Algebraic backgrounds and the $B - L$ symmetry

Fabien Besnard
EPF
fabien.besnard@epf.fr

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Fixing notations

- Real even spectral triple $S = (\mathcal{A}, \mathcal{H}, D, J, \chi)$. ($\mathcal{A} \subset \mathcal{B}(\mathcal{H})$).
- In indefinite signature $\mathcal{H} \to \mathcal{K}$ *pre*-Krein space, $\dagger \to \times$.
- Let $a_i, b_i \in \mathcal{A}$. Then a NC 1-form is

$$\omega = \sum_i a_i [D, b_i].$$

- The $\mathcal{A}$-bimodule of NC 1-forms is written Ω^1_D. Let $\omega \in \Omega^1_D$ be selfadjoint, then the *fluctuated Dirac* D_ω is

$$D_\omega = D + \omega + J\omega J^{-1}.$$

- $\{D_\omega\}$ is the bosonic configuration space of both Connes-Lott and Connes-Chamseddine theories.

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Important question because the equality

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for a well-chosen NC algebra is one of the main motivations for the NCG approach to the SM.

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Two definitions for $\mathrm{Aut}(S)$ (depending on the books):

1. $UU^\times = 1, U\mathcal{A}U^{-1} = \mathcal{A}, U\chi = \chi U, JU = UJ$.
2. same $+ UD = DU$.

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The second one seems more logical, but with it one gets $\mathrm{Isom}(M, g)$ instead of $\mathrm{Diff}(M)$.
The first one then must be right... Right ?

Let us apply it to the case of the canonical ST over a manifold. Then:

1. $\mathrm{Aut}(S) = \mathrm{Diff}(M) \ltimes \Gamma(\mathrm{Spin}(n))$ for $n \leq 4$.
2. $\mathrm{Aut}(S) \supsetneq \mathrm{Diff}(M) \ltimes \Gamma(\mathrm{Spin}(n))$ for $n \geq 6$.
   (Example: multiplication by $\sin t\gamma_1\gamma_2 + \cos t\gamma_3 \ldots \gamma_6 \notin Spin(n)$.)

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Troubling: aren't we allowed to do GR with Spectral Triples in dim ≥ 6 ?
But there is worse...
The bosonic configuration space of the Spectral Standard Model is

$$\mathcal{C} = \{D_0 + \omega + J\omega J^{-1} | \omega \in \Omega^1_{D_0}, \omega^\times = \omega\}$$

It is clearly not invariant under $\mathrm{Aut}(S)$ according to the first definition. (But it is according to the second one.)
What about the SSM coupled with gravity ? This time

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There is something wrong since the very beginning !

Consider a *parallelizable* manifold, and[1]

1. A trivial bundle $M \times S$, $S = \mathbb{C}^4$,
2. gamma matrices $\gamma_a \in \mathrm{End}(S)$ (in a representation s.t. $\gamma_a^\dagger = \pm\gamma_a$),
3. $\chi = \gamma_5$,
4. $J = \gamma_2 \circ c.c$,
5. "spinor metric" $H_S(\psi, \psi') = \psi^\dagger \gamma_0 \psi'$.

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Then every tetrad $e = (e_a)$ defines at the same time a metric g_e such that e is pseudo-orthonormal, a g_e-spin structure with rep $\rho_e : \mathbb{C}\ell TM \to \mathrm{End}(S)$ s.t $\rho_e(e_a) = \gamma_a$, and so a Dirac operator $D(e) = i \sum \pm\gamma_a \nabla^e_{e_a}$.

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Let $\Gamma = \mathrm{Span}(\gamma_a | a = 0, \dots, 3)$. Then:

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$\Rightarrow \Omega^1$ should be a background structure while D should not.

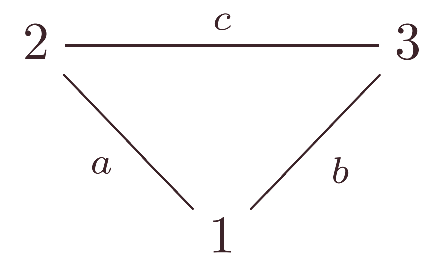
[1] Here $n = 1 + 3$.

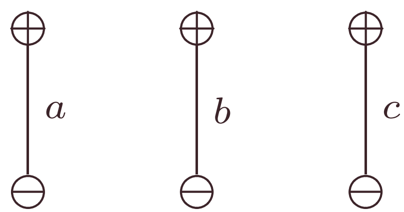
Algebraic backgrounds

- An *algebraic background* $\mathcal{B} = (\mathcal{A}, \mathcal{K}, \pi, J, \chi, \Omega^1)$ is a ST - D + an odd bimodule $\Omega^1 \subset \mathrm{End}(\mathcal{K})$.
- A *compatible Dirac operator* on a background $\mathcal{B}$ is an operator D such that:
  - $D^\times = D$, $D\chi = -\chi D$, $JD = DJ$,
  - $[D, \pi(a)] \in \Omega^1$ for all $a \in \mathcal{A}$.
- A compatible Dirac is *regular* if $\Omega^1_D = \Omega^1$.
- An automorphism of $\mathcal{B}$ is an operator U such that:
  - $U^\times U = 1$,
  - $U\chi = \chi U$,
  - $UJ = JU$,
  - $U\pi(\mathcal{A})U^{-1} = \pi(\mathcal{A})$,
  - $U\Omega^1 U^{-1} = \Omega^1$.

The case of a finite graph

$G = (V, E)$ finite graph with weight function $\delta : E \to \mathbb{R}_+^*$





Connes' distance formula reproduces the geodesic distance with the "split graph" ST:

- $\mathcal{A} = \mathbb{R}^V$, $\tilde{E} := E \times \{-;+\}$, $H = L^2(\tilde{E}) = \mathbb{C}^E \otimes \mathbb{C}^2$ + canonical $\langle ., . \rangle$.
- $\pi(a)F(e, \pm) = a(e^\pm)F(e, \pm) = \bigoplus_{e \in E} \begin{pmatrix} a(e^-) & 0 \\ 0 & a(e^+) \end{pmatrix}$.
- $D = \bigoplus_{e \in E} \frac{1}{\delta_e} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $\chi = \bigoplus_{e \in E} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $J = c.c.$, KO dim 0

Split graph background: D out, $\Omega^1 = \Omega^1_D = \{\bigoplus_e \begin{pmatrix} 0 & \omega_e^+ \\ \omega_e^- & 0 \end{pmatrix}\}$ in.

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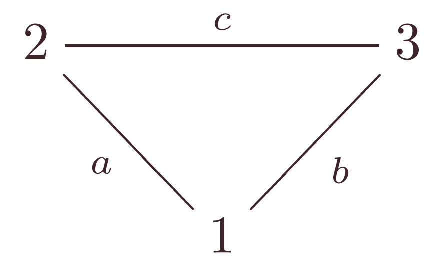

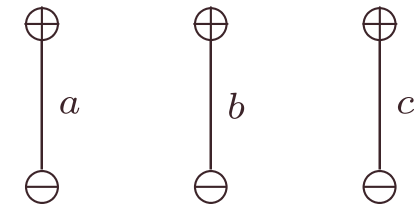

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Split graph background: D out, $\Omega^1 = \Omega^1_D = \{\bigoplus_e \begin{pmatrix} 0 & \omega_e^+ \\ \omega_e^- & 0 \end{pmatrix}\}$ in.

1. $\mathrm{Out}(\mathcal{A}, \mathcal{H}, J, \chi) = \mathrm{Perm}(V)$, + large config space,
2. $\mathrm{Out}(\mathcal{A}, \mathcal{H}, J, \chi, \Omega^1) = \mathrm{Aut}(G)$, config space $\simeq \{w : E \to \mathbb{R} \cup \{\infty\}\}$,
3. $\mathrm{Out}(\mathcal{A}, \mathcal{H}, J, \chi, D) = \mathrm{Isom}(G)$, no config space.

The canonical background $\mathcal{B}(M)$ of a parallelizable manifold is constructed like before thanks to an origin metric g_0 of signature (p, q), only needed to define

$$(\Psi, \Psi') = \int_M H_S(\Psi_x, \Psi'_x)\text{vol}_{g_0}$$

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Let $\theta : M \to M$ be a diffeo and $\Sigma : M \to \mathrm{Spin}(p, q)^0 \subset \mathrm{End}(S)$, then

$$V_\theta : \Psi \mapsto \sqrt{\frac{\mathrm{vol}_{\theta^* g_0}}{\mathrm{vol}_{g_0}}}\Psi \circ \theta^{-1}, \text{ and } U_\Sigma : \Psi \mapsto \Sigma\Psi$$

are automorphisms of $\mathcal{B}(M)$. Moreover, they generate $\mathrm{Aut}(\mathcal{B}_M)$.

$$\Rightarrow \mathrm{Aut}\mathcal{B}_M = \text{symmetry group of (tetradic) GR.}$$

- Let r be a field of invertible matrices: acts on tetrads $e \mapsto r \cdot e$.
- $S_r \in \mathrm{End}(\mathcal{K})$ is defined by $\Psi \mapsto |\det r|^{-1/2}\Psi$

Theorem The regular Dirac operators of the canonical background $\mathcal{B}(M)$ are

$$D = \delta_r + \zeta$$

where $\delta_r = S_r D(r \cdot e_0) S_r^{-1}$ and ζ is a multiplication operator $(\zeta\Psi)_x = \zeta_x \Psi_x$, s.t. $\zeta_x^\times = \zeta_x$, ζ commutes with J and anticommutes with χ.

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- In $1+3$ dim, there is a single centralizing pseudo-vector field.
- The spaces of δ_r's and ζ's are separately invariant under automorphisms.
- They are “orthogonal”.

$\Rightarrow$ centralizing fields can be removed safely.

$\mathcal{B}_{SM} = \mathcal{B}(M)\hat{\otimes}\mathcal{B}_F$ where $\mathcal{B}_F = (\mathcal{A}_F, \mathcal{K}_F, \pi_F, J_F, \chi_F, \Omega^1_F)$:

- $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$,
- $\mathcal{K}_F = \mathcal{K}_R \oplus \mathcal{K}_L \oplus \mathcal{K}_{\bar{R}} \oplus \mathcal{K}_{\bar{L}}$, $K_\sigma = \mathbb{C}^2 \otimes (\mathbb{C} \oplus \mathbb{C}^3_{\text{color}}) \otimes \mathbb{C}^3_{\text{gen}}$,
- Finite Krein product $(\psi, \psi') = \psi^\dagger \chi_F \psi$, with $\chi_F = [1_R, -1_L, -1_{\bar{R}}, 1_{\bar{L}}]$,
- $J_F = \begin{pmatrix} 0 & -1_{antipart} \\ 1_{part} & 0 \end{pmatrix} \circ c.c.$,
- $\pi_F(\lambda, q, m) = [\tilde{q}_\lambda, \tilde{q}, \lambda 1_2 \oplus 1_2 \otimes m, \lambda 1_2 \oplus 1_2 \otimes m] \otimes 1_3$, where $q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^* \end{pmatrix}$ and $\tilde{q} = q \oplus q \otimes 1_3 \simeq q \otimes 1_4$.
- $\Omega^1_F = \{ \begin{pmatrix} 0 & Y_0^\dagger \tilde{q}_1 & 0 & 0 \\ \tilde{q}_2 Y_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, q_1, q_2 \in \mathbb{H}\}$, where $Y_0 = \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix} \oplus \begin{pmatrix} 1_3 \otimes Y_u & 0 \\ 0 & 1_3 \otimes Y_d \end{pmatrix}$.

Choice of Ω^1_F constrained by: 1) odd $\mathcal{A}_F$-$\mathcal{A}_F$-bimodule, 2) non-vanishing config space, and 3) first-order condition: $[\Omega^1_F, J_F \pi_F(\mathcal{A}_F) J_F^{-1}] = 0$

Automorphisms of the SM background

For $T \in \mathrm{End}(\mathcal{K})$ define $T^o = JT^\times J^{-1}$. Then for $u \in U(\mathcal{A})$, let
$\Upsilon(u) = uJuJ^{-1} = u(u^{-1})^o$.
Υ is a group homomorphism from $U(\mathcal{A})$ into $\mathrm{Aut}(\mathcal{B}_M)$.

- True because of first-order condition and $[\pi(\mathcal{A}_F), \pi(\mathcal{A}_F)^o] = 0$.
- Only needs the “weak order 1 cond.” $\pi(u)^o\Omega^1\pi(u^{-1})^o = \Omega^1$.
- $\Upsilon(U(\mathcal{A})) =$ group of local gauge transf. $M \to U(1) \times SU(2) \times U(3)$.

Th: If $\pi_1(M) = \{1\}, Y_0$ is invertible and M_ν, M_e (resp. M_u, M_d) have no common eigenvector, then $\mathrm{Aut}(\mathcal{B}_{SM})$ is generated by

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Remarks:

1. There are counterexamples without the topological hypothesis.
2. $\mathrm{Aut}(S_{SM})$ is larger than $\mathrm{Aut}(B_{SM})$ if D is not fixed[2], and *smaller*[3] if it is.

[2] Ex: $U = [A, B, A^*, B^*]$ with arbitrary unitary matrices A, B commuting with $\mathcal{A}$.
[3] Only constant gauge transformations !

Th: The compatible Dirac operators are

$$\Phi(q) + \Phi(q)^o + \sigma(M)$$

where $q \in \mathbb{H}$, $M =$ symmetric matrix acting on generations and

$$\Phi(q) = \begin{pmatrix} 0 & -Y_0^\dagger \tilde{q}^\dagger & 0 & 0 \\ \tilde{q} Y_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \sigma(M) = \begin{pmatrix} 0 & 0 & -p_\nu \otimes M^\dagger & 0 \\ 0 & 0 & 0 & 0 \\ p_\nu \otimes M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $p_\nu = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the projection on the space spanned by ν.

The configuration space of $\mathcal{B}_F$

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where $q \in \mathbb{H}$, $M =$ symmetric matrix acting on generations and

$$\Phi(q) = \begin{pmatrix} 0 & -Y_0^\dagger \tilde{q}^\dagger & 0 & 0 \\ \tilde{q} Y_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \sigma(M) = \begin{pmatrix} 0 & 0 & -p_\nu \otimes M^\dagger & 0 \\ 0 & 0 & 0 & 0 \\ p_\nu \otimes M & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where $p_\nu = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is the projection on the space spanned by ν.

- The $\Phi(q) + \Phi(q)^o$ part can be obtained by the “fluctuation formalism”.
- The $\sigma(M)$ part cannot.
- The latter is the one that had been put by hand (with only 1 dof) to correct the Higgs mass prediction.

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$$D = \delta_r \hat{\otimes} 1 + \zeta_g + \zeta_X + \zeta_{B-L} + \zeta_H + \zeta_\sigma + \zeta_{\text{other}}$$

1. The ζ_{other} part contains centralizing fields which act on generations only.

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Conclusion:

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Conclusion:

$\Rightarrow$ we can freely include from 0 to 6 $\sigma-$ fields, but we need at least one to have neutrino oscillations.

$\Rightarrow$ we can throw away ζ_{other} without harm.

$\Rightarrow$ We keep some centralizing fields, and throw some others away: not pretty...

$\Rightarrow$ There is no known action in Lorentzian signature for these fields...

$\Rightarrow$ But the Euclidean SA could be applied !

Just replace $\mathcal{A}_F$ by $\mathcal{A}_F^{\text{ext}} = \mathbb{C} \oplus \mathcal{A}_F = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with

$$\pi_F^{\text{ext}}(\lambda, \mu, q, m) = [\tilde{q}_\lambda, \tilde{q}, \mu 1_2 \oplus 1_2 \otimes m, \mu 1_2 \oplus 1_2 \otimes m] \otimes 1_3$$

and Ω_F^1 by

$$(\Omega_F^1)^{\text{ext}} \ni \begin{pmatrix} 0 & Y_0^\dagger \tilde{q}_1 & z_1 p_\nu \otimes M_0^\dagger & 0 \\ \tilde{q}_2 Y_0 & 0 & 0 & 0 \\ z_2 p_\nu \otimes M_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, z_1, z_2 \in \mathbb{C}, q_1, q_2 \in \mathbb{H}$$

- Only satisfies *weak* order 1 condition.
- The compatible finite Dirac are $\Phi(q) + \Phi(q)^o + \sigma(zM_0)$.
- $\mathcal{B}_{\text{SM}}^{\text{ext}} = \mathcal{B}(M) \hat{\otimes} \mathcal{B}_F^{\text{ext}}$ has the same automorphism group as $\mathcal{B}_{\text{SM}}$.
- Its configuration space contains: SM fields + anomalous $X + Z'_{B-L}$ + 1 complex scalar $\sigma(zM_0)$, + flavour changing ζ_{other}.
- All fields apart from ζ_{other}, are now fluctuations.

$\Rightarrow$ The Connes-Lott action can be used on this model.

Let $\mathcal{B} = (\mathcal{A}, \ldots, \Omega^1)$ be a background. Then the J-*symmetrized background* is $\hat{\mathcal{B}} = (\hat{\mathcal{A}}, \ldots, \hat{\Omega}^1)$ with

- $\mathcal{A} = [\pi(\mathcal{A}), \pi(\mathcal{A})^o]$, $\hat{\pi} = \mathrm{Id}$, $\Omega^1 = [\Omega^1, (\Omega^1)^o]$,
- same J, same χ.

Let D be a Dirac and $\hat{\omega} = \sum_i \hat{a}_i[D, \hat{b}_i] \in \hat{\Omega}^1$. Then define $d_D\hat{\omega} = \sum_i [D, \hat{a}_i][D, \hat{b}_i]$.

- Well-defined up to a “junk 2-form”.
- The curvature of $\hat{\omega}$ is $\rho_D(\hat{\omega}) = d_D\hat{\omega} + \hat{\omega}^2$.
- For an AC background $\mathcal{B}$, the Connes-Lott action

$$S_{\mathrm{CL}}(D_\omega) = -\int_M \mathrm{Tr}\left\{P_{\mathrm{junk}^\perp}(\rho_D(\hat{\omega}))^\times P_{\mathrm{junk}^\perp}(\rho_D(\hat{\omega})\right\} \mathrm{vol}_g$$

  is gauge-invariant.
- If C_1 holds, the space of fluctuated Diracs D_ω is gauge-invariant.
- It is gauge-invariant by accident in the case of $\mathcal{B}_{SM}^{\mathrm{ext}}$.

One gets

$$\begin{aligned}\mathcal{L}_{\mathrm{CL}} &= -160\frac{N}{3}\mathbb{F}^{Y}_{\mu\nu}\mathbb{F}^{Y\,\mu\nu} - 32N\mathbb{F}^{W}_{\mu\nu a}\mathbb{F}^{W\,\mu\nu a} - 32N\mathbb{F}^{C}_{\mu\nu a}\mathbb{F}^{C\,\mu\nu a}\\ &\quad -\frac{64}{3}NF^{Z'}_{\mu\nu}F^{Z'\,\mu\nu} - \frac{128}{3}N\mathbb{F}^{Y}_{\mu\nu}F^{Z'\,\mu\nu} + 16a|D_\mu H|^2 + 8b|D_\mu z|^2\\ &\quad -8V_0(|H|^2-1)^2 - 8W_0(|z|^2-1)^2 - 16K(|H|^2-1)(|z|^2-1)\end{aligned}$$

Normalization of kinetic terms: $\mathbb{B}^{Y}_{\mu} = \frac{1}{2}g_Y Y_\mu$, $\mathbb{B}^{W\,a}_{\mu} = \frac{1}{2}g_w W^a_\mu$, $\mathbb{B}^{C\,a}_{\mu} = \frac{1}{2}g_s G^a_\mu$, $Z'_\mu = \frac{1}{2}g_{Z'}\hat{Z}'_\mu$, $H = k\tilde{H}$, $z = l\tilde{z}$, with

$$g_w^2 = g_s^2 = \frac{5}{3}g_Y^2 = \frac{2}{3}g_{Z'}^2 = \frac{1}{32N}, \quad \kappa = 64\tfrac{N}{3}g_Y g_{Z'} = \sqrt{\tfrac{2}{5}}$$

$$k^2 = \frac{1}{16a}, \qquad l^2 = \tfrac{1}{8b}$$

$$\begin{aligned}M_W^2 &= \frac{1}{k^2}g_w^2\\ &= \frac{1}{4}\frac{1}{32N}32\mathrm{Tr}(Y_eY_e^\dagger + Y_\nu Y_\nu^\dagger + 3M_u + 3M_d)\\ &= \frac{1}{4N}\sum \text{squared Dirac masses of fermions}\end{aligned}$$

In particular for $N = 3$, one obtains $M_{\mathrm{top}} \leq 2M_W$.

- Spectral Triples are inadequate for theories including gravity because. . .
- . . . obviously you have to allow D to vary.
- . . . but even if you do, things go astray:
  - The configuration space is too large.
  - The automorphism group almost never comes out right.
  - The differential structure is lost with D.

With the algebraic background framework:

- The differential structure stays in the picture through Ω^1.
- It keeps the config space under control (though it's still larger than in GR).
- symmetries exactly correspond to those of tetradic GR,
- variable=Dirac operator (fluctuations are not needed anymore),
- there are unexpected payoffs: Z'_{B-L} and σ !

Work to do:

- What are the SA prediction with this model (Euclidean signature) ?
- What is the exact role of the centralizing fields ?
- Hint towards a link with unimodularity: in Pati-Salam X is the only centralizing gauge field.

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