_{k=0}^{L} (-1)^k \int_{\Delta_k(t)} ds\; \sigma_{s_1}(B)\cdots\sigma_{s_k}(B) \;+\; (-1)^{L+1} R_{L+1}(A,B,t)$$

where

$$\Delta_k(t) := \{s = (s_1,\ldots,s_k) \in \mathbb{R}_+^k \mid 0 \leq s_k \leq s_{k-1} \leq \cdots \leq s_2 \leq s_1 \leq t\}$$

$$\sigma_s(B) := e^{s\,A}\, B\, e^{-s\,A}$$

$$R_{L+1}(A,B,t) := e^{-t\,A} \int_{\Delta_{L+1}(t)} ds\; \sigma_{s_1}(B)\cdots\sigma_{s_L}(B)\, e^{s_{L+1}\,A}\, B\, e^{-s_{L+1}\,(A+B)}$$

Duhamel–Volterra series

Decomposition of $\mathcal{P}(x, \partial, \xi) = e^{-ix\cdot\xi}\, P_{|U}(x, \partial)\, e^{ix\cdot\xi}$
in homogeneous polynomial in ξ: $\mathcal{P}(\xi) := \sum_\ell \mathcal{P}_\ell(\xi)$ (unbounded operator)
Choose
$A = \mathcal{P}_p$ = principal symbol of $P_{|U}$ (bounded operator)
$B_t = \sum_{\ell=0}^{p-1} t^{1-\ell/p}\, \mathcal{P}_\ell$
$v \in \mathbb{C}^N$
Checking all constraints: perturbation of bounded by unbounded!

$$e^{-(A+B_t)} = \sum_{k=0}^{L} (-1)^k f_k(\xi)[B_t^{\otimes^k}]\, v \;+\; (-1)^{L+1} R_{L+1}(\mathcal{P}_p, B_t, 1) \tag{1}$$

where
$$f_k(\xi)[C_1 \otimes \cdots \otimes C_k] := \int_{\Delta_k} \mathrm{d}s\; \sigma_{s_1}(C_1) \cdots \sigma_{s_k}(C_k)$$

for $C_i \in M_N[\xi, \partial]$ (polynomials in ξ and ∂ with matrix coefficients
$f_k(\xi)[C_1 \otimes \cdots \otimes C_k]$ = M_N-valued functions acting on constant sections in $\mathbb{C}^N$

Interest of (1): t appears as a polynomial

About Heat coefficients

More generally

Theorem

In the asymptotics of the diagonal heat kernel, the coefficients computed by the parametrix, the Duhamel or resolvent series are the same

Independent on the existence of the asymptotics!

Laplace type operator $P = -(u^{\mu\nu}\partial_\mu\partial_\nu + v^\nu\partial_\nu + w)$

Generalisation: ∇: covariant derivative on V

$$P := -(|g|^{-1/2}\nabla_\mu |g|^{1/2} u^{\mu\nu}\nabla_\nu + p^\mu \nabla_\mu + q)$$

where $u^{\mu\nu}, p^\mu, q$ are sections of $\mathrm{End}(V)$

Restriction: $u^{\mu\nu} = g^{\mu\nu} u$, $u \in M_N\mathbb{C})$ strictly positive matrix

$$\boxed{P = -(g^{\mu\nu} u \nabla_\mu\nabla_\nu + L^\mu \nabla_\mu + q)}$$

P is strongly elliptic with principal symbol $\sigma(x,\xi) = \|\xi\|^2 u$

Interest of restriction: Appearance of Gaussian integrals in ξ

$$f_k(\xi)[B_1 \otimes \cdots \otimes B_k] = \int_{\Delta_k} \mathrm{d}s\, e^{-(1-s_1)\,H}\, B_1\, e^{-(s_1-s_2)\,H} \cdots B_k\, e^{-s_k H}, \quad H = \|\xi\|^2 u$$

$$a_0(x) = \frac{\sqrt{|g_x|}}{(4\pi)^d}\,\mathrm{tr}[b(x)\, u(x)^{-d/2}]$$

$P = -(g^{\mu\nu} u \nabla_\mu \nabla_\nu + L^\mu \nabla_\nu + q)$

$\mathrm{Tr}\,[b\, e^{-tP}] \underset{t\downarrow 0}{\sim} t^{-d/2} \sum_{r=0}^{\infty} a_r\, t^{r/2}, \qquad a_r(x) := (2\pi)^{-d}\, \mathrm{tr}\,[b(x)\, \mathcal{R}_r(x)]$

Explicit formulae for all $\mathcal{R}_r(x)$

$$\mathcal{R}_0(x) = |g|^{-1/2} \int_{\mathbb{R}^d} \mathrm{d}\xi\, f_0(\xi)[1]$$
$$\mathcal{R}_2(x) = |g|^{-1/2} \int_{\mathbb{R}^d} \mathrm{d}\xi\, \big(f_2(\xi)[K \otimes K] - f_1(\xi)[P]\big)$$

$K = L$ modulo a contraction of Cristoffel symbol

The steps:

$u = \sum_\ell r_\ell E_\ell$, r_ℓ : eigenvalues, E_ℓ: spectral projections

1) Universal objects: For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ and $b_i \in M_N(\mathbb{C})$

$$X_{\alpha,k}[b_1 \otimes \cdots \otimes b_k] := I_{\alpha,k}(r_0, \ldots, r_k)\, E_0 b_1 E_1 \cdots E_{k-1} b_k E_k$$

$$I_{\alpha,k}(r_0, \ldots, r_k) := \int_{\Delta_k} \mathrm{d}s \, \Big[\sum_{\ell=0}^{k} (s_\ell - s_{\ell+1})\, r_\ell \Big]^{-\alpha}$$

Plenty of relations like the expansion/reduction process

$$\mathcal{E}_k[b_1 \otimes \cdots \otimes b_k] := u \otimes b_1 \otimes \cdots \otimes b_k + b_1 \otimes u \otimes \cdots \otimes b_k + \cdots + b_1 \otimes \cdots \otimes b_k \otimes u$$

$$X_{\alpha+1,k+1} \circ \mathcal{E}_k = X_{\alpha,k}$$

The steps

2) Total covariant derivative
$\widehat{\nabla}$: combines the (gauge) connection ∇ on V (and gauge potential A_μ) with Levi-Civita covariant derivative ${}^g\nabla$

3) Convert ingredients in normal coordinates

4) Compatibility with derivations

The results for Laplace type operator

Locally $a_r(b,P)(x) := (4\pi)^{-d/2}\,\mathrm{tr}\,[b(x)\,\mathcal{R}_r(x)]$ with $\mathcal{R}_r \in C^\infty(End(V))$

Theorem (I.-Masson)

$$\mathcal{R}_0 = X_{d/2,0}[\mathbb{1}] = u^{-d/2}$$

$$\mathcal{R}_2 =$$

$$+\tfrac{1}{6}\mathfrak{R}\, X_{d/2,1}[u] + X_{d/2,1}[q] - \tfrac{1}{2}(d+2)\, g^{\mu\nu}\, X_{d/2+2,3}[u \otimes \widehat{\nabla}^2_{\mu\nu} u \otimes u]$$

$$-\tfrac{1}{2}(d+1)\, g^{\mu\nu}\, X_{d/2+1,2}[\widehat{\nabla}_\mu u \otimes \widehat{\nabla}_\nu u] + \tfrac{1}{2}(d+2)\, g^{\mu\nu}\, X_{d/2+2,3}[\widehat{\nabla}_\mu u \otimes u \otimes \widehat{\nabla}_\nu u]$$

$$+\tfrac{1}{2}(d+2)(d+4)\, g^{\mu\nu}\, X_{d/2+3,4}[u \otimes \widehat{\nabla}_\mu u \otimes \widehat{\nabla}_\nu u \otimes u]$$

$$+\tfrac{1}{2}\, X_{d/2+1,2}[p^\mu \otimes \widehat{\nabla}_\mu u] - X_{d/2+1,2}[u \otimes \widehat{\nabla}_\mu p^\mu]$$

$$-\tfrac{1}{2}(d+2)\, X_{d/2+2,3}[p^\mu \otimes \widehat{\nabla}_\mu u \otimes u] - \tfrac{1}{2}\, X_{d/2+1,2}[\widehat{\nabla}_\mu u \otimes p^\mu]$$

$$+\tfrac{1}{2}(d+2)\, X_{d/2+2,3}[u \otimes \widehat{\nabla}_\mu u \otimes p^\mu] - \tfrac{1}{2}\, g_{\mu\nu}\, X_{d/2+1,2}[p^\mu \otimes p^\nu]$$

If $u = \mathbb{1}$, $\mathcal{R}_2 = +\tfrac{1}{6}\mathfrak{R} + q$ (for $p^\mu = 0$)

When $u \neq \mathbb{1}$, some coefficients depends on d

The results for Laplace type operator

For $\mathcal{R}_4$: Thousands terms generated!
Code written from scratch by Thierry Masson
$\mathcal{R}_4$ decomposes into 5 natural summands:

$$\mathcal{R}_{4,4} = +\tfrac{1}{12}\, X_1[u]\, F^{\nu_1\nu_2} F_{\nu_1\nu_2}$$

$$\begin{aligned}\mathcal{R}_{4,3} = +\tfrac{1}{6}(d-2)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_2[u \otimes (\widehat{\nabla}_{\nu_4} u)]\, (\widehat{\nabla}_{\nu_1} F_{\nu_2\nu_3}) \\ - d\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_3[u \otimes u \otimes (\widehat{\nabla}_{\nu_4} u)]\, (\widehat{\nabla}_{\nu_1} F_{\nu_2\nu_3}) \\ + 2(d+2)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_4[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_4} u)]\, (\widehat{\nabla}_{\nu_1} F_{\nu_2\nu_3})\end{aligned}$$

Field strength $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

The results for Laplace type operator

$$
\begin{aligned}
\mathcal{R}_{4,2} =\ & \\
& - d\, X_3[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)]\, F^{\nu_1\nu_2} - 4\, X_4[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)]\, F^{\nu_1\nu_2} \\
& + \tfrac{1}{2}(d+2)(d+4)\, X_5[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u]\, F^{\nu_1\nu_2} \\
& + 2(d+4)\, X_5[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u]\, F^{\nu_1\nu_2} \\
& + 4(d+4)\, X_5[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u)]\, F^{\nu_1\nu_2} \\
& + (d+4)(d+6)\, X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)]\, F^{\nu_1\nu_2} \\
& + (d+4)(d+6)\, X_6[u \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u)]\, F^{\nu_1\nu_2}
\end{aligned}
$$

$$\mathcal{R}_{4,1} = 0$$

$$G^{\nu_1\nu_2\nu_3\nu_4} := \tfrac{1}{4}\left(g^{\nu_1\nu_2}g^{\nu_3\nu_4} + g^{\nu_1\nu_3}g^{\nu_2\nu_4} + g^{\nu_1\nu_4}g^{\nu_2\nu_3}\right)$$
$$\widehat{\Delta} = g^{\mu\nu}\,\tfrac{1}{2}(\widehat{\nabla}_\mu\widehat{\nabla}_\nu + \widehat{\nabla}_\nu\widehat{\nabla}_\mu)$$

$$\mathcal{R}_{4,0} =$$
$$+\tfrac{1}{180}\,|\mathrm{Riem}|^2\,X_1[u] - \tfrac{1}{180}\,|\mathrm{Ric}|^2\,X_1[u] + \tfrac{1}{30}\,(\widehat{\Delta}\mathfrak{R})\,X_1[u] + \tfrac{1}{72}\,\mathfrak{R}^2\,X_1[u]$$
$$-\tfrac{1}{12}(d-2)\,g^{\nu_1\nu_2}(\widehat{\nabla}_{\nu_2}\mathfrak{R})\,X_2[u\otimes(\widehat{\nabla}_{\nu_1}u)] + \tfrac{1}{2}(d+2)\,g^{\nu_1\nu_2}(\widehat{\nabla}_{\nu_2}\mathfrak{R})\,X_4[u\otimes u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)]$$
$$-\tfrac{1}{3}d\,\mathrm{Ric}^{\nu_1\nu_2}\,X_3[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)] + \tfrac{1}{3}(d+2)(d+3)\,\mathrm{Ric}^{\nu_1\nu_2}\,X_4[u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)]$$
$$-\tfrac{1}{12}d(d+2)\,\mathrm{Ric}^{\nu_1\nu_2}\,X_4[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)]$$
$$-(d+2)(d+4)\,\mathrm{Ric}^{\nu_1\nu_2}\,X_5[u\otimes u\otimes u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)]$$
$$+\tfrac{1}{4}(d+2)(d+4)\,\mathrm{Ric}^{\nu_1\nu_2}\,X_5[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes u\otimes(\widehat{\nabla}_{\nu_2}u)\otimes u]$$
$$-\tfrac{1}{6}d\,g^{\nu_1\nu_2}\mathfrak{R}\,X_3[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)] + \tfrac{1}{12}d(d+2)\,g^{\nu_1\nu_2}\mathfrak{R}\,X_4[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)\otimes u]$$
$$-2(d+4)(d^2+10d+28)\,G^{\nu_1\nu_2\nu_3\nu_4}\,X_5[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)\otimes(\widehat{\nabla}_{\nu_3}u)\otimes(\widehat{\nabla}_{\nu_4}u)]$$
$$+4(d+4)(d+6)\,G^{\nu_1\nu_2\nu_3\nu_4}\,X_6[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)\otimes(\widehat{\nabla}_{\nu_3}u)\otimes(\widehat{\nabla}_{\nu_4}u)\otimes u]$$
$$+2(d+4)(d+6)^2\,G^{\nu_1\nu_2\nu_3\nu_4}\,X_6[u\otimes(\widehat{\nabla}_{\nu_1}u)\otimes(\widehat{\nabla}_{\nu_2}u)\otimes(\widehat{\nabla}_{\nu_3}u)\otimes u\otimes(\widehat{\nabla}_{\nu_4}u)]$$

The results for Laplace type operator

$$+ 2(d+4)^2(d+6)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$- 4(d+4)(d^2+8d+28)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_6[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$- 2(d+2)(d+4)(d+6)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_6[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$+ 8(d+4)(d^2+10d+32)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$+ 4(d+4)(d^2+6d+16)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes (\widehat{\nabla}_{\nu_3} u) \otimes u \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$- 32(d+4)^2\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u) \otimes u]$$
$$+ 8(d+4)^2(d+6)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$+ 4(d+4)^2(d+6)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$+ 2d(d+4)(d+6)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes (\widehat{\nabla}_{\nu_3} u) \otimes u \otimes (\widehat{\nabla}_{\nu_4} u)]$$
$$- 16(d+4)(d+6)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes u \otimes (\widehat{\nabla}_{\nu_2} u) \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u) \otimes u]$$
$$+ 2(d+4)(d+6)(d+8)\, G^{\nu_1\nu_2\nu_3\nu_4}\, X_7[u \otimes (\widehat{\nabla}_{\nu_1} u) \otimes (\widehat{\nabla}_{\nu_2} u) \otimes u \otimes (\widehat{\nabla}_{\nu_3} u) \otimes (\widehat{\nabla}_{\nu_4} u) \otimes u]$$

$$+ \;\ldots\ldots$$

The results for $P := -(|g|^{-1/2}\nabla_\mu |g|^{1/2} u^{\mu\nu}\nabla_\nu + p^\mu\nabla_\mu + q)$

$\mathcal{R}_{4,0}$ has 180 terms

Other presentations are possible due to reduction process

When $u = \mathbb{1}$, all shrinks to 8 terms:

$$\mathcal{R}_4 = +\tfrac{1}{180}\,|\mathrm{Riem}|^2\,X_1[u] - \tfrac{1}{180}\,|\mathrm{Ric}|^2\,X_1[u] + \tfrac{1}{30}\,(\widehat{\Delta}\mathfrak{R})\,X_1[u] + \tfrac{1}{72}\,\mathfrak{R}^2\,X_1[u] \\ + \tfrac{1}{12}\,X_1[u]\,F^{\nu_1\nu_2}F_{\nu_1\nu_2} + \tfrac{1}{6}\,\mathfrak{R}\,X_1[q] + X_2[q\otimes q] + X_3[u\otimes(\widehat{\Delta}q)\otimes u]$$

III: Application to noncommutative torus

If $\theta = p/q$ is rational, $\mathcal{A} \simeq \Gamma(A)$
A fiber bundle in $M_q(\mathbb{C})$ over a 2-torus $\mathbb{T}^2$
Choose $0 < k \in \mathcal{A}$

Define P on $\mathcal{H}$:

$$\begin{aligned} P &:= k\,\Delta\,k \\ &= -g^{\mu\nu}\big[k^2\delta_\mu\delta_\nu - 2k(\delta_\nu k)\,\delta_\mu - k(\delta_\mu\delta_\nu k)\big] \end{aligned}$$

$$k^2 \longleftrightarrow u$$

$$\delta_\mu \longleftrightarrow \nabla_\mu$$

III: The noncommutative torus

In dimension d, with $u = k^2$

Theorem (I.–Masson)

$$
\begin{aligned}
\mathcal{R}_2 = g^{\nu_1\nu_2}\Big(& -(d+2)\,X_3[u\otimes(\delta_{\nu_1}k)(\delta_{\nu_2}k)\otimes u] + \tfrac{1}{2}(d^2+2d+8)\,X_4[u\otimes k(\delta_{\nu_1}k)\otimes(\delta_{\nu_2}k)k\otimes u] \\
& -(d-2)\,X_4[k(\delta_{\nu_1}k)\otimes u\otimes(\delta_{\nu_2}k)k\otimes u] - 2d\,X_4[k(\delta_{\nu_1}k)\otimes(\delta_{\nu_2}k)k\otimes u\otimes u] \\
& -(d-2)\,X_4[u\otimes k(\delta_{\nu_1}k)\otimes u\otimes(\delta_{\nu_2}k)k] + 4\,X_4[k(\delta_{\nu_1}k)\otimes u\otimes u\otimes(\delta_{\nu_2}k)k] \\
& -2d\,X_4[u\otimes u\otimes k(\delta_{\nu_1}k)\otimes(\delta_{\nu_2}k)k] + \tfrac{1}{2}(d+2)^2\,X_4[u\otimes k(\delta_{\nu_1}k)\otimes k(\delta_{\nu_2}k)\otimes u] \\
& -(d+2)\,X_4[k(\delta_{\nu_1}k)\otimes u\otimes k(\delta_{\nu_2}k)\otimes u] - 2(d+2)\,X_4[k(\delta_{\nu_1}k)\otimes k(\delta_{\nu_2}k)\otimes u\otimes u] \\
& -2(d+2)\,X_4[u\otimes u\otimes(\delta_{\nu_1}k)k\otimes(\delta_{\nu_2}k)k] - (d+2)\,X_4[u\otimes(\delta_{\nu_1}k)k\otimes u\otimes(\delta_{\nu_2}k)k] \\
& + \tfrac{1}{2}(d+2)(d+4)\,X_4[u\otimes(\delta_{\nu_1}k)k\otimes k(\delta_{\nu_2}k)\otimes u]\Big) \\
& \\
& + \tfrac{1}{2}d\,X_3[u\otimes k(\Delta k)\otimes u] - 2\,X_3[k(\Delta k)\otimes u\otimes u] \\
& - 2\,X_3[u\otimes u\otimes(\Delta k)k] + \tfrac{1}{2}d\,X_3[u\otimes(\Delta k)k\otimes u]
\end{aligned}
$$

III: The noncommutative torus

$$
\begin{aligned}
\mathcal{R}_4 = & -12(d+2)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_6[u \otimes u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
& + 2(d^2+4d+8)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
& + 4(d^2+6d+16)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
& + 2(d^2+4d+8)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
& -12(d+2)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u \otimes u] \\
& + 2(d^2+6d+12)\, g^{\nu_1\nu_2} g^{\nu_3\nu_4}\, X_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
& -12(d+2)\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, X_6[u \otimes u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
& -4d\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, X_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
& + 4(d^2+6d+16)\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, X_6[u \otimes u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
& -4d\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, X_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u] \\
& -12(d+2)\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, X_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u \otimes u \otimes u] \\
& + 8\, g^{\nu_1\nu_4} g^{\nu_2\nu_3}\, X_6[u \otimes (\delta_{\nu_1} k)(\delta_{\nu_2} k) \otimes u \otimes u \otimes (\delta_{\nu_3} k)(\delta_{\nu_4} k) \otimes u] \\
& + \cdots\cdots
\end{aligned}
$$

$\mathcal{R}_4$ has a presentation with 3527 terms before simplification

Comparison with previous works

Intensive activities since 2011:

Connes–Tretkoff, Gayral–Iochum–Vassilevich
Connes–Moscovici, Lesch–Moscovici, Fathizadeh–Khalkhali
Dabrowski–Sitarz, Liu
Connes–Fatizadeh ($\mathcal{R}_4$ for $d = 2$: 134 pages, tiny fontsize for formulae)
Floricel–Ghorbanpour–Khalkhali

Present method:

purely classical geometry + functional analysis
avoids Connes' pseudodifferential calculus on nc-tori
valid in arbitrary dimension, arbitrary metric
keep tracks of the origin of terms, especially Riemann curvature tensor

Goal:

Application to some spectral triples

Problem: How to extract geometric notions from this?

IV: And the torsion?

Following Pfäffle–Stephan-Hanish, 2010- ....

(M, g), closed and spin, dimension $d \geq 3$
A connection on TM is orthogonal when

$$\partial_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Lemma (É. Cartan)

For any orthogonal connection ∇ there exist
a unique vector field V
a unique 3-form T
a unique $(3, 0)$-tensor field S with $S_x \in T_3(T_x M)$ for any $x \in M$

$$\nabla_X Y = \nabla^g_X Y + \langle X, Y\rangle V - \langle V, Y\rangle X + T(X, Y, \cdot)^\sharp + S(X, Y, \cdot)^\sharp$$

$T(X, Y, \cdot)^\sharp$ or $S(X, Y, \cdot)^\sharp$ are defined by
$\left\langle T(X, Y, \cdot)^\sharp, Z\right\rangle := T(X, Y, Z)$

IV: And the torsion?

$S \in$ Cartan-type torsion tensors

$$T_3(T_xM) := \Big\{ A \in T_x^*M \otimes T_x^*M \otimes T_x^*M$$

$$A(X,Y,Z) + A(Y,Z,X) + A(Z,X,Y) = 0$$

$$A(X,Y,Z) = -A(X,Z,Y)$$

$$\sum_{i=1}^{d} A(e_i, e_i, \cdot) = 0 \text{ for any orthonormal basis } e_1, \dots, e_n \text{ of } T_xM$$

$$\Big\}$$

Dirac operator with torsion

Choose a spin structure and extend ∇ to spinor bundle ΣM
Dirac operator associated to ∇ is

$$D\psi = D^g\psi + \tfrac{3}{2} T\cdot\psi - \tfrac{d-1}{2} V\cdot\psi$$

"$\cdot$": Clifford multiplication
D is symmetric if and only if $V = 0$
Cartan-type torsion S does not contribute to D

Theorem (Lichnerowicz formula)

$$\begin{aligned} D^*D\psi \;=\;& \widetilde{\Delta}\psi + \tfrac{1}{4}\mathfrak{R}^g\psi + \tfrac{3}{2} dT\cdot\psi - \tfrac{3}{4}\|T\|^2\psi \\ &+\tfrac{d-1}{2}\,\mathrm{div}^g(V)\,\psi + \left(\tfrac{d-1}{2}\right)^2(2-d)\,|V|^2\,\psi \\ &+3(d-1)\Big(T\cdot V\cdot\psi + (V\lrcorner T)\cdot\psi\Big) \end{aligned}$$

$\widetilde{\Delta}$: *Laplacian associated to the connection*
$\widetilde{\nabla}_X Y := \nabla^g_X Y + (d-1)\,(\langle X, Y\rangle V - \langle V, Y\rangle X) + 3T(X, Y, \cdot)^\sharp$

A parenthesis: Torsion in dimension 3

If $d = 3$, totally antisymmetric torsion has only one component:

$$\mathcal{D} \to \mathcal{D}_\phi := \mathcal{D} + \phi, \quad \phi \text{ is real function}$$

$$a_2(\mathcal{D}_\phi^2) = (4\pi)^{-3/2} \int_M \mathrm{dvol}_g \,(-\tfrac{1}{6}\mathfrak{R} + 4\phi^2)$$
$$a_4(\mathcal{D}_\phi^2) = (4\pi)^{-3/2} \int_M \tfrac{8}{3}\, \mathrm{dvol}_g \,(\nabla_i\phi)\,(\nabla^i\phi)$$

(Weyl tensor and Gaus–Bonnet integrand vanish)

Generalisation to spectral triple of dimension 3

Following Sitarz–Zajac

In $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, regular with simple dimension spectrum, dimension 3

$$\mathcal{D} \to \mathcal{D}_\Phi := \mathcal{D} + \Phi + J\Phi J^{-1}, \qquad \Phi = \Phi^* \in \mathcal{A}$$

$$\mathrm{Tr}[f(|\mathcal{D}|/\Lambda] \sim_{\Lambda\to\infty} \sum_{j=1}^{d} f_{d-j}\, \Lambda^{d-j} \fint |\mathcal{D}|^{-d+j} + f(0)\, \zeta_{\mathcal{D}}(0) + \cdots$$

Theorem ($\Phi \geq 0$, First terms of spectral action for $\mathcal{D}_\Phi$)

$$\fint |\mathcal{D}_\Phi|^{-3} = \fint |\mathcal{D}|^{-3}$$

$$\fint |\mathcal{D}_\Phi|^{-2} = \fint |\mathcal{D}|^{-2} - 4 \fint \Phi\, F\, |\mathcal{D}|^{-3}, \qquad F := \mathcal{D}|\mathcal{D}|^{-1}$$

$$\fint |\mathcal{D}_\Phi|^{-1} = \fint |\mathcal{D}|^{-1} - 2 \fint \Phi\, F\, |\mathcal{D}|^{-2} + 2 \fint (\Phi^2 + \Phi J \Phi J^{-1})\, |\mathcal{D}|^{-3}$$

$$\zeta_{\mathcal{D}_\Phi}(0) - \zeta_{\mathcal{D}}(0) = \cdots$$

$\to$ Application to $SU_q(2)$

Bosonic spectral action

Assume $d = 4$

Theorem (Pfäffle–Stephan)

$$a_2(D^* D) = -\frac{1}{48\,\pi^2} \int_M dvolg\, \widetilde{R}$$

$$\widetilde{\mathfrak{R}} = \mathfrak{R}^g + 18\ \mathrm{div}^g(V) - 54\,|V|^2 - 9\,\|T\|^2$$

$$a_4(D^* D) = \frac{11}{720}\,\chi(M) - \frac{1}{320\,\pi^2} \int_M dvol_g\,\|W^g\|^2 - \frac{3}{32\,\pi^2} \int_M dvol_g\,\left(\|\delta T\|^2 + \|d(V^\flat)\|^2\right)$$

$\widetilde{\mathfrak{R}}$: scalar curvature of modified connection $\widetilde{\nabla}$
$\chi(M)$: Euler characteristics of M
$W^g = weyl(riem^g)$: Weyl curvature of Levi-Civita connection ∇^g

Manifolds with boundaries

I–Levy–Vassilevich (2010):

Generalisation to compact manifolds M with boundaries

Chiral bag boundary conditions:

$$\not{D}_T = \not{D} + T \qquad \text{with } T \text{ totally antisymmetric}$$

$$\chi_\theta := -i\, e^{\theta\gamma_5}\gamma_5\gamma_4, \qquad \theta \in \mathbb{R}$$

$$\tfrac{1}{2}(\mathbb{1} - \chi_\theta)\, \psi_{|\text{boundary}} = 0$$

$\not{D}_T$ is selfadjoint

Possible to adjust an algebra to get a spectral triple

Computation of spectral action for $\not{D}_T$ when $\theta = 0$:

$a_1 = 0$

Formulae for a_2, a_3 and a_4

Particle Physics

Matter: encoded in Hermitian vector bundle $\mathcal{H} \to M$ with $\nabla^{\mathcal{H}}$
Dirac operator $D_{\mathcal{H}}$ on $\mathcal{E} := \Sigma M \otimes \mathcal{H}$ is

$$D_{\mathcal{H}}(\psi \otimes \chi) := \sum_{i=1}^{4} \Big((e_i \cdot \nabla_{e_i}\psi) \otimes \chi + (e_i \cdot \psi) \otimes (\nabla^{\mathcal{H}}_{e_i}\chi) \Big)$$

Choose $\Phi \in \mathrm{End}(\mathcal{H})$: *Higgs endomorphism*
Chamseddine–Connes Dirac operator on $\mathcal{E}$

$$\mathcal{D}_\Phi\,(\psi \otimes \chi) := D_{\mathcal{H}}(\psi \otimes \chi) + (\omega^g \cdot \psi) \otimes (\Phi\,\chi)$$

ω^g: volume form on ΣM

Assume $\mathcal{H} = \mathcal{H}^r \oplus \mathcal{H}^\ell$ with $\dim \mathcal{H}^r \neq \dim \mathcal{H}^\ell$

γ: corresponding chirality operator
Also ω^g induces a grading of $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ with $\omega^g|_{\Sigma^\pm M} = \pm \mathbb{1}_{\Sigma^\pm M}$

Consider $\mathcal{E}^+ := (\Sigma^+ M \otimes \mathcal{H}^r) \oplus (\Sigma^- M \otimes \mathcal{H}^\ell)$
Orthogonal projection $P : \mathcal{E} \to \mathcal{E}^+$

$$P = \tfrac{1}{2}\left(\mathbb{1}_{\Sigma M} \otimes \mathbb{1}_{\mathcal{H}} + \omega^g \otimes \gamma\right)$$

Particle Physics: Chamseddine–Connes spectral action

Consider the operator

$$P\,\mathcal{D}_\Phi{}^*\mathcal{D}_\Phi$$

Theorem (Pfäffle–Stephan)

$$a_0 = \tfrac{1}{8\,\pi^2}\ \mathrm{rk}\,(\mathcal{H})\,vol\,(M)$$

$$a_2 = -\tfrac{\mathrm{rk}(\mathcal{H})}{96\,\pi^2}\int_M \Big(\,\widetilde{R}\,\omega^g + \frac{\mathrm{tr}_{\mathcal{H}}(\gamma)}{\mathrm{rk}(\mathcal{H})}\,\widetilde{C}_H\Big) - \tfrac{1}{8\,\pi^2}\int_M \mathrm{tr}_{\mathcal{H}}(\Phi^2)\,dx$$

Holst term (4-form)

$$\widetilde{C}_H := 18\,dT - 18\,\langle T, *V^\flat\rangle\,\omega^g$$

Barbero–Immirzi parameter $= -\frac{\mathrm{tr}_{\mathcal{H}}(\gamma)}{\mathrm{rk}(\mathcal{H})}$

Chamseddine–Connes spectral action

Theorem

$$a_4 = + \frac{11\ \mathrm{rk}(\mathcal{H})}{1440}\,\chi(M) - \frac{\mathrm{tr}_{\mathcal{H}}(\gamma)}{96}\,p_1(M) - \frac{\mathrm{rk}(\mathcal{H})}{640\,\pi^2}\int_M \|W^g\|^2\,dx$$
$$- \frac{3\ \mathrm{rk}(\mathcal{H})}{64\,\pi^2}\int_M \left(\|\delta T\|^2 + \|d(V^\flat)\|^2\right) \quad + \frac{\mathrm{tr}_{\mathcal{H}}(\gamma)}{1152\,\pi^2}\int_M \widetilde{\mathfrak{R}}\,\widetilde{C}_H$$
$$+ \frac{1}{16\,\pi^2}\int_M \left(\mathrm{tr}_{\mathcal{H}}([\nabla^{\mathcal{H}},\Phi]^2) + \mathrm{tr}_{\mathcal{H}}(\Phi^4) + \tfrac{1}{6}\left(\mathfrak{R}^g - 9\|T\|^2\right)\mathrm{tr}_{\mathcal{H}}(\Phi^2)\right)\,dx$$
$$+ \frac{1}{96\,\pi^2}\int_M \mathrm{tr}_{\mathcal{H}}(\Phi^2\,\gamma)\,\widetilde{C}_H + \frac{5}{192\,\pi^2}\int_M \mathrm{tr}_{\mathcal{H}}(\Omega^{\mathcal{H}}\Omega^{\mathcal{H}})\,dx$$
$$+ \frac{1}{64\,\pi^2}\int_M \mathrm{tr}_{\mathcal{H}}(*\Omega^{\mathcal{H}}\Omega^{\mathcal{H}}\gamma)\,dx$$

with notations $[\nabla^{\mathcal{H}},\Phi]^2 = \sum_i [\nabla^{\mathcal{H}}_{e_i},\Phi][\nabla^{\mathcal{H}}_{e_i},\Phi]$
$\Omega^{\mathcal{H}}\Omega^{\mathcal{H}} = \sum_{i,j}\Omega^{\mathcal{H}}_{ij}\Omega^{\mathcal{H}}_{ij}$, $\quad *\Omega^{\mathcal{H}}\,\Omega^{\mathcal{H}} = \sum_{i,j}(*\Omega^{\mathcal{H}}_{ij})\,\Omega^{\mathcal{H}}_{ij}$

$p_1(M) := -(1/16\pi^2)\int_M \mathrm{dvol}_g\,\langle riem,\ \mathbb{1}_{\Lambda^2}\otimes *riem\rangle$: first Pontryaging class

Chamseddine–Connes spectral action

En résumé

If torsion exists, then

Torsion couples with Higgs and fermions

Derivative of $T \to$ torsion becomes dynamical

$\int_M \widetilde{\mathcal{R}}\, \widetilde{C}_H$: not a topological term
Barbero–Immirzi parameter depends on particle content

Critical point of spectral action with non-zero torsion

Analogous notion in noncommutative geometry?

The end

If you understood everything I said,
it's probably that I misspoke!

Thank you