Noncommutative inner geometry of the Standard Model

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A non-commutative C*-algebra is commonly regarded as the algebra of continuous functions on a 'quantum space'. Its smooth and metric structures can be described in terms of a spectral triple which involves an analogue of the Dirac operator. The Standard Model of fundamental particles in physics can be described as the almost commutative geometry, the inner part of which can be interpreted as a quantum analogue of the de-Rham-Hodge spectral triple

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Goal

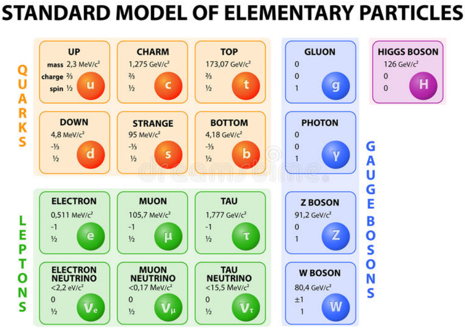
Unveil the geometric nature of the flavour multiplet of fermions in the Standard Model (acronym: SM) of fundamental particles

Plan

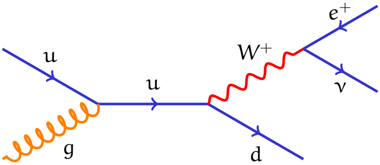
1. *Dirac spinors, de Rham forms and Clifford fields.*
2. *quantum analogue of ①.*
3. *application to noncommutative SM (acronym: νSM).*

Proviso: *quantum = noncommutative (acronym: NC)*

(Unreasonably) successful Standard Model



& interactions



$$2\times(3+1)\times2\times2\times3=96$$

'flavors'

governed by:

Lagrangian

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)$$

$$- Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu$$

$$+ \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H$$

$$- 2M^2\alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \beta_h\left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h - g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \frac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2\right.$$

$$\left.+4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) - gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H)\right.$$

$$\left.+W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + M\,(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+)$$

$$- ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2 W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right)$$

$$- \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) + \frac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^-$$

$$- g^2 s_w^2 A_\mu A_\mu\phi^+\phi^- + \frac{1}{2}ig_s\,\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial + m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_w A_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right)$$

$$+ \frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{d}^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2 - 1 - \gamma^5)d^\lambda_j) + (\bar{u}^\lambda_j\gamma^\mu(1 - \frac{8}{3}s_w^2 + \gamma^5)u^\lambda_j)\} + \frac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)U^{lep}{}_{\lambda\kappa}e^\kappa) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right)$$

$$+ \frac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\kappa U^{lep\dagger}{}_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_e(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1+\gamma^5)e^\kappa) + \frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_e(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1+\gamma^5)\nu^\kappa)\right.\right.$$

$$\left.-m^\kappa_\nu(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right) - \frac{g}{2}\frac{m^\lambda_\nu}{M}H(\bar{\nu}^\lambda\nu^\lambda) - \frac{g}{2}\frac{m^\lambda_e}{M}H(\bar{e}^\lambda e^\lambda) + \frac{ig}{2}\frac{m^\lambda_\nu}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda) - \frac{ig}{2}\frac{m^\lambda_e}{M}\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda) - \frac{1}{4}\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa - \frac{1}{4}\overline{\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa}$$

$$+ \frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right) + \frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right) - \frac{g}{2}\frac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \frac{g}{2}\frac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j)$$

$$+ \frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j)$$

Conceptually/Geometrically:

$U(1) \times SU(2) \times SU(3)$

Φ *gauge fields (bosons)*
minimally coupled to
matter fields (fermions)
& Higgs field (boson)

$\mathcal{M}$ *connection*
($\sim$ multiplet of vectors)
on (a multiplet of) spinors;
& a doublet of scalars

2nd quantization with gauge fixing,
spontaneous symmetry breaking,
regularization & perturbative renormalization

However unexplained:

- contents of particles (especially 3 families)
- several parameters,
- not included the 4th known interaction: **gravitation**
  $+$ its fundamental symmetry: **general relativity (diffeomorphisms)**
- & more

Spectral Triple

NC geometry by A. Connes et.al. is primarily based on **algebras** rather than *groups*, and it enriches the Gelfand-Naimark ♠ SF

$$\boxed{topological\ spaces \longleftrightarrow commutative\ C^* - algebras}$$

& Serre-Swan

$$\boxed{vector\ bundles \longleftrightarrow modules}$$

equivalences, by encoding *smoothness*, *calculus* and *metric* structure in terms of **spectral triples** (acronym: ST), ↫

$$(A, H, D)$$

which consists of a $*$-algebra A of operators on Hilbert space H and $D = D^\dagger$ on H, such that

$$[D, A] \subset \mathcal{B}(H), \quad (D - i)^{-1} \in \mathcal{K}(H).$$

ST 2

A ST is *even* if $\exists$ a $\mathbb{Z}_2$-grading χ of H, $\chi^2 = 1$, $\chi^\dagger = \chi$,

$$[\chi, A] = 0, \quad \{\chi, D\} = 0.$$

A ST is *real* if $\exists$ a real structure, i.e. antiunitary J on H, such that denoting B' the commutant of $B \subset \mathcal{B}(H)$,

$$JAJ^{-1} \subset A', \quad \text{(order 0 condition)}. \tag{1}$$

In addition, we call

$$JAJ^{-1} \subset [D, A]', \quad \text{(order 1 condition)} \tag{2}$$

and

$$J[D, A]J^{-1} \subset [D, A]', \quad \text{(order 2 condition)}. \tag{3}$$

Denote $\boxed{Cl_D(A)}$ the algebra generated by $A \cup [D, A]$ & call its elements *quantum Clifford fields*.

Such J permits right actions

$$\triangleleft\, b := Jb^* J^{-1} \ \text{ on } \ H,$$

so that (1), (2), (3) mean that H contains densely a

$$A - A, \quad A - Cl_D(A) \quad \text{and} \quad Cl_D(A) - Cl_D(A)$$

bimodule, resp.

Canonical ST

Prototype: the *canonical* ST on a spin manifold M (dim(M) even)

$$(C^\infty(M), L^2(S_M), \not{D}),$$

where $C^\infty(M)$ is the algebra of smooth complex functions on M, S_M is rank$_{\mathbb{C}} = 2^n$ Dirac bundle on M whose sections $\Gamma^\infty(S_M) =: \Gamma$ (**spinor fields**) carry a faithful irrep γ of the algebra of sections $\Gamma^\infty(\mathcal{C}\ell_M)$ (**Clifford fields**) of the Clifford bundle (& completions)

$$\gamma : \Gamma(\mathcal{C}\ell_M) \underset{\approx}{\to} \mathrm{End}_{C(M)}\, \Gamma(S_M) \subset B(L^2(S_M)) \qquad (4)$$

and $\not{D}$ is the usual Dirac operator on M:

$$\not{D} = \check{\gamma} \circ \tilde{\nabla} \qquad (= \sum_j^n \gamma^j \tilde{\nabla}_j \quad \text{locally}), \qquad (5)$$

with $\tilde{\nabla} : \Gamma \to \Omega_M \otimes \Gamma$ the spin connection & $\check{\gamma} : \Gamma(\mathcal{C}\ell_M) \otimes \Gamma \to \Gamma$.

$\looparrowright$ Now $\approx$ in (4) means

$\Gamma(S_M)$ is a Morita equivalence $\Gamma(\mathcal{C}\ell_M) - C(M)$ bimodule

and this exactly characterizes spin$_c$ manifolds M [Plymen].

Canonical ST 2

Due to $[\not{D}, a] = \gamma(da)$ for $a \in C^\infty(M)$, indeed

$$\boxed{\mathcal{C}\ell_{\not{D}}(C^\infty(M)) \approx \Gamma^\infty(\mathcal{C}\ell(M)).}$$

Next, $\exists$ a "chiral" $\mathbb{Z}_2$-grading χ_S of $L^2(S)$

& also $\exists$ real structure J_S that satisfies order 0 and 1 condition, but obviously not the order 2 since it implements Morita eqv.

$$J_S\,\mathcal{C}\ell_{\not{D}}(C(M))J_S^{-1} = C(M)';$$

this precisely characterizes spin manifolds.

J_S satisfies also

$$J_S^2 = \epsilon\,\mathrm{id}_H, \quad J_S\not{D} = \epsilon'\not{D}J_S, \quad J_S\chi_S = \epsilon''\chi_S J_S, \tag{6}$$

where $\epsilon, \epsilon', \epsilon'' \in \{\pm 1\}$.

$\looparrowright$ The canonical ST fully encodes the geometric data on M, that can be indeed reconstructed [Connes]. $\looparrowleft$

But it is not the only natural ST on a Riemannian manifold M, $\exists$

de Rham-Hodge ST

$$(C^\infty(M), L^2(\Omega(M)), d + d^*),$$

where $\Omega(M)$ is the space of de Rham differential forms on M, d is the exterior derivative and d^* its adjoint with respect to the hermitian product (from the metric g) on M.

The operator $d + d^*$ is Dirac-type:

$$d + d^* = \lambda \circ \nabla, \tag{7}$$

where ∇ is the Levi-Civita connection and the representation

$$\lambda : \Gamma(\mathcal{C}\ell(M)) \to \mathrm{End}_{C(M)}\Omega(M), \quad \lambda(v) = v \wedge - v \lrcorner, \ \ v \in T^*M \tag{8}$$

is equivalent to the left regular self-representation of $\Gamma(\mathcal{C}\ell(M))$.

Clearly $[d + d^*, a] = \lambda(da)$ so again

$$\boxed{\mathcal{C}\ell_{d+d^*}(C^\infty(M)) \approx \Gamma^\infty(\mathcal{C}\ell(M)).}$$

There is also an anti-representation ($\approx$ right regular one of $\Gamma(\mathcal{C}\ell(M))$)

$$\rho : \Gamma(\mathcal{C}\ell(M)) \to \mathrm{End}_{C(M)}\Omega(M), \ \rho(v) = (v \wedge + v \lrcorner)\chi_\Omega,$$

where χ_Ω is the grading ± 1 on even/odd forms.

Furthermore, since λ_v and $\rho_{v'}$ commute, $\Omega(M)$ is a $\Gamma(\mathcal{C}\ell(M))$-$\Gamma(\mathcal{C}\ell(M))$ bimodule, equivalent to $\Gamma(\mathcal{C}\ell(M))$, and thus

$\Omega(M)$ is a Morita equivalence $\Gamma(\mathcal{C}\ell(M)) - \Gamma(\mathcal{C}\ell(M))$ bimodule

which characterizes $\Omega(M)$ up to $\otimes$ with a complex line bundle. ♠

Besides the parity grading χ_Ω there is another

$$\chi'_\Omega = \text{normalized Hodge star.}$$

With them (as well known):

$$\text{index}(d + d^*) = \text{Euler, resp., signature of } M.$$

On any M $\exists$ also a real structure on $\Omega(M)$

$$J_\Omega := c.c,$$

which satisfies the order 0 and 1 conditions but <u>not</u> order 2, and so <u>can not</u> implement the $\Gamma(\mathcal{C}\ell(M))$ self-Morita equivalence.

But $\exists$ another J'_Ω on $\Omega(M)$: the main anti-involution $\circ\, c.c.$,

$$J'_\Omega = (-)^{k(k-1)/2} \circ c.c. \quad \text{on} \quad \Omega^k(M), \tag{9}$$

that interchanges the actions λ and ρ.

It satisfies all the order 0, 1 and 2 conditions and <u>does implement</u> the $\Gamma(\mathcal{C}\ell(M)$-$\Gamma(\mathcal{C}\ell(M))$ self-Morita equivalence (!).

Clearly $(J'_\Omega)^2 = \epsilon = +1$, & also $\epsilon'' = 1$ for χ_Ω
but for χ'_Ω instead of a sign ϵ'' we need the other grading

$$J'_\Omega \chi'_\Omega = \epsilon'' \chi_\Omega\, \chi'_\Omega J'_\Omega. \tag{10}$$

However not even this works for ϵ'; we need:

$$\nu J'_\Omega D\!\!\!/ = \epsilon' D\!\!\!/ J'_\Omega \nu, \quad \text{where} \quad \nu = (-)^{k(k+1)/2}. \tag{11}$$

digression: twisted real structure

This fits [Brzezinski,Ciccoli,LD,Sitarz]; [LD,Magee]:
Let $\nu \in B(H)$ with $\nu^{-1} \in B(H)$, such that $Ad_\nu \in Aut A$.
By *ν-twisted 1st (resp. 2nd) order condition* we mean that $\forall a, b \in A$

$$[[D,a], JbJ^{-1}]_{Ad^2_\nu} = 0,$$

$$[[D,a], [D, JbJ^{-1}]_{Ad^2_\nu}]_{Ad^2_\nu} = 0,$$

while by *ν-twisted ϵ' (resp. ϵ'') condition* we mean that

$$DJ\nu = \epsilon' \nu JD, \quad \gamma J\nu = \epsilon'' \nu J\gamma \quad \text{with } \epsilon', \epsilon'' \in \{+,-\}.$$

Actually, J'_Ω is *mildly twisted* as $\nu^2 = 1$ (so plain order conditions).
Another example: conformal rescaling hDh by $JAJ \ni h > 0$.
In a project with A. Sitarz yet a wider extension: multi-twisting
i.e. $D = \sum_j D_j$ with ν_j1O, $\forall j$.
Then we are closed under the product of S.T.,
and include 'asymmetric NC torus' ([LD, Sitarz], [Khalkhali...]),
matrix conformally rescaled D [Khalkhali,Sitarz])
& partially rescaled D by $\omega \in \Omega_D(A)$ on S^1-bundles [LD, Sitarz].

digression^2: real/modular twisted ST

①. In [Martinetti, Landi, Lizzi...] real twisted ST (with $[Da - \rho(a)D] \in B(H)$).

②. In "Crossed product" [Bruno, Thierry] another real modular S.T..

In [Brzezinski, L.D., Sitarz] a method to "untwist" 1. (in some cases) to our framework.

Also there a table with 3 twist types (for conformal case): ① fits.

A. Magee checks if ②. also fits (?) or if it can be "untwisted" (?).

νSM: A_F

The underlying arena of νSM by Connes et.al. is

ordinary (spin) manifold M $\times$ a finite quantum space F,

described by the algebra $C^\infty(M) \otimes A_F$, where

$$\boxed{A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).}$$

The Hilbert space is

$$L^2(S) \otimes H_F,$$

where

$$\boxed{H_F = \mathbb{C}^{96} =: H_f \otimes \mathbb{C}^3,}$$

with $\mathbb{C}^3$ corresponding to $g = 3$ generations, and

νSM: H_f

$$H_f = \mathbb{C}^{32} \simeq M_{8\times 4}(\mathbb{C})$$

with basis labelled by particles and antiparticles, we arrange as

$$\begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \\ \bar{\nu}_R & \bar{e}_R & \bar{\nu}_L & \bar{e}_L \\ \bar{u}_R^1 & \bar{d}_R^1 & \bar{u}_L^1 & \bar{d}_L^1 \\ \bar{u}_R^2 & \bar{d}_R^2 & \bar{u}_L^2 & \bar{d}_L^2 \\ \bar{u}_R^3 & \bar{d}_R^3 & \bar{u}_L^3 & \bar{d}_L^3 \end{bmatrix}$$

(1,2,3=colors).

νSM: γ_F

The action of $(\lambda, q, m) \in A_F$, diagonal in generations, on H_f is:

$$\begin{bmatrix} \left[\begin{array}{cc|cc} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{q}} \\ 0 & 0 & & \end{array}\right] & 0_4 \\ 0_4 & \left[\begin{array}{c|ccc} \lambda & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & m & \\ 0 & & & \end{array}\right] \end{bmatrix} \blacktriangleright \tag{12}$$

The grading is $\gamma_S \otimes \gamma_F$, where

$$\gamma_F = \begin{bmatrix} 1_2 & \\ & -1_2 \end{bmatrix} \otimes 1_3 \tag{13}$$

on leptons and opposite on quarks.

νSM: J_F & D_F

The real conjugation is $J = J_S \otimes J_F$, where J_F on H_f is

$$J_F \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_2^* \\ v_1^* \end{bmatrix} \ . \tag{14}$$

Finally, the Dirac operator is $D = \not{D} \otimes \mathrm{id} + \gamma_S \otimes D_F$, where

$$D_F \simeq \left(D_l \oplus D_q^{(3)} \right) \oplus 0^{(16g)} \ (+\,\text{part commuting with}\, A_F) \ . \tag{15}$$

Here $g = 3$ denotes the number of generations, and $D_l, D_q \in M_{4g}$ acting on leptons and quarks, respectively, as

$$D_l = \left(\begin{array}{cc|cc} 0 & 0 & \Upsilon_\nu & 0 \\ 0 & 0 & 0 & \Upsilon_e \\ \hline \Upsilon_\nu^* & 0 & 0 & 0 \\ 0 & \Upsilon_e^* & 0 & 0 \end{array} \right), \quad D_q = \left(\begin{array}{cc|cc} 0 & 0 & \Upsilon_u & 0 \\ 0 & 0 & 0 & \Upsilon_d \\ \hline \Upsilon_u^* & 0 & 0 & 0 \\ 0 & \Upsilon_{d*} & 0 & 0 \end{array} \right), \tag{16}$$

with (unitarily diagonalizable) Υ's $\in M_g$.

νSM

With all that:

• $\mathcal{G} := \{U = uJuJ^{-1} \,|\, u \in A, det\, U = 1\} \simeq U(1) \times SU(2) \times SU(3)$ (S.M. gauge group)

• all the fundamental fermions in H have the correct S.M. charges w.r.t. $\mathcal{G}$ (broken to $U(1)_{em} \times SU(3)$)

• the 1-forms $a[D, b]$, $a, b \in A$ yield the S.M. gauge fields $A_\mu, W^\pm, Z, G_\mu$ (from the part $\not{D}$ of D), plus the complex scalar (weak doublet) Higgs field (from the part D_F of D).

SOME MERITS:

• gauge & Higgs field as a connection,

• explains why only the fundamental reps of $\mathcal{G}$,

• a simple spectral action Tr$f(D/\Lambda)$ reproduces (besides gravity) the bosonic part of $\mathcal{L}_{SM}$ as the lowest terms of expansion in Λ, & and $< \phi, D\phi >$ the (Wick-rotated) fermionic part

• couples to gravity on M

Internal geometry of νSM (1)

The above "almost commutative" geometry is described by a ST

$$(C^\infty(M), L^2(S), \not{D}) \times (A_F, H_f, D_F),$$

that is a product of the 'external' canonical ST on spin manifold M with the 'internal' *finite* ST.

What is the geometric meaning of (A_F, H_f, D_F) **?**

Does it also correspond to a (noncommutative) spin manifold ?
Are the elements of H_f 'Dirac spinors' in some sense ?

Def

A spectral triple (A, H, D) is called spin$_c$
if H contains densely a Morita equivalence $\mathcal{C}\ell_D(A)$-A bimodule and it is called spin if the right action of $a \in A$ is Ja^*J^{-1}.
Furthermore the elements of H are called quantum Dirac spinors ("charged" or "neutral", respectively).

Answer: **'NO'** [Farnsworth, PhD], c.f. [FD'A, LD], unless ... ♠

Internal geometry of νSM (2)

But may be (A, H, D) is some analogue of de-Rham forms?

Def

A spectral triple (A, H, D) *is called* **Hodge**$_c$ *if* H *contains densely a Morita equivalence* $\mathcal{C}\ell_D(A)$*-*$\mathcal{C}\ell_D(A)$ *bimodule, and* **Hodge** *if the right* $\mathcal{C}\ell_D(A)$*-action is implemented by* J*. Furthermore we then say that* H *consists of, respectively, complex or real quantum de Rham forms.*

After a scrupulous analysis, first for 1g in [LD,FD'A,AS]:

Theorem (LD,AS)

Assume that both Υ_e *and* Υ_ν *have 3 distinct non-zero eigenvalues and in the eigenbasis of* Υ_e *no matrix element of unitary* U_l *which diagonalizes* Υ_ν *is of modulus* 1*, and analogously for* Υ_u*,* Υ_d *&* U_q*. Let also any eigenvalue of* Υ_ν *be distinct from any eigenvalue of* Υ_u*, and any eigenvalue of* Υ_e *be distinct from any eigenvalue of* Υ_d*. Then the Hodge property holds for* D_F*.*

Proof: partial condition

In the basis where

$$D_F \simeq \left(D_l \oplus D_q^3\right) \oplus \boxed{0^{16g}} + (\text{smth.} \in A_F') \ . \qquad (17)$$

denoting $\mathbf{n} := M_n$,

$$\pi(A_F) \simeq (\mathbf{1} \oplus \tilde{\mathbf{1}} \oplus \mathbf{2})^{4g} \oplus \boxed{(\mathbf{1} \oplus \mathbf{3})^{4g}}. \qquad (18)$$

Therefore $\mathcal{C}\ell$ contains $\boxed{\mathbf{1}^{4g} \oplus \mathbf{3}^{4g}}$, so $\mathcal{C}\ell'$ must contain $\mathbf{4g} \oplus \tilde{\mathbf{4g}}^3$ and if the Hodge duality holds so must $\mathcal{C}\ell$.
But the other algebras besides $\mathbf{1}^{4g} \oplus \mathbf{3}^{4g}$ that $\mathcal{C}\ell$ contains are generated by $(\mathbf{1} \oplus \tilde{\mathbf{1}} \oplus \mathbf{2})^{g}$ & D_l and $(\mathbf{1} \oplus \tilde{\mathbf{1}} \oplus \mathbf{2})^{3g}$ & D_q^3 ([P,S]).
Thus the only possibility that the Hodge condition holds is when these two algebras are exactly $\mathbf{4g}$ and $\tilde{\mathbf{4g}}^3$ $(\approx \tilde{\mathbf{4g}})$.
For that check if the only matrix that commutes with them is $\mathbb{C}\,1$.
Start with leptons:

Proof: partial condition 2

A matrix that commutes with $(\mathbf{1} \oplus \tilde{\mathbf{1}} \oplus \mathbf{2})$ has a form
$P_1 \oplus P_2 \oplus (1_2 \otimes P_3)$, where $P_1, P_2, P_3 \in \mathbf{g}$.
If it also commutes with D_l then:

$$P_1 \Upsilon_\nu = \Upsilon_\nu P_3, \qquad P_2 \Upsilon_e = \Upsilon_e P_3,$$
$$P_3 \Upsilon_\nu^* = \Upsilon_\nu^* P_1, \qquad P_3 \Upsilon_e^* = \Upsilon_e^* P_1.$$

But Υ_ν should be invertible (as otherwise $\exists$ solution $P_1 \neq 1$).
Similarly for Υ_e and P_2.

Moreover Υs are normal so we infer that P_1 & P_3 must commute with $\Upsilon_\nu \Upsilon_\nu^*$ whereas P_2 & P_3 must commute with $\Upsilon_e \Upsilon_e^*$.
Therefore in order $P_1 = P_3 = P_2 \sim 1$, by Schur's lemma,
the pair $\Upsilon_\nu \Upsilon_\nu^*$ and $\Upsilon_e \Upsilon_e^*$ should also generate the full algebra M_g.

Proof: partial condition 3

The latter condition, by Burnside theorem means (for $g = 3$) that $\Upsilon_\nu \Upsilon_\nu^*$ and $\Upsilon_e \Upsilon_e^*$ do not share a common eigenvector.

Since this is $U(3)$ invariant issue (inessential for the algebra action) w.l.o.g. we can assume that say Υ_e is diagonal, while Υ_ν is diagonalized by some $U_l \in U(3)$. Then U_l should not map any the basis vectors to another basis vector.
Assuming that both Υ_e and Υ_ν have 3 distinct ($\neq 0$) eigenvalues, we only need that in the eigenbasis of Υ_e no matrix element of U_l is of modulus 1 (or that some row and some column has two zeros).

Similar arguments hold for quarks: to assure that the algebra generated by $(\mathbf{1} \oplus \tilde{\mathbf{1}} \oplus \mathbf{2})$ & D_q is the full $\mathbf{4g}$ it suffices that (diagonal) Υ_u has 3 distinct $\neq 0$ eigenvalues and that invertible Υ_d is unitarily diagonalized by $U_q \in U(3)$ with properties like U_l.

Proof: full condition

This was only a partial condition for the Hodge property;
we need still that $(\mathbf{1} \oplus \tilde{\mathbf{1}} \oplus \mathbf{2})^{(2)}$ & $D_l \oplus D_q$ generate full $\mathbf{4g} \oplus \tilde{\mathbf{4g}}$, which imposes certain requirements that relate D_l and D_q.

If not, i.e. generate only a SUBalgebra, then there would exist a matrix in $\mathbf{8g}$ that commutes with both, and which w.l.g. can be taken hermitian (as D_l & D_q are such) of the form

$$\begin{pmatrix} c_1 1_{4g} & Q \\ Q^* & c_2 1_{4g} \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{R}$ and

$$0 \neq Q = Q_1 \oplus Q_2 \oplus (1_2 \otimes Q_3)$$

with each $Q_1, Q_2, Q_3 \in \mathbf{g}$.

Proof: full condition 2

We get:

$$D_l Q = Q D_q, \qquad D_q Q = Q D_l,$$

which leads to:

$$\Upsilon_\nu Q_3 = Q_1 \Upsilon_u, \Upsilon_e Q_3 = Q_2 \Upsilon_d, \Upsilon_\nu^* Q_1 = Q_3 \Upsilon_u^*, \Upsilon_e^* Q_2 = Q_3 \Upsilon_d^*,$$

and by simple manipulations

$$\begin{aligned} (\Upsilon_\nu \Upsilon_\nu^*) Q_1 &= Q_1 (\Upsilon_u \Upsilon_u^*), \qquad & (\Upsilon_e \Upsilon_e^*) Q_2 &= Q_2 (\Upsilon_d \Upsilon_d^*) \\ (\Upsilon_\nu^* \Upsilon_\nu) Q_3 &= Q_3 (\Upsilon_u^* \Upsilon_u), \qquad & (\Upsilon_e^* \Upsilon_e) Q_3 &= Q_3 (\Upsilon_d^* \Upsilon_d) \ . \end{aligned}$$

Thus in order $(\mathbf{1} \oplus \tilde{\mathbf{1}} \oplus \mathbf{2})^{(2)}$ & $D_l \oplus D_q$ generate full $\mathbf{4g} \oplus \tilde{\mathbf{4g}}$, it suffices then that $Q_1 = Q_2 = Q_3 = 0$ are the only solutions.

Due to the diagonal form of the mixing matrices Υ_e & Υ_u, and unitary diagonalizability of Υ_ν & Υ_d, this holds when any eigenvalue of Υ_ν is distinct from any eigenvalue of Υ_u, and any eigenvalue of Υ_e is distinct from any eigenvalue of Υ_d.

Experimental data ?

• Leptons: $\Upsilon_e = \delta_l^{\downarrow} = \mathrm{diag}(m_e < m_\mu < m_\tau)$, with $0 < m_e$ and $\Upsilon_\nu = U_l \delta_l^{\uparrow} U_l^*$, with $\delta_l^{\uparrow} = \mathrm{diag}(m_{\nu_e} < m_{\nu_\mu} < m_{\nu_\tau})$ &

$$U_l = U_{\mathrm{PMNS}} = \begin{bmatrix} 0.82 \pm 0.01 & 0.54 \pm 0.02 & -0.15 \pm 0.03 \\ -0.35 \pm 0.06 & 0.70 \pm 0.06 & 0.62 \pm 0.06 \\ 0.44 \pm 0.06 & -0.45 \pm 0.06 & 0.77 \pm 0.06 \end{bmatrix}.$$

• Quarks: $\Upsilon_u = \delta_q^{\uparrow}$, $\Upsilon_d = U_q \delta_q^{\downarrow} U_q^*$, with both $\delta_q^{\uparrow}$, $\delta_q^{\downarrow}$ diagonal with different positive masses &
$U_q = U_{\mathrm{CKM}} = \delta_q^{\uparrow}$, parametrized by
$\theta_{12} = 13.04 \pm 0.05$, $\theta_{23} = 2.38 \pm 0.06$, $\theta_{13} = 0.201 \pm 0.011$ and
$\delta_{13} = 1.20 \pm 0.08$,

satisfy 1st part of our conditions.

Full Hodge duality

Finally, also the lepton masses are different from quark masses, so all (but $m_{\nu_e} \neq 0$?) our conditions are satisfied, so:

YES !

Main result (LD,AS)

Provided there is no massless neutrino, the Standard Model satisfies the internal quantum Hodge condition and the flavor multiplet of fundamental fermions constitutes quantum de-Rham forms.

This adds mainly to the conceptual significance of the (noncommutative) geometry of S.M, which as stressed by Connes brings a message about the geometric nature of the space-time ...

Geometric conclusions

The νSM interprets geometry of the SM as the gravity on the product $M \times F$ of a (Riemannian) manifold M with a finite noncommutative 'internal' space F.
The multiplet of fundamental fermions that constitute H_F, each a Dirac spinor on M, corresponds just to fields on F .

We show that the geometric nature of this flavor multiplet is not a quantum analogue of Dirac spinors, but of de-Rham forms on F.
I.e. not only the 2nd O.C. but in fact the Hodge property holds: $\mathcal{C}\ell_D(A)' = J\mathcal{C}\ell_D(A)J$ in the full experimental range of values of CKM and PMNS coefficients.

♠ Can grasp other features of SM with other type of structures ? Jordan algebras ?

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