Towards Noncommutative Fibre Bundles

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based on joint work with
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Goals

- Develop a framework for investigations of noncommutative analogs of compact fibre bundles.
- Build a class of interesting examples.
- Include both topological (using C^*-algebraic objects) and differential (using purely algebraic objects) aspects.
- Go beyond principal bundles.
- Save whatever possible from local triviality.

Motivating classical example

- Principal bundle
$$G \to M \to B$$
G compact Lie group

  G acts freely on M

  $B \cong M/G$ the orbit space

- $H \leq G$ subgroup
$$G/H \to M/H \to B$$
locally trivial bundle

  with fibre the homogeneous space G/H

P algebra
C coalgebra

$\varrho : P \to P \otimes C$ coalgebra coaction

$P^{\mathrm{co}\, C} := \{b \in P \mid \forall p \in P \ \varrho(bp) = b\varrho(p)\}$ subalgebra of *coinvariants*

Set $B = P^{\mathrm{co}\, C}$. The extension of algebras $B \subseteq P$ is said to be *coalgebra-Galois* or *C-Galois* if the *canonical Galois map*

$$\mathrm{can} : P \underset{B}{\otimes} P \longrightarrow P \otimes C, \qquad p \otimes q \longmapsto p\varrho(q),$$

is bijective.

A coalgebra-Galois extension has an additional symmetry arising from the *canonical entwining map*:

$$\psi : C \otimes P \longmapsto P \otimes C, \qquad c \otimes p \mapsto \mathrm{can}\left(\mathrm{can}^{-1}(1 \otimes c)\, p\right).$$

ψ is a device which records transfer of the right P-module structure on $P \otimes_B P$ to $P \otimes C$ in such a way that the canonical Galois map is a homomorphism of P-bimodules. Explicitly, we have

$$(p \otimes c)\cdot q := \mathrm{can}\left(\mathrm{can}^{-1}(p \otimes c)\, q\right) = p\psi(c \otimes q), \quad c \in C,\ p, q \in P,$$

A C-Galois extension $B \subseteq P$ is said to be *copointed* (or *e-copointed*) if

$$\varrho(1) = 1 \otimes e,$$

for a (necessarily) group-like element $e \in C$. We have

$$B = P_e^{\mathrm{co}\, C} := \{b \in P \mid \varrho(b) = b \otimes e\}.$$

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An e-pointed C-Galois extension $B \subseteq P$ is called *principal coalgebra extension* provided:

- the canonical entwining map is bijective, and
- P is a C-equivariantly projective left B-module, i.e. there exists a left B-module and right C-comodule splitting of the multiplication map $B \otimes P \to P$.

If ψ is bijective, then P is also a left C-comodule with the coaction $\lambda : P \to C \otimes P$ such that

$$\lambda(p) = \psi^{-1}(p \otimes e), \quad \text{for all } p \in P.$$

Given a principal coalgebra extension $B \subseteq P$, as above, we consider a coalgebra D and a coalgebra morphism

$$\pi : C \to D.$$

Since $e \in C$ group-like, hence $\bar{e} := \pi(e)$ group-like in D.
Coaction ϱ of C on P gives rise to coaction $\bar{\varrho}$ of D on P,

$$\bar{\varrho} : P \longmapsto P \otimes D, \qquad \bar{\varrho} = (\mathrm{id} \otimes \pi) \circ \varrho,$$

and then one may consider the $\bar{e}$-coinvariants,

$$A := P_{\bar{e}}^{\mathrm{co}\, D} = \{a \in P \mid \bar{\varrho}(a) = a \otimes \bar{e}\}.$$

NC-bundles with homogeneous fibres 2

A is left B-submodule of P that contains B,
A plays the role of the total space of the bundle with homogeneous fibres X, where

$$X := C_{\bar{e}}^{\mathrm{co}\, D} = \{x \in C \mid (\mathrm{id} \otimes \pi) \circ \Delta(x) = x \otimes \bar{e}\}.$$

At this level of generality, X does not need to be algebra, but it is a left C coideal, i.e. $\Delta(X) \subseteq C \otimes X$, which reflects the homogeneity of the underlying object. This, in particular, allows us to consider the cotensor products:

NC-bundles with homogeneous fibres 3

$$P \,\square_C X := \left\{ \sum_i p_i \otimes x_i \in P \otimes X \mid \sum_i \varrho(p_i) \otimes x_i = \sum_i p_i \otimes \Delta(x_i) \right\}$$

$$P \,\square_D X := \left\{ \sum_i p_i \otimes x_i \in P \otimes X \mid \sum_i \bar{\varrho}(p_i) \otimes x_i = \sum_i p_i \otimes ((\pi \otimes \mathrm{id}) \circ \Delta(x_i)) \right\}$$

We have $P \,\square_C X \subseteq P \,\square_D X$.

Since all elements of B are coinvariant, the left B-action on $P \otimes X$ restricts to the actions on $P \,\square_C X$ and $P \,\square_D X$.

Theorem 1. Let $B \subseteq P$ be a principal coalgebra C-extension. Let $\pi : C \to D$ be a coalgebra morphism and A and X the coinvariants of the induced coactions, as above. Then

- The coaction ϱ restricts to the isomorphism of left B-modules

$$A \cong P \,\square_C X.$$

- A is a projective left B-module.
- The canonical map $\mathrm{can} : P \otimes_B P \to P \otimes C$ restricts to the isomorphism of left P-modules

$$P \underset{B}{\otimes} A \cong P \otimes X.$$

- $\mathrm{can}: P \otimes_B P \to P \otimes C$ restricts to the isomorphism

$$\overline{A} \underset{B}{\otimes} A \cong {}^{\mathrm{co}\, D}(P \otimes X)_{\bar{e}},$$

where

$$\overline{A} = \{a \in P \mid (\pi \otimes \mathrm{id}) \circ \lambda(a) = \bar{e} \otimes a\},$$

and the coinvariants on the RHS are calculated with respect to the left coaction $\Lambda : P \otimes X \to D \otimes P \otimes X$ such that

$$\Lambda = (\pi \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\psi^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta).$$

Comments.

- The first statement says that A is a module of sections of a fibre bundle associated to the principal bundle represented by P and heuristically A is fibered by X.
- The second statement means that we can indeed interpret A as (sections of) a bundle over (the space represented by) B.
- It is possible for A to be an algebra if some additional conditions of technical nature are imposed. In particular, this is also the case and even more can be said when there is a background symmetry encoded by a Hopf algebra, as below.
- Under some additional assumptions (see below) we have $\overline{A} = A$.

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Theorem 2. In addition, assume that H is a Hopf algebra with a bijective antipode such that

- P is a right H-comodule algebra with coaction $\delta : P \to P \otimes H$;
- C is a right H-module coalgebra, that is the right H-action on C satisfies the conditions, for all $h \in H$ and $c \in C$,

$$\Delta_C(c \cdot h) = \Delta_C(c) \cdot \Delta_H(h) \quad \text{and} \quad \varepsilon_C(c \cdot h) = \varepsilon_C(c)\varepsilon_H(h);$$

- D is a right H-module and $\pi : C \to D$ is a right H-module homomorphism;
- the canonical Galois map is a right P-module homomorphism, when $P \otimes C$ is equipped with the diagonal right P-action,

$$(p \otimes c) \cdot q = (p \otimes c) \cdot \delta(q), \qquad p, q \in P \text{ and } c \in C.$$

Then

- A is a subalgebra of P containing B.
- The canonical Galois map restricts to the isomorphism:

$$A \underset{B}{\otimes} A \cong P \,\square_D X.$$

Comments.

- The situation of $H = C$, $\varrho = \delta$ and $e = 1_H$, so that $\bar{e} = \pi(1_H)$, is a special case.
- Both Theorems 1 and 2 are purely algebraic. However, C^*-algebras pop up naturally in examples. It is not obvious how to axiomatize the relationship between the purely algebraic setting and its analytic, C^*-algebraic counterpart.

The quantum flag manifold 1

From the principal bundle

$$U(2) \longrightarrow SU(3) \longrightarrow \mathbb{C}P^2$$

taking fiber-by-fiber quotient by $\mathbb{T}^2$, the maximal torus of $U(2)$, we get

$$\mathbb{C}P^1 \longrightarrow SU(3)/\mathbb{T}^2 \longrightarrow \mathbb{C}P^2$$

with $SU(3)/\mathbb{T}^2$ the full flag manifold of $SU(3)$.

This classical setting admits different noncommutative deformations.

We start with the quantum principal bundle

$$U_q(2) \longrightarrow SU_q(3) \longrightarrow \mathbb{C}P^2_q$$

Considering the classical subgroup $\mathbb{T}^2$ of $U_q(2)$ we get a noncommutative fibre bundle:

$$\mathbb{C}P^1_q \longrightarrow SU_q(3)/\mathbb{T}^2 \longrightarrow \mathbb{C}P^2_q$$

$\mathbb{C}P^1_q$ and $\mathbb{C}P^2_q$ are the quantum complex projective spaces of Vaksman and Soibelman

Note: In this case, we have a Hopf algebra coaction of $\mathcal{O}(\mathbb{T}^2)$.

A. Carotenuto and R. Ó Buachalla (work in progress) produced a sweeping generalization of this example to a large class of quantum flag manifolds.

The quantum flag manifold 3

Another application of Theorem 2, with the background symmetry provided by $H = C = U_q(2)$, yields the following noncommutative fibre bundle:

$$\mathbb{C}P^1_{q,s} \longrightarrow FM_{q,s} \longrightarrow \mathbb{C}P^2_q$$

$FM_{q,s}$ the full flag manifold of $SU_q(3)$

$\mathbb{C}P^1_{q,s}$ the generic (ie two-parameter deformation) of the quantum complex projective 1-space

Note: In this case, we only have a coalgebra coaction of $\mathcal{O}(\mathbb{T}^2)$.

The quantum twistor bundle 1 (Mikkelsen, Sz)

We start with the quantum instanton bundle of Bonechi, Ciccoli, Dąbrowski and Tarlini

$$SU_q(2) \longrightarrow S_q^7 \longrightarrow S_q^4$$

$SU_q(2)$ quantum $SU(2)$ group of Woronowicz

S_q^7 the quantum 7-sphere of Vaksman and Soibelman

S_q^4 the quantum 4-sphere of Bonechi, Ciccoli and Tarlini, with $C(S_q^4)$ the minimal unitization of the compacts

The quantum twistor bundle 2 (Mikkelsen, Sz)

Applying Theorem 2 with $D = U(1)$, the maximal torus of $C = SU_q(2)$, and $H = U_q(4)$, we get the quantum twistor bundle

$$\mathbb{C}P^1_q \longrightarrow \mathbb{C}P^3_q \longrightarrow S^4_q$$

$\mathbb{C}P^1_q$ the standard Podleś sphere

$\mathbb{C}P^3_q$ quantum complex projective 3-space, same as the one constructed by Vaksman and Soibelman on the C^*-algebra level, but different from it on the polynomial algebra level

Note: Starting from different quantum instanton bundles (several constructions exist in the literature), one may obtain different quantum twistor bundles.

Final comments

- Large classes of interesting examples are currently under construction, including quantum flag manifolds (Carotenuto and Ó Buachalla) and quantum lens spaces (Mikkelsen).
- There is a very interesting but not yet fully understood interplay between the purely algebraic and the C^*-algebraic setting. This problem needs further investigations.
- In the C^*-algebraic setting, several natural questions arise. For example, existence of a K-theoretic Gysin sequence for quantum sphere bundles.
- The question of going beyond fibres with homogeneous spaces remains wide open.

References

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