The Levi-Civita connection in noncommutative Riemannian geometry: the quantum group case

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Horizon Europe Weeks meeting on

'Noncommutative Geometry and Physics: Quantum Spacetimes'

Krakow

01.12.2021

Bicovariant Bimodules (Hopf Modules)

(H, Δ, ϵ, S) Hopf algebra over a field $\Bbbk$ with invertible antipode.
Sweedler's notation: $\Delta(a) =: a_1 \otimes a_2 \in H \otimes H$

Definition (${}^H_H\mathcal{M}^H_H$)

A bicovariant H-bimodule M is

1. an H-bimodule: $M \in {}_H\mathcal{M}_H$
2. an H-bicomodule: $(M, \lambda_M, \delta_M) \in {}^H\mathcal{M}^H$
3. compatibility: $\lambda_M(a \cdot m \cdot b) = \Delta(a) \cdot \lambda_M(m) \cdot \Delta(b)$, $\delta_M(a \cdot m \cdot b) = \Delta(a) \cdot \delta_M(m) \cdot \Delta(b)$

Sweedler's notation $\lambda_M(m) =: m_{-1} \otimes m_0 \in H \otimes M$ and $\delta_M(m) =: m_0 \otimes m_1 \in M \otimes H$.
Then 3. reads

$$(a \cdot m \cdot b)_{-1} \otimes (a \cdot m \cdot b)_0 = a_1 m_{-1} b_1 \otimes a_2 \cdot m_0 \cdot b_2$$
$$(a \cdot m \cdot b)_0 \otimes (a \cdot m \cdot b)_1 = a_1 \cdot m_0 \cdot b_1 \otimes a_2 m_1 b_2$$

Woronowicz Category: $\otimes_H, \mathrm{Hom}_H$

Theorem

$({}^H_H\mathcal{M}^H_H, \otimes_H, H)$ *is a monoidal category which is*

1. *closed monoidal with respect to the internal* Hom*-functor* $\mathrm{Hom}_H(M, N)$
2. *braided monoidal with respect to the braiding* $\sigma^{\mathcal{W}}_{M,N} \colon M \otimes_H N \to N \otimes_H M$ *defined by* $\sigma^{\mathcal{W}}_{M,N}(m \otimes_H n) := {}^{\alpha}n \otimes_H {}_{\alpha}m := m_{-2} n_0 S(n_1) \otimes_H S(m_{-1}) m_0 n_2$

Theorem (Majid)

$$\otimes_{\sigma^{\mathcal{W}}} \colon \mathrm{Hom}_H(M, M') \otimes_H \mathrm{Hom}_H(N, N') \to \mathrm{Hom}_H(M \otimes_H N, M' \otimes_H N')$$

defined by
$(\phi \otimes_{\sigma^{\mathcal{W}}} \psi)(m \otimes_H n) := \phi({}^{\alpha}m) \otimes_H ({}_{\alpha}\psi)(n) = \phi(\psi_{-2} m_0 S(m_1)) \otimes_H S(\psi_{-1})\psi_0(m_2 n)$ *is*

1. *a morphism in* ${}^H_H\mathcal{M}^H_H$
2. *an associative operation*
3. $\phi \otimes_{\sigma^{\mathcal{W}}} \psi = (\phi \otimes_{\sigma^{\mathcal{W}}} \mathrm{id}_{N'}) \circ (\mathrm{id}_M \otimes_{\sigma^{\mathcal{W}}} \psi)$

Monoidal Structure

Theorem

$({}^H_H\mathcal{M}^H_H, \otimes_{\mathbb{k}}, \mathbb{k})$ *is a monoidal category.*

Proof.

$M, N \in {}^H_H\mathcal{M}^H_H$, then $M \otimes N \in {}^H_H\mathcal{M}^H_H$ via

$$\begin{aligned} a \cdot (m \otimes n) \cdot b :=& (a \cdot m) \otimes (n \cdot b) \\ \lambda_{M \otimes N}(m \otimes n) :=& m_{-1} n_{-1} \otimes (m_0 \otimes n_0) \\ \delta_{M \otimes N}(m \otimes n) :=& (m_0 \otimes n_0) \otimes m_1 n_1 \end{aligned}$$

□

- It is *not* closed monoidal
- It is *not* braided monoidal

Closed Structure

For $M, N \in {}^H_H\mathcal{M}^H_H$ and $\phi\colon \mathrm{Hom}_{\Bbbk}(M, N)$ define $(a \cdot \phi \cdot b) \in \mathrm{Hom}_{\Bbbk}(M, N)$ via

$$(a \cdot \phi \cdot b)(m) := a \cdot \phi(b \cdot m),$$

$\lambda^{\mathrm{Ad}}\colon \mathrm{Hom}_{\Bbbk}(M, N) \to \mathrm{Hom}_{\Bbbk}(M, H \otimes N)$, $\delta^{\mathrm{Ad}}\colon \mathrm{Hom}_{\Bbbk}(M, N) \to \mathrm{Hom}_{\Bbbk}(M, N \otimes H)$ via

$$\begin{aligned}\lambda^{\mathrm{Ad}}(\phi)(m) :=& \phi(m_0)_{-1}\overline{S}(m_{-1}) \otimes \phi(m_0)_0\\ \delta^{\mathrm{Ad}}(\phi)(m) :=& \phi(m_0)_0 \otimes \phi(m_0)_1 S(m_1).\end{aligned}$$

We call $\mathrm{HOM}_{\Bbbk}(M, N) := (\lambda^{\mathrm{Ad}})^{-1}(H \otimes \mathrm{Hom}_{\Bbbk}(M, N)) \cap (\delta^{\mathrm{Ad}})^{-1}(\mathrm{Hom}_{\Bbbk}(M, N) \otimes H)$ the rational morphisms.

Theorem (Caenepeel-Guedenon '05)

$({}^H_H\mathcal{M}^H_H, \mathrm{HOM}_{\Bbbk})$ *is closed category, i.e.* $\mathrm{HOM}_{\Bbbk}(M, N) \in {}^H_H\mathcal{M}^H_H$ *for all* $M, N \in {}^H_H\mathcal{M}^H_H$.

Proposition

If M, N are in ${}^H_H\mathcal{M}^H_H$ we have $\mathrm{Hom}_H(M, N) \subseteq \mathrm{HOM}_{\Bbbk}(M, N)$ *and* $\mathrm{Con}_H(M) \subseteq \mathrm{HOM}_{\Bbbk}(M, M \otimes_H \Omega)$.

Lifting of Woronowicz Braiding

For $M, N \in {}^H_H\mathcal{M}^H_H$ we define $\sigma_{M,N} \colon M \otimes N \to N \otimes M$ via

$$\sigma_{M,N}(m \otimes n) := {}^{\alpha}n \otimes {}_{\alpha}m := m_{-2}n_0 S(m_{-1}n_1) \otimes m_0 n_2.$$

Lemma

$\sigma_{M,N}$ is a morphism in ${}^H\mathcal{M}^H_H$, satisfying the hexagon equation

$$\sigma_{M\otimes N,O} = (\sigma_{M,O} \otimes \mathrm{id}_N) \circ (\mathrm{id}_M \otimes \sigma_{N,O})$$

and lifts the Woronowicz braiding

$$\begin{array}{ccc} M \otimes N & \xrightarrow{\sigma_{M,N}} & N \otimes M \\ \otimes_H \downarrow & & \downarrow \otimes_H \\ M \otimes_H N & \xrightarrow{\sigma^{\mathcal{W}}_{M,N}} & N \otimes_H M \end{array}$$

Note that σ is *not* a braiding

Braided Tensor Product of Rational Morphisms

For $M, M', N, N' \in {}^H_H\mathcal{M}^H_H$ we define an operation

$$\otimes_\sigma \colon \mathrm{HOM}_\Bbbk(M, M') \otimes \mathrm{HOM}_\Bbbk(N, N') \to \mathrm{HOM}_\Bbbk(M \otimes N, M' \otimes N')$$

$$(\phi \otimes_\sigma \psi)(m \otimes n) := \phi({}^\alpha m) \otimes ({}_\alpha\psi)(n) = \phi(\psi_{-2} m_0 S(\psi_{-1} m_1)) \otimes \psi_0(m_2 n),$$

where $\phi \in \mathrm{HOM}_\Bbbk(M, M')$, $\psi \in \mathrm{HOM}_\Bbbk(N, N')$.

Theorem

The map $\otimes_\sigma$

1. *is a morphism of bicovariant H-bimodules*
2. *is an associative operation*
3. *decomposes as* $\phi \otimes_\sigma \psi = (\phi \otimes_\sigma \mathrm{id}_{N'}) \circ (\mathrm{id}_M \otimes_\sigma \psi)$
4. *descends to*

$$\hat{\otimes} \colon \mathrm{HOM}_\Bbbk(M, M') \otimes \mathrm{HOM}_\Bbbk(N, N') \to \mathrm{HOM}_\Bbbk(M \otimes_H N, M' \otimes_H N')$$

an associative morphism in ${}^H_H\mathcal{M}^H_H$ *such that* $\phi \hat{\otimes} \psi = \phi \otimes_{\sigma^{\mathcal{W}}} \psi$ *for* $\phi \in \mathrm{Hom}_H(M, M')$ *and* $\psi \in \mathrm{Hom}_H(N, N')$

Bicovariant Differential Calculi

Definition

A FODC (Ω, d) on H is said to be bicovariant if Ω is a bicovariant H-bimodule and the coactions are compatible with the differential d.

Theorem (Woronowicz)

$$\left\{\textit{Bicovariant FODC on } H\right\} \xleftrightarrow{1:1} \left\{\mathcal{R} \subseteq \ker \epsilon \subseteq H \textit{ right ideal s.t. } \mathrm{Ad}(\mathcal{R}) \subseteq \mathcal{R} \otimes H \right\}$$

Diagonalizable Braiding

Fix (Ω, d) bicovariant FODC on H.
Assume that $\sigma^{\mathcal{W}} \colon \Omega \otimes_H \Omega \to \Omega \otimes_H \Omega$ is diagonalizable with eigenspaces V_λ. Then

1. $\Omega^{\otimes_H 2} \cong \Omega^{\otimes_S 2} \oplus \Omega^{\wedge 2}$ in ${}^H_H\mathcal{M}^H_H$, where
   1. $\Omega^{\otimes_S 2} := V_1 = \{\omega \in \Omega^{\otimes_H 2} \mid \sigma^{\mathcal{W}}(\omega) = \omega\}$ "$\sigma^{\mathcal{W}}$-symmetric 2-tensors"
   2. $\Omega^{\wedge 2} := \bigoplus_{\lambda \neq 1} V_\lambda$ "2-forms"
2. $P_{\mathrm{sym}} := \prod_{\lambda \neq 1} \frac{\sigma^{\mathcal{W}} - \lambda \cdot \mathrm{id}}{1 - \lambda} \colon \Omega^{\otimes_H 2} \to \Omega^{\otimes_S 2}$ and $P_\wedge := \mathrm{id} - P_{\mathrm{sym}} \colon \Omega^{\otimes_H 2} \to \Omega^{\wedge 2}$ are projections

Vector fields $\mathfrak{X} := \mathrm{Hom}_H(\Omega, H)$ with (non-degenerate) dual pairing $\langle \cdot, \cdot \rangle \colon \mathfrak{X} \otimes_H \Omega \to H$
Extend "onion-like" $\langle X_1 \otimes_H \ldots \otimes_H X_k, \omega_1 \otimes_H \ldots \otimes_H \omega_k \rangle := \langle X_1 \langle \ldots \langle X_k, \omega_1 \rangle \ldots \rangle, \omega_K \rangle$
Morphism $A \colon \Omega^{\otimes_H k} \to \Omega^{\otimes_H k}$ in ${}^H_H\mathcal{M}^H_H$ induces morphism $A^* \colon \mathfrak{X}^{\otimes_H k} \to \mathfrak{X}^{\otimes_H k}$ via
$\langle A^*(X_1 \otimes_H \ldots \otimes_H X_k), \omega_1 \otimes_H \ldots \otimes_H \omega_k \rangle = \langle X_1 \otimes_H \ldots \otimes_H X_k, A(\omega_1 \otimes_H \ldots \otimes_H \omega_k) \rangle$

3. $(\sigma^{\mathcal{W}}_{\Omega,\Omega})^* = \sigma^{\mathcal{W}}_{\mathfrak{X},\mathfrak{X}}$
4. $\sigma^{\mathcal{W}}_{\mathfrak{X},\mathfrak{X}}$ is diagonalizable with eigenspaces V^*_λ
5. $\langle \cdot, \cdot \rangle_S \colon \mathfrak{X}^{\otimes_S 2} \otimes_H \Omega^{\otimes_S 2} \to H$ is a non-degenerate dual pairing

Connections

M, N bicovariant H-bimodules, (Ω, d) bicovariant FODC on H

1. On M there is an H-bicovariant right connection

$$\mathrm{d}^M \colon M \to M \otimes \Omega$$
$$m \mapsto m_0 S(m_1) \otimes_H \mathrm{d}m_2$$

2. For every right connection $\nabla \colon M \to M \otimes_H \Omega$ there is a unique $\phi \in \mathrm{Hom}_H(M, M \otimes_H \Omega)$ such that $\nabla = \mathrm{d}^M + \phi$
3. All connections on M are rational morphisms via $\delta^{\mathrm{Ad}}(\nabla) = \mathrm{d}^M \otimes 1 + \delta^{\mathrm{Ad}}(\phi)$ and $\lambda^{\mathrm{Ad}}(\nabla) = 1 \otimes \mathrm{d}^M + \lambda^{\mathrm{Ad}}(\phi)$
4. On Ω there is a torsion-free right connection $\nabla^c \colon \Omega \to \Omega \otimes_H \Omega$ determined by $\nabla^c(\omega^i) := -\omega^j \otimes_H \omega^k \mathrm{C}^i_{kj}$, where $\{\omega^i\}$ is a basis of ${}^{\mathrm{co}H}\Omega$ and C^i_{kj} are the structure constants $\mathrm{d}\omega^i = \mathrm{C}^i_{kj} \omega^j \wedge \omega^k$

Sum of Connections

Theorem (Sum of Connections, Aschieri-W.)

$\nabla^M \oplus \nabla^N := \sigma_{23}^{\mathcal{W}} \circ (\nabla^M \hat{\otimes} \mathrm{id}) + \mathrm{id} \hat{\otimes} \nabla^N \colon M \otimes_H N \to (M \otimes_H N) \otimes_H \Omega$ *is a right connection on* $M \otimes_H N$.

Explicitly

$$(\nabla^M \oplus \nabla^N)(m \otimes_H n) = \sigma_{23}^{\mathcal{W}}(\nabla^M(m_0 S(m_1)) \otimes_H m_2 n) + \nabla^N{}_{-2} m_0 S(\nabla^N{}_{-1} m_1) \otimes_H \nabla^N_0 (m_2 n)$$

Generalizes

1. classical sum of connection
2. sum of bimodule connections (Michor-DuboisViolette)
3. braided-symmetric sum of connections with $(H, \mathcal{R})$ triangular

Notice that

$$\nabla^M \oplus \nabla^N = \sigma_{23}^{\mathcal{W}} \circ ((\nabla^M - \mathrm{d}^M) \otimes_{\sigma^{\mathcal{W}}} \mathrm{id}_N) + \mathrm{id}_M \otimes_{\sigma^{\mathcal{W}}} (\nabla^N - \mathrm{d}^N) + \mathrm{d}^{M \otimes_H N}$$

Levi-Civita Connections

Definition (Metric)

An element $\mathbf{g} = \mathbf{g}^i \otimes_H \mathbf{g}_i \in \mathfrak{X}^{\otimes_H 2}$ is said to be a (pseudo-Riemannian) metric if

1. $\sigma^{\mathcal{W}}(\mathbf{g}) = \mathbf{g}$, i.e. if $\mathbf{g} \in \mathfrak{X}^{\otimes_S 2}$
2. $\mathbf{g}^\# \colon \Omega^1 \ni \omega \mapsto \mathbf{g}^i \langle \mathbf{g}_i, \omega \rangle \in \mathfrak{X}$ is a right H-linear isomorphism

If $\sigma^{\mathcal{W}}$ is diagonalizable every right connection $\nabla \colon \Omega^{\otimes_H 2} \to \Omega^{\otimes_H 2} \otimes_H \Omega^1$ determines and is determined by right connections

$$\begin{aligned}\nabla_S &= (P_{\mathrm{sym}} \otimes_H \mathrm{id}) \circ \nabla|_{\Omega^{\otimes_S 2}} \colon \Omega^{\otimes_S 2} \to \Omega^{\otimes_S 2} \otimes_H \Omega^1,\\ \nabla_\wedge &= (P_\wedge \otimes_H \mathrm{id}) \circ \nabla|_{\Omega^{\wedge 2}} \colon \Omega^{\wedge 2} \to \Omega^{\wedge 2} \otimes_H \Omega^1\end{aligned}$$

and right H-linear maps

$$\begin{aligned}\nabla_{12} &= (P_{\mathrm{sym}} \otimes_H \mathrm{id}) \circ \nabla|_{\Omega^{\wedge 2}} \colon \Omega^{\wedge 2} \to \Omega^{\otimes_S 2} \otimes_H \Omega^1,\\ \nabla_{21} &= (P_\wedge \otimes_H \mathrm{id}) \circ \nabla|_{\Omega^{\otimes_S 2}} \colon \Omega^{\otimes_S 2} \to \Omega^{\wedge 2} \otimes_H \Omega^1\end{aligned}$$

Definition (Levi-Civita)

A right connection $\nabla \colon \Omega^1 \to \Omega^1 \otimes_H \Omega^1$ is called Levi-Civita, if $\mathrm{Tor}^{\nabla} = 0$ and $\nabla_S(\mathbf{g}) = 0$.

Characterization

Let $\mathbf{g} \in \mathfrak{X}^{\otimes_S 2}$ be a pseudo-Riemannian metric. Define

$$\Phi_{\mathbf{g}} \colon \mathrm{Hom}_H(\Omega^1, \Omega^1 \otimes_S \Omega^1) \to \mathrm{Hom}_H(\Omega^1 \otimes_S \Omega^1, \Omega^1)$$

for any $\phi \in \mathrm{Hom}_H(\Omega^1, \Omega^1 \otimes_S \Omega^1)$ by

$$\Phi_{\mathbf{g}}(\phi) := \langle \mathbf{g}, \cdot \otimes_H \cdot \rangle \circ \phi_S,$$

where $\phi_S := (P_{\mathrm{sym}} \otimes_H \mathrm{id}_{\Omega^1}) \circ (\phi \oplus \phi)|_{\Omega^1 \otimes_S \Omega^1} \colon \Omega^1 \otimes_S \Omega^1 \to (\Omega^1 \otimes_S \Omega^1) \otimes_H \Omega^1$.

Theorem

If $\Phi_{\mathbf{g}}$ is an isomorphism of $\mathbb{k}$-modules then

$$\nabla^{\mathrm{LC}} := \nabla^c + \Phi_{\mathbf{g}}^{-1}\left(\left(\mathrm{d} \circ \langle \mathbf{g}, \cdot \otimes_H \cdot \rangle - \langle \mathbf{g}, \nabla^c(\cdot \otimes_H \cdot) \rangle\right)\Big|_{\Omega^1 \otimes_S \Omega^1}\right)$$

is the unique Levi-Civita connection for $\mathbf{g}$.

Existence and Uniqueness Results

We call a pseudo Riemannian metric $\mathbf{g}$

1. central, if $\mathbf{g} \cdot a = a \cdot \mathbf{g}$ for all $a \in H$
2. quasi-central, if $\mathbf{g} = f \cdot \mathbf{g}_c$ for a central metric $\mathbf{g}_c$ and an invertible $f \in H$

Theorem (Aschieri-W.)

For every quasi-central metric there is a unique Levi-Civita connection.

Proof.

- central case:

$$
\begin{array}{ccc}
\mathrm{Hom}_H(\Omega^1, \Omega^1 \otimes_S \Omega^1) \cong (\Omega^1 \otimes_S \Omega^1) \otimes_H \mathfrak{X} & \xrightarrow{\mathrm{id}_{\Omega^1 \otimes_S \Omega^1} \otimes_\sigma \mathcal{W}\, \mathbf{g}^{\#-1}} & (\Omega^1 \otimes_S \Omega^1) \otimes_H \Omega^1 \\
\Big\downarrow {\scriptstyle \frac{1}{2}\Phi_{\mathbf{g}}} & & \Big\downarrow {\scriptstyle P^{23}_{\mathrm{sym}}} \\
\mathrm{Hom}_H(\Omega^1 \otimes_S \Omega^1, \Omega^1) \cong \Omega^1 \otimes_H (\Omega^1 \otimes_S \Omega^1)^* & \xleftarrow{\mathrm{id}_{\Omega^1} \otimes_\sigma \mathcal{W}\, \mathbf{g}^{\#2}} & \Omega^1 \otimes_H (\Omega^1 \otimes_S \Omega^1)
\end{array}
$$

- quasi-central case: $\Phi_{f\mathbf{g}_c} = \ell_f \circ \Phi_{\mathbf{g}_c}$

□

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Thank you for your attention!