Some comments on Morita equivalence and spectral triples

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References

Talk based on:

- FD & L. Dabrowski,
  The SM in NCG and Morita equivalence,
  J. Noncommut. Geom. 10 (2016) 551–578.

- L. Dabrowski, FD & A. Sitarz,
  The SM in NCG: fundamental fermions as internal forms,
  Lett. Math. Phys. 108 (2018) 1323.

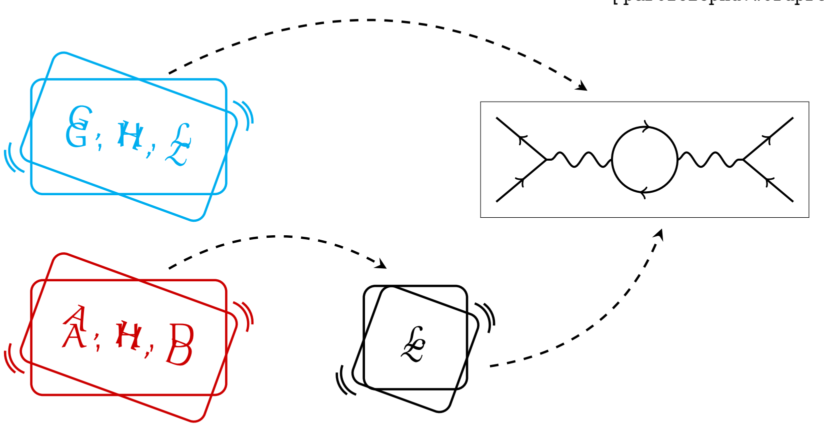
- [L. Dabrowski & A. Sitarz,
  Fermion masses, mass-mixing and the almost commutative geometry of the SM,
  JHEP 68 (2019).]

- L. Dabrowski, FD & A. Magee,
  in preparation.

Prelude

"A theory is a black box that we can shake to make predictions of physical observables."

[particlephd.wordpress.com]



- Classical Yang-Mills Theory: $\mathrm{G} =$ Lie group, $\mathrm{H} =$ Hilbert space, $\mathcal{L} =$ Lagrangian
- Noncommutative Geometry: $\mathrm{A} = *$-algebra, $\mathrm{H} =$ Hilbert space, $\mathrm{D} =$ Dirac operator

The NCG approach

$$\boxed{\mathrm{A}\,,\ \mathrm{H}\,,\ \mathrm{D}}$$

Two main goals:

☑ derive the Standard Model (the complicated Lagrangian) from simple geometric data;

☑ get some clues on unification with gravity.

Advantages:

- The Lagrangian is not postulated but derived from the geometry;
- One gets for free the Higgs field (in the Standard Model case)...
- ...and a theory coupled with (classical) gravity.

D encodes the free parameters of the theory $\rightsquigarrow$ constrains on D?

Spinors or differential forms?

In the Standard Model of elementary particles, the representation π of the gauge group $U(1) \times SU(2) \times SU(3)$ and is dictated by the experiments. One has:

$$H = \text{spinors} \otimes \mathbb{C}^{32} \otimes \underbrace{\mathbb{C}^{g}}_{g \text{ generations}}$$

The representation π is highly non-trivial. Where does it come from?

Consider the inclusion (in block form):

$$U(1) \times SU(2) \times SU(3) \to SU(5)\,, \qquad (x, y, z) \mapsto \begin{array}{cc} \overset{2\times 2}{\downarrow} & \overset{3\times 3}{\downarrow} \\ \multicolumn{2}{c}{\begin{pmatrix} x^3 y & \\ & \bar{x}^2 z \end{pmatrix}} \end{array}$$

It turns out that π is the restriction of the natural representation of $SU(5)$ on $\underbrace{\bigwedge^{\bullet}\mathbb{C}^5}_{\dim = 32}$.

Is H given by some kind of noncommutative differential forms?

Spectral triples (again!)

Definition

A unital spectral triple (A, H, D) is the datum of:

(i) a (real or complex) unital $*$-algebra A of bounded operators on a (separable) complex Hilbert space H,

(ii) a selfadjoint operator D on H with compact resolvent,

such that $a \cdot \mathrm{Dom}(D) \subset \mathrm{Dom} D$ and $[D, a] \in \mathcal{B}(H)$ for all $a \in A$.
It is called

- even if $\exists\, \gamma = \gamma^*$ on H s.t. $\gamma^2 = 1$, $\gamma D = -D\gamma$ and $[\gamma, a] = 0\ \forall\, a \in A$;
- real if $\exists$ an antilinear isometry J on H s.t.

$$J^2 = \pm 1, \qquad JD = \pm DJ, \qquad \text{and (only in the even case)} \quad J\gamma = \pm\gamma J$$

and $\forall\, a, b \in A$:

$$\underset{\text{(reality)}}{[a, JbJ^{-1}] = 0} \qquad \underset{\text{(1st order)}}{[[D, a], JbJ^{-1}] = 0}$$

Examples of spectral triples

Let: (M, g) = compact oriented Riemannian manifold without boundary, $E \to M$ herm. vector bundle equipped with a unitary Clifford action $c : C^\infty(M, T^*_{\mathbb{C}}M \otimes E) \to C^\infty(M, E)$ and a connection ∇^E compatible with g. Then:

$$A = C(M) \qquad H = L^2(M, E) \qquad D = c \circ \nabla^E$$

is a spectral triple.

Two main examples belonging to this class:

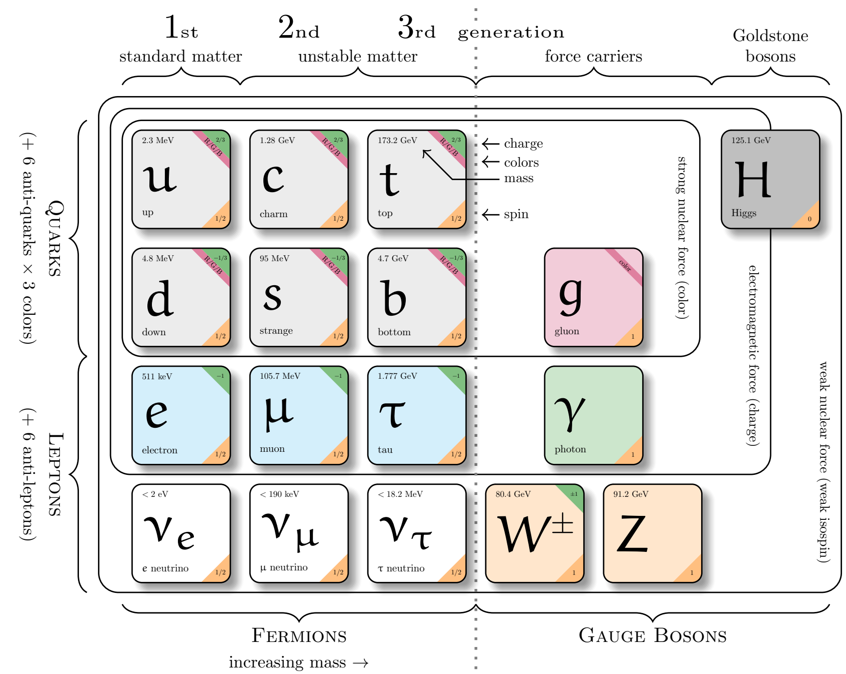
- the Hodge operator $D = d + d^*$ on $E = \bigwedge^{\mathrm{even}} T^*_{\mathbb{C}}M \oplus \bigwedge^{\mathrm{odd}} T^*_{\mathbb{C}}M$ (always even);
- the Dirac operator $D = \not{D}$ on the spinor bundle E (if M is a spin manifold).

Remarks

In both examples, (1) there exists a real structure J,
(2) H carries commuting representations of $C(M)$ and $\mathcal{C}\ell(M, g)$.
In the former, (3) H carries two commuting representation of $\mathcal{C}\ell(M, g)$.

The Standard Model

[Picture from www.texample.net/tikz]



Particles in a box...

We arrange particles in a 4×4 matrix:

$$\begin{bmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{bmatrix}$$

So, for example the unit vectors:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

represent a right-handed electron a mix right-handed neutrino/left-handed electron.

Internal degrees of freedom are encoded in the Hilbert space

$$\underset{\text{(particles)}}{M_4(\mathbb{C})} \oplus \underset{\text{(antiparticles)}}{M_4(\mathbb{C})} \simeq \mathbb{C}^{32}$$

The Standard Model spectral triple

The underlying space is

$$\underset{\text{(spin manifold)}}{M} \times \underset{\text{(finite nc space)}}{F}$$

with finite-dim. spectral triple $(A_F, H_F, D_F, \gamma_F, J_F)$ given by:

- $H_F \simeq \mathbb{C}^{32g} \rightsquigarrow$ internal degrees of freedom of the elementary fermions. Total nr:

$$\underset{\text{(weak isospin)}}{2} \times \underset{\text{(lepton + quark in 3 colors)}}{4} \times \underset{\text{(L,R chirality)}}{2} \times \underset{\text{(particle or antiparticle)}}{2} \times \underset{\text{(generations)}}{g} = 32g$$

- $\gamma_F =$ chirality operator
- $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
- $J_F =$ charge conjugation
- D_F encodes the free parameters of the theory.

Dirac vs. Hodge

From the example of Hodge operator, we learn:

Real spectral triple $\longleftrightarrow$ oriented Riemannian manifold

How to algebraically characterize Dirac's Dirac operator? Take $M = \mathbb{R}^4$:

$\psi \in \Omega^\bullet(M)$ has 16 components. A Dirac spinor $\psi \in L^2(M, S)$ has 4 components.

Both carry a rep. of $C_0(M)$ and $\mathcal{C}\ell_{4,0}(\mathbb{R})$, but only the latter satisfies the following:

If a bounded operator commutes with $C_0(M)$ and all γ^μ's, then it is a function.

This completely characterizes Dirac spinors.

Theorem

1. A closed oriented Riem. manifold M admits a spin^c structure iff $\exists$ a Morita equivalence $C(M)$-$\mathcal{C}\ell(M)$ bimodule Σ, with $\mathcal{C}\ell(M)$ the algebra of sections of the Clifford bundle.
2. $\Sigma = C^0$ sections of the spinor bundle $S \to M$ (Dirac spinors in the conventional sense).

Once we have S, we can introduce the Dirac operator D of the spin^c structure:

3. M is a spin manifold iff $\exists$ a real structure J on $L^2(M, S)$.

Dirac vs. Hodge in NCG

Let (A, H, D, J) be a finite-dimensional real spectral triple. We set:

$\xi^\circ := J\xi^* J^{-1}, \quad \forall\, \xi \in \mathrm{End}_{\mathbb{C}}(H)$

If B is a subset of $\mathrm{End}_{\mathbb{C}}(H)$, we call:

$B^\circ := \{\xi^\circ : \xi \in B\};$

$B' := \{\xi \in \mathrm{End}_{\mathbb{C}}(H) : [\xi, \eta] = 0 \; \forall\, \eta \in B\}$ (the commutant of B)

Finally, we define the A-bimodule of 1-forms and the Clifford algebra as:

$\Omega^1_D(A) :=$ complex vector subspace of $\mathrm{End}_{\mathbb{C}}(H)$ spanned by $a[D, b]$, $a, b \in A$;

$\mathcal{C}\ell_D(A) :=$ complex $*$-subalgebra of $\mathrm{End}_{\mathbb{C}}(H)$ generated by A and $\Omega^1_D(A)$.

Remark

Reality + 1st order cond. are equivalent to: $A^\circ \subseteq \mathcal{C}\ell_D(A)'$ $(\star)$

The 2nd order condition is equivalent to: $\mathcal{C}\ell_D(A)^\circ \subseteq \mathcal{C}\ell_D(A)'$ $(\star\star)$

We say that the spectral triple has the Dirac property if $(\star)$ / Hodge property if $(\star\star)$ is an equality.

In the Standard Model, can we use the condition(s)

Dirac	**Hodge**
$\mathcal{C}\ell_D(A)' = A^\circ$	$\mathcal{C}\ell_D(A)' = \mathcal{C}\ell_D(A)^\circ$

to "select" good Dirac operators?

Theorem [FD & L. Dabrowski]

In order to satisfy the Dirac condition, one must modify the operator studied by A. Connes and A. Chamseddine. As a byproduct one gets:

→ a new scalar field;

→ a field coupling leptons with quarks.

Physical implications are discussed in:

M. Kurkov and F. Lizzi, Phys. Rev. D 97 (2018) 085024.

Finite-dimensional spectral triples

Let (A, H, D) be a finite-dim. spectral triple and J an antilinear isometry satisfying the reality condition. From the structure theorem for finite-dimensional C^*-algebras:

$$A_{\mathbb{C}} \simeq \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}) .$$

Call P_i the unit of the summand $M_{n_i}(\mathbb{C})$, $Q_i := JP_iJ^{-1}$ and

$$H_{kl} := P_kQ_lH , \qquad D_{ij,kl} := P_iQ_jDP_kQ_l .$$

We can decompose the Dirac operator into four pieces:

$$D = D_0 + D_1 + D_2 + D_R ,$$

where

$$D_0 := \sum_{i,j,k\,:\,i\neq k} D_{ij,kj} , \qquad D_1 := \sum_{i,j,l\,:\,j\neq l} D_{ij,il} ,$$

$$D_2 := \sum_{\substack{i,j,l,k \\ i\neq k,j\neq l}} D_{ij,kl} , \qquad D_R := \sum_{i,j} D_{ij,ij} .$$

Lemma

$D_0 + D_2 \in \Omega^1_D(A)$.

Lemma (1st order)

D satisfies the 1st order condition if and only if:

$$D_2 = 0 , \qquad D_1 \in A' \qquad \text{and} \qquad D_R \text{ satisfies the 1st order condition.}$$

Corollary

If (A, H, D, J) is a finite-dimensional real spectral triple and $D_R \in A'$, then

$$\Omega^1_D = \Omega^1_{D_0}$$

and $\mathcal{C}\ell_D(A)$ is generated by $A_{\mathbb{C}}$ and D_0.

Lemma (2nd order)

Let (A, H, D, J) be a finite-dimensional real spectral triple with $D_R \in A'$.
The 2nd order condition is satisfied if and only if $[D_0, D_1] = 0$.

The Hodge property

Lemma

Let (A, H, D, J) be finite-dim. real and $B \subseteq \mathrm{End}_{\mathbb{C}}(H)$ a unital complex $*$-algebra s.t.:

$$B' = B^\circ \qquad \text{and} \qquad \mathcal{C}\ell_D(A) \subseteq B .$$

The following are equivalent:

(a) the Hodge property holds; (b) $\mathcal{C}\ell_D(A)' \subseteq B^\circ$; (c) $\mathcal{C}\ell_D(A) = B$.

For the Standard Model, the strategy to classify D's satisfying the Hodge property is:

1. Define a suitable B independent of D and containing $\mathcal{C}\ell_D(A)$ (for any D).
2. Check that $B^\circ \subseteq B'$.
3. Prove that $B^\circ = B'$ (it is enough to show that $\dim B = \dim B'$).
4. Find under what conditions on D one has $\mathcal{C}\ell_D(A)' \subseteq B^\circ$.

Given $(A_F, H_F, J_F, \gamma_F)$ of the νSM, the D's satisfying 2nd order belong to a set union of four (intersecting) vector spaces. For almost all of these D's, the Hodge property is satisfied (in the above parameter space, such a property fails only on a measure zero subset).

Dirac $\times$ Dirac is Dirac

All spectral triples here are (unital) real and finite-dimensional.

The product $S := (A, H, D, J)$ of two spectral triples $S_1 := (A_1, H_1, D_1, \gamma_1, J_1)$ and $S_2 := (A_2, H_2, D_2, J_2)$ is given by

$$A = A_1 \otimes A_2\,, \qquad H = H_1 \otimes H_2\,, \qquad D = D_1 \otimes 1 + \gamma_1 \otimes D_2\,, \qquad J = J_1 \otimes J_2\,.$$

Lemma

If $\gamma_1 \in \mathcal{C}\ell_{D_1}(A_1)$, then

$$\mathcal{C}\ell_D(A) = \mathcal{C}\ell_{D_1}(A_1) \otimes \mathcal{C}\ell_{D_2}(A_2)$$

If S_1 is Dirac $\Rightarrow\ \gamma_1 \in A_1' = J_1 \mathcal{C}\ell_{D_1}(A_1) J_1^{-1}\ \Rightarrow\ \gamma_1 = \pm J_1 \gamma_1 J_1^{-1} \in \mathcal{C}\ell_{D_1}(A_1)$.

From this observation, the above Lemma and the commutation theorem for tensor products of (von Neumann) algebras, we get:

Proposition

If S_1 and S_2 are Dirac, their product S is Dirac.

Hodge $\times$ Hodge is Hodge?

More tricky. In general, $\mathcal{C}\ell_D(A)$ is a graded tensor product of $\mathcal{C}\ell_{D_1}(A_1)$ and $\mathcal{C}\ell_{D_2}(A_2)$.

Assume both triples are even. The degree of $v \in H_1$ and $a \in \mathrm{End}_{\mathbb{C}}(H_1)$ is

$$|v| = \begin{cases} 0 & \text{if } \gamma_1(v) = +v \\ 1 & \text{if } \gamma_1(v) = -v \end{cases} \qquad |a| = \begin{cases} 0 & \text{if } \gamma_1 a = a\gamma_1 \\ 1 & \text{if } \gamma_1 a = -a\gamma_1 \end{cases}\,.$$

The same for the second spectral triple.

For all homogeneous $a \in \mathrm{End}_{\mathbb{C}}(H_1)$, $b \in \mathrm{End}_{\mathbb{C}}(H_2)$, define $a \odot b$ and $a \odot' b \in \mathrm{End}_{\mathbb{C}}(H)$ by

$$(a \odot b)(v \otimes w) = (-1)^{|b|\,|v|} av \otimes bw \qquad (a \odot' b)(v \otimes w) = (-1)^{|a|\,|w|} av \otimes bw$$

for all homogeneous $v \in H_1$ and $w \in H_2$. Then

Commutation theorem for graded tensor products

Let $B_i \subseteq \mathrm{End}_{\mathbb{C}}(H_i)$ be a unital $*$-subalgebra, $i = 1, 2$. Then $(B_1 \odot B_2)' = B_1' \odot' B_2'$.

Since $D = D_1 \odot 1 + 1 \odot D_2$, one has $\mathcal{C}\ell_D(A) = \mathcal{C}\ell_{D_1}(A_1) \odot \mathcal{C}\ell_{D_2}(A_2)$.

If we change the real structure to

$$J(v \otimes w) = (-1)^{|v|\,|w|} J_1(v) \otimes J_2(w)$$

(so $J \neq J_1 \otimes J_2$) then the product of two Hodge spectral triples is Hodge.

Thank you for your attention.