What is Braided Quantum Field Theory?

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Noncommutative Gauge Theory

- After over 20 years of intensive work, there are still many open general problems in the description and quantization of noncommutative gauge theories (e.g. those arising in string theory with non-constant Poisson or twisted Poisson structures)

- Failure of Leibniz rule: $\mathrm{d}(f \star g) \neq \mathrm{d}f \star g + f \star \mathrm{d}g$ obstructs a good noncommutative differential calculus, and in particular closure of gauge transformations: $[\delta^\star_{\lambda_1}, \delta^\star_{\lambda_2}]A \neq \delta^\star_{[\lambda_1 \overset{\star}{,} \lambda_2]}A$

- ***L_∞*-algebras** offer a natural arena for systematic constructions of noncommutative gauge theories that deal with these issues — but so far not understood beyond "semi-classical (Poisson) level"
(Blumenhagen, Brunner, Kupriyanov & Lüst '18; Kupriyanov & Sz '21)

Braided Field Theory

- In some cases this can be rectified by deforming the L_∞-algebra itself: **Braided L_∞-algebras** construct braided field theories equivariant under a triangular Hopf algebra action, with braided noncommutative fields (Dimitrijević Ćirić, Giotopoulos, Radovanović & Sz '21)

- Notion of braided gauge symmetry is not new — kinematical aspects of this idea have appeared before (Brzezinski & Majid '92; ...) — ideas and techniques borrowed from twisted noncommutative gravity (Aschieri *et al.* '05; ...)

- Explicit realizations in physics? Look at Hopf algebraic symmetries of string amplitudes ... (Asakawa, Mori & Watamura '08); Braided deformations underlie AdS/CFT dual gauge theories to Yang-Baxter deformations of $\mathrm{AdS}_5 \times S^5$ string σ-models (van Tongeren '15)

Braided Quantum Field Theory

- **Quantization?** Oeckl's algebraic approach to braided QFT based on braided Wick's Theorem and Gaussian integration — but does not treat theories with gauge symmetries (Oeckl '99)

- **Goals:** Apply modern incarnation of Batalin-Vilkovisky (BV) quantization (à la Costello-Gwilliam) to conventional noncommutative field theories

  Develop braided version which completely captures perturbative braided QFT with explicit computations of correlation functions

- Avoid functional analytic complications of continuum field theories $\Longrightarrow$ work with *fuzzy* field theories (i.e. finite-dimensional, algebraic BV formalism)

Outline

- BV Quantization
- Example: Scalar Field Theory on the Fuzzy Sphere
- Braided BV Formalism
- Example: Braided Scalar Field Theory on the Fuzzy Torus

with Hans Nguyen and Alexander Schenkel [arXiv: 2107.02532]

Free BV Field Theory $(E, Q_0, \langle -,-\rangle)$

- Graded vector space

  $E \;=\; \cdots \oplus E^{-1} \oplus E^{0} \oplus E^{1} \oplus \cdots \;=\; \text{ghosts} \oplus \text{fields} \oplus \text{antifields}$

  $Q_0 : E \longrightarrow E$ differential of degree 1 $(Q_0^2 \;=\; 0)$

  $\langle -,-\rangle : E \otimes E \longrightarrow \mathbb{C}$ non-degenerate graded antisymmetric of degree -1 and Q_0-invariant (-1-shifted symplectic structure)

- Describes *derived* space of free fields

- Observables $\big(\mathrm{Sym}\, E^* \simeq \mathrm{Sym}\, E[1], Q_0, \{-,-\}\big)$: Shifted Poisson bracket $\{\varphi,\psi\} \;=\; \langle\varphi,\psi\rangle\,\mathbb{1}$ for $\varphi,\psi \in E[1]$ defines a P_0-algebra:

$$
\begin{aligned}
-Q_0\{\varphi,\psi\} &= \{Q_0\varphi,\psi\} + (-1)^{|\varphi|}\,\{\varphi, Q_0\psi\} && \text{Leibniz rule}\\
\{\varphi,\psi\} &= (-1)^{|\varphi|\,|\psi|}\,\{\psi,\varphi\} && \text{symmetric}\\
\{\varphi,\{\psi,\chi\}\} &= \pm\{\psi,\{\chi,\varphi\}\} \pm \{\chi,\{\varphi,\psi\}\} && \text{Jacobi identity}\\
\{\varphi,\psi\,\chi\} &= \{\varphi,\psi\}\,\chi \pm \psi\,\{\varphi,\chi\} && \text{Leibniz rule}
\end{aligned}
$$

L_∞-Algebras

- Extend cochain complex $(E[-1], Q_0)$ by antisymmetric maps $\{\ell_n : E[-1]^{\otimes n} \longrightarrow E[-1]\}_{n\geq 2}$ to form an L_∞-algebra:

$$Q_0\ell_2(v,w) \;=\; \ell_2(Q_0 v, w) \pm \ell_2(v, Q_0 w) \qquad \text{Leibniz rule}$$

$$\ell_2(v,\ell_2(w,u)) + \text{cyclic} \;=\; (Q_0 \circ \ell_3 \pm \ell_3 \circ Q_0)(v,w,u) \quad \text{Jacobi up to homotopy}$$

  plus "higher homotopy Jacobi identities"

- *Cyclic* with respect to pairing $\langle -,- \rangle : E[-1] \otimes E[-1] \longrightarrow \mathbb{C}$:

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n) \rangle \;=\; \pm \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1}) \rangle$$

- (Cyclic) L_∞-algebras are homotopy coherent generalizations of (quadratic) Lie algebras

- Extended L_∞-algebra on $(\mathrm{Sym}\, E[1]) \otimes E[-1]$:

$$\ell_n^{\rm ext}(a_1 \otimes v_1, \dots, a_n \otimes v_n) \;=\; \pm\, a_1 \cdots a_n \otimes \ell_n(v_1, \dots, v_n)$$

$$\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle_{\rm ext} \;=\; \pm\, a_1\, a_2\, \langle v_1, v_2 \rangle$$

Interacting BV Field Theory

- Interactions $I \in \mathrm{Sym}\, E[1]$ incorporated by choosing dual bases $\varepsilon_\alpha \in E[-1]$, $\varrho^\alpha \in E[-1]^* \simeq E[2]$ and 'contracted coordinate functions' $\mathsf{a} = \varrho^\alpha \otimes \varepsilon_\alpha \in (\mathrm{Sym}\, E[1]) \otimes E[-1]$

- Homotopy Maurer-Cartan Action:

$$\lambda\, I = \sum_{n\geq 2} \frac{\lambda^{n-1}}{(n+1)!}\, \langle \mathsf{a}, \ell_n^{\mathrm{ext}}(\mathsf{a},\dots,\mathsf{a})\rangle_{\mathrm{ext}} \in \mathrm{Sym}\, E[1]$$

$$S_{\mathrm{BV}} = \langle \mathsf{a}, Q_0(\mathsf{a})\rangle_{\mathrm{ext}} + \lambda\, I = \text{BV action}$$

- (Classical) Master Equation: $Q_0(\lambda\, I) + \frac{1}{2}\,\{\lambda\, I, \lambda\, I\} = 0$

- $Q_{\mathrm{int}}^2 = 0$ where $Q_{\mathrm{int}} = Q_0 + \{\lambda\, I, -\}$

- Defines P_0-algebra $\big(\mathrm{Sym}\, E[1], Q_{\mathrm{int}}, \{-,-\}\big)$ of observables for interacting BV field theory

Quantum BV Field Theory

- BV Laplacian $\Delta_{\rm BV} : \mathrm{Sym}\, E[1] \longrightarrow (\mathrm{Sym} E[1])[1]$:

$$\Delta_{\rm BV}(\mathbb{1}) \ = \ 0 \ = \ \Delta_{\rm BV}(\varphi) \quad , \quad \Delta_{\rm BV}(\varphi\,\psi) \ = \ \{\varphi,\psi\}$$

$$\Delta_{\rm BV}(a\,b) \ = \ \Delta_{\rm BV}(a)\,b + (-1)^{|a|}\,a\,\Delta_{\rm BV}(b) + \{a,b\}$$

  Implements Gaussian integration/Wick's Theorem

- Satisfies $Q_0\,\Delta_{\rm BV} + \Delta_{\rm BV}\,Q_0 \ = \ 0 \ , \ \Delta_{\rm BV}^2 \ = \ 0 \ , \ \Delta_{\rm BV}(\lambda\, I) \ = \ 0$

- $Q_{\rm BV}^2 \ = \ 0$ where $Q_{\rm BV} \ = \ Q_{\rm int} + \hbar\,\Delta_{\rm BV} \ = \ Q_0 + \{\lambda\, I, -\} + \hbar\,\Delta_{\rm BV}$

- Quantum observables $\big(\mathrm{Sym}\, E[1], Q_{\rm BV}\big)$ (E_0-algebra) for interacting BV field theory

Homological Perturbation Theory

- Propagators determine strong deformation retracts of $E^* \simeq E[1]$:

$$\begin{tikzcd} \big(H^\bullet(E[1]),0\big) \arrow[r, shift left, "\iota"] & \big(E[1],Q_0\big) \arrow[l, shift left, "\pi"] \arrow[loop above, "\gamma"] \end{tikzcd} \qquad \begin{array}{c} \pi\,\iota = \mathbb{1}\ ,\ \iota\,\pi - \mathbb{1} = Q_0\,\gamma + \gamma\,Q_0 \\ \gamma^2 = 0\ ,\ \gamma\,\iota = 0\ ,\ \pi\,\gamma = 0 \end{array}$$

- Observables: $\begin{tikzcd} \big(\mathrm{Sym}\,H^\bullet(E[1]),0\big) \arrow[r, shift left, "\mathcal{I}"] & \big(\mathrm{Sym}\,E[1],Q_0\big) \arrow[l, shift left, "\Pi"] \arrow[loop above, "\Gamma"] \end{tikzcd}$

- Homological Perturbation Lemma: With $\delta = \{\lambda\,I, -\} + \hbar\,\Delta_{\rm BV}$, there is a strong deformation retract

$$\begin{tikzcd} \big(\mathrm{Sym}\,H^\bullet(E[1]),\widetilde{\delta}\,\big) \arrow[r, shift left, "\widetilde{\mathcal{I}}"] & \big(\mathrm{Sym}\,E[1],Q_{\rm BV}\big) \arrow[l, shift left, "\widetilde{\Pi}"] \arrow[loop above, "\widetilde{\Gamma}"] \end{tikzcd}$$

where $\widetilde{\Pi} = \Pi\,(\mathbb{1} - \delta\,\Gamma)^{-1}\,\delta\,\Gamma = \Pi \circ \sum_{k=1}^{\infty} (\delta\,\Gamma)^k$

- $\widetilde{\Pi}(\varphi_1\cdots\varphi_n) \in \mathrm{Sym}\,H^\bullet(E[1])$ are n-point correlation functions on space of vacua $H^\bullet(E)$ of the field theory

Scalar Field Theory on the Fuzzy Sphere

- Fuzzy sphere: $A = (j) \otimes (j)^* \simeq \mathrm{Mat}(N)$ for spin $j = \frac{N-1}{2}$ irrep of $su(2)$, with generators $[X_i, X_j] = \mathrm{i}\, r_N\, \epsilon_{ijk}\, X_k$, $X_i\, X_i = \mathbb{1}$, $X_i^* = X_i$

- Free BV field theory: $E = E^0 \oplus E^1$ with $E^0 = E^1 = A$

$$Q_0 = \Delta + m^2 \quad \text{with} \quad \Delta(a) = \tfrac{1}{r_N^2}\, [X_i, [X_i, a]] \quad \text{(fuzzy Laplacian)}$$

$$\langle \varphi, \psi \rangle = (-1)^{|\varphi|}\, \tfrac{4\pi}{N}\, \mathrm{Tr}(\varphi\, \psi)$$

- Fuzzy spherical harmonics $Y_j^J \in A$ satisfy

$$\Delta(Y_j^J) = J\,(J+1)\, Y_j^J \quad , \quad \tfrac{4\pi}{N}\, \mathrm{Tr}(Y_j^{J\,*}\, Y_{j'}^{J'}) = \delta_{JJ'}\, \delta_{jj'}$$

- L_∞-algebra: For any $n \geq 2$, choose $\ell_n : E[-1]^{\otimes n} \longrightarrow E[-1]$ as

$$\ell_n(\varphi_1, \dots, \varphi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \varphi_{\sigma(1)} \cdots \varphi_{\sigma(n)}$$

- Interactions: $\lambda\, I = \frac{\lambda^{n-1}}{(n+1)!} \sum_{\{J_i, j_i\}} I^{J_0 \cdots J_n}_{j_0 \cdots j_n}\, Y_{j_0}^{J_0\,*} \cdots Y_{j_n}^{J_n\,*} \in \mathrm{Sym}\, E[1]$

  $I^{J_0 \cdots J_n}_{j_0 \cdots j_n} = \langle Y_{j_0}^{J_0}, \ell_n(Y_{j_1}^{J_1}, \dots, Y_{j_n}^{J_n}) \rangle \in \mathbb{C}$ symmetric under neighbour swaps (Wigner $3j$ and $6j$ symbols (Chu, Madore & Steinacker '01; ...))

Scalar Field Theory on the Fuzzy Sphere

- Deformation retract: $H^\bullet(E[1]) = 0$ for $m^2 > 0$:

$$\begin{tikzcd} (0,0) \arrow[r, shift left, "0"] & (E[1], Q_0) \arrow[l, shift left, "0"] \arrow[loop above, "-G"] \end{tikzcd} \qquad G = Q_0^{-1} = (\Delta + m^2)^{-1}$$

- Correlation functions: $\begin{tikzcd} (\mathbb{C}, 0) \arrow[r, shift left, "\widetilde{\mathcal{I}}"] & (\mathrm{Sym}\, E[1], Q_{\rm BV}) \arrow[l, shift left, "\widetilde{\Pi}"] \arrow[loop above, "\widetilde{\Gamma}"] \end{tikzcd}$

  Only $\Pi(\mathbb{1}) = 1$ is non-zero (because $\pi = 0$)

- **Example:** 2-point function at 1-loop in ϕ^4-theory ($n = 3$):

$$\widetilde{\Pi}(\varphi_1\,\varphi_2) = \Pi\big(\delta\,\Gamma(\varphi_1\,\varphi_2) + (\delta\,\Gamma)^2(\varphi_1\,\varphi_2) + (\delta\,\Gamma)^3(\varphi_1\,\varphi_2)\big)$$

$$= -\hbar\,\langle\varphi_1, G(\varphi_2)\rangle$$

$$- \frac{\lambda^2\,\hbar^2}{2} \sum_{\{J_i, j_i\}} \frac{I^{J_1\,JJJ_2}_{j_1 jjj_2}}{J\,(J+1) + m^2}\,\langle Y^{J_1}_{j_1}{}^*, G(\varphi_1)\rangle\,\langle Y^{J_2}_{j_2}{}^*, G(\varphi_2)\rangle + O(\lambda^4)$$

- Receives both planar and non-planar loop corrections as in conventional perturbation theory (Chu, Madore & Steinacker '01), due to L_∞-structure of I

Representations of Triangular Hopf Algebras

- **Idea:** BV formalism/L_∞-algebras are defined in the category of vector spaces, but the definitions make sense in any (closed abelian) symmetric monoidal category (with non-trivial braiding isomorphism) and define braided BV formalism/braided L_∞-algebras
- In particular, there is a subcategory of vector spaces which are (left) modules for a fixed given triangular Hopf algebra H (morphisms are H-equivariant maps)
- Universal R-matrix: $R \ = \ R^\alpha \otimes R_\alpha \in H \otimes H$ is *triangular* if
$$R^{-1} \ = \ R_{21} \ = \ R_\alpha \otimes R^\alpha$$
- Braiding isomorphism $\tau_R : V \otimes W \longrightarrow W \otimes V$:
$$\tau_R(v \otimes w) \ = \ (R_\alpha \triangleright w) \otimes (R^\alpha \triangleright v)$$
Symmetric if R is triangular: $\tau_R^2 \ = \ \mathbb{1}$

Free Braided BV Field Theory $(E, Q_0, \langle -,- \rangle)$

- $\mathbb{Z}$-graded H-module E

  $Q_0 : E \longrightarrow E$ H-equivariant differential of degree 1

  $\langle -,- \rangle : E \otimes E \longrightarrow \mathbb{C}$ H-invariant non-degenerate braided graded antisymmetric of degree -1 and Q_0-invariant:

  $$\langle \varphi, \psi \rangle = -(-1)^{|\varphi|\,|\psi|} \, \langle R_\alpha \triangleright \psi, R^\alpha \triangleright \varphi \rangle$$

- Braided symmetric algebra $\mathrm{Sym}_R E[1]$:

  $$\varphi\,\psi = (-1)^{|\varphi|\,|\psi|} \, (R_\alpha \triangleright \psi)\,(R^\alpha \triangleright \varphi)$$

- Observables $\big(\mathrm{Sym}_R E[1], Q_0, \{-,-\}\big)$ defines a braided P_0-algebra:

$$-Q_0\{\varphi,\psi\} = \{Q_0\varphi,\psi\} + (-1)^{|\varphi|}\,\{\varphi, Q_0\psi\} \qquad \text{Leibniz rule}$$

$$\{\varphi,\psi\} = (-1)^{|\varphi|\,|\psi|}\,\{R_\alpha \triangleright \psi, R^\alpha \triangleright \varphi\} \qquad \text{braided symmetric}$$

$$\begin{aligned}\{\varphi,\{\psi,\chi\}\} &= \pm\,\{R_\alpha \triangleright \psi, \{R_\beta \triangleright \chi, R^\beta\,R^\alpha \triangleright \varphi\}\} \\ &\quad \pm\,\{R_\beta\,R_\alpha \triangleright \chi, \{R^\beta \triangleright \varphi, R^\alpha \triangleright \psi\}\} \qquad \text{braided Jacobi identity}\end{aligned}$$

$$\{\varphi,\psi\,\chi\} = \{\varphi,\psi\}\,\chi \pm (R_\alpha \triangleright \psi)\,\{R^\alpha \triangleright \varphi, \chi\} \qquad \text{braided Leibniz rule}$$

Braided L_∞-Algebras

- Extend cochain complex $(E[-1], Q_0)$ by H-equivariant braided antisymmetric maps $\{\ell_n : E[-1]^{\otimes n} \longrightarrow E[-1]\}_{n\geq 2}$ to form a braided L_∞-algebra:

$$\ell_n(\dots, v, v', \dots) \ = \ -(-1)^{|v|\,|v'|}\ \ell_n(\dots, R_\alpha \triangleright v', R^\alpha \triangleright v, \dots)$$

  plus braided homotopy Jacobi identities (unchanged for $n = 2$)

- Braided L_∞-algebras are homotopy coherent generalizations of braided Lie algebras (Woronowicz '89; Majid '93; ...)

- Braided cyclic with respect to $\langle -, - \rangle : E[-1] \otimes E[-1] \longrightarrow \mathbb{C}$:

$$\langle v_0, \ell_n(v_1, \dots, v_n)\rangle \ = \ \pm \langle R_{\alpha_0} \cdots R_{\alpha_{n-1}} \triangleright v_n, \ell_n(R^{\alpha_0} \triangleright v_0, \dots, R^{\alpha_{n-1}} \triangleright v_{n-1})\rangle$$

- Extended braided L_∞-algebra $\{Q_0, \ell_n^{\rm ext}\}$ on $(\mathrm{Sym}_R E[1]) \otimes E[-1]$:

$$\langle a_1 \otimes v_1, a_2 \otimes v_2\rangle_{\rm ext} \ = \ \pm\, a_1\, (R_\alpha \triangleright a_2)\, \langle R^\alpha \triangleright v_1, v_2\rangle \quad \text{etc.}$$

Braided Quantum Field Theory

- Interactions: With $\mathsf{a} = \varrho^\alpha \otimes \varepsilon_\alpha \in (\mathrm{Sym}_R E[1]) \otimes E[-1]$:

$$\lambda\, I \ = \ \sum_{n\geq 2} \frac{\lambda^{n-1}}{(n+1)!}\, \langle \mathsf{a}, \ell_n^{\rm ext}(\mathsf{a},\dots,\mathsf{a})\rangle_{\rm ext} \ \in \ \mathrm{Sym}_R E[1]$$

- Braided P_0-algebra $\big(\mathrm{Sym}_R E[1], Q_{\rm int} \ = \ Q_0 + \{\lambda\, I, -\}, \{-,-\}\big)$ of observables for interacting braided BV field theory

- Braided BV Laplacian $\Delta_{\rm BV} : \mathrm{Sym}_R E[1] \longrightarrow (\mathrm{Sym}_R E[1])[1]$:

$$\Delta_{\rm BV}(\varphi_1\cdots\varphi_n) \ = \ \sum_{i<j} \pm \langle \varphi_i, R_{\alpha_{i+1}}\cdots R_{\alpha_{j-1}} \triangleright \varphi_j\rangle \\ \times \ \varphi_1\cdots\varphi_{i-1}\, \widehat{\varphi}_i\, (R^{\alpha_{i+1}} \triangleright \varphi_{i+1})\cdots(R^{\alpha_{j-1}} \triangleright \varphi_{j-1})\, \widehat{\varphi}_j\, \varphi_{j+1}\cdots\varphi_n$$

  Implements *braided* Gaussian integration/Wick's Theorem (Oeckl '99)

- Braided E_0-algebra $\big(\mathrm{Sym}_R E[1], Q_{\rm BV} \ = \ Q_{\rm int} + \hbar\, \Delta_{\rm BV}\big)$ of quantum observables for interacting braided BV field theory.

Braided Quantum Field Theory

- Braided strong deformation retract:

$$\begin{tikzcd} \big(H^\bullet(E[1]),0\big) \arrow[r, shift left, "\iota"] & \big(E[1],Q_0\big) \arrow[l, shift left, "\pi"] \arrow[loop above, "\gamma"] \end{tikzcd} \qquad \begin{aligned} \pi,\iota &= H\text{-equivariant} \\ \gamma &= H\text{-invariant} \end{aligned}$$

- Applying Homological Perturbation Lemma to H-invariant $\delta \ = \ \{\lambda\, I, -\} + \hbar\, \Delta_{\rm BV}$ gives braided strong deformation retract

$$\begin{tikzcd} \big(\mathrm{Sym}_R H^\bullet(E[1]),\widetilde{\delta}\,\big) \arrow[r, shift left, "\widetilde{\mathcal{I}}"] & \big(\mathrm{Sym}_R E[1],Q_{\rm BV}\big) \arrow[l, shift left, "\widetilde{\Pi}"] \arrow[loop above, "\widetilde{\Gamma}"] \end{tikzcd}$$

where $\widetilde{\Pi} \ = \ \Pi \circ \sum\limits_{k=1}^\infty (\delta\,\Gamma)^k$

- Braided homological perturbation theory computes correlation functions of braided quantum field theory

Braided Scalar Field Theory on the Fuzzy Torus

- Fuzzy torus $A \simeq \mathrm{Mat}(N)$: $a = \sum_{i,j\in\mathbb{Z}_N} a_{ij}\, U^i\, V^j$ with generators obeying:

  $$U\,U^* = V\,V^* = \mathbb{1} \quad , \quad U\,V = q\,V\,U \quad , \quad U^N = V^N = \mathbb{1}$$

  where $q = \mathrm{e}^{2\pi\,\mathrm{i}\,/N}$; $\mathrm{Tr}(a) = a_{00}$ defines a trace on A

- Group Hopf algebra $H = \mathbb{C}[\mathbb{Z}_N^2]$ acts on A:

  $$\underline{k} \triangleright U = q^{k_1}\, U \quad , \quad \underline{k} \triangleright V = q^{k_2}\, V \qquad \text{where} \quad \underline{k} = (k_1, k_2) \in \mathbb{Z}_N^2$$

- Triangular R-matrix: $R = \frac{1}{N^2} \sum_{\underline{s},\underline{t}\in\mathbb{Z}_N^2} q^{\underline{s}\wedge\underline{t}}\, \underline{s} \otimes \underline{t} = R^\alpha \otimes R_\alpha \in H \otimes H$

  A is a braided commutative H-module algebra: $a\,b = (R_\alpha \triangleright b)\,(R^\alpha \triangleright a)$

- Free braided BV field theory: $E = E^0 \oplus E^1$ with $E^0 = E^1 = A$

  $$Q_0 = \Delta + m^2 \ , \ \Delta(a) = -\frac{1}{(q^{1/2}-q^{-1/2})^2} \big(\big[U, [U^*, a]\big] + \big[V, [V^*, a]\big]\big)$$

  $$\langle \varphi, \psi \rangle = (-1)^{|\varphi|}\, \mathrm{Tr}(\varphi\, \psi)$$

Braided Scalar Field Theory on the Fuzzy Torus

- Fuzzy plane waves $e_{\underline{k}} = U^{k_1} V^{k_2} \in A$ satisfy

$$\Delta(e_{\underline{k}}) = \big([k_1]_q^2 + [k_2]_q^2\big)\, e_{\underline{k}} \quad , \quad \mathrm{Tr}(e_{\underline{k}}^* \, e_{\underline{l}}) = \delta_{\underline{k},\underline{l}}$$

where $[n]_q = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}$ (q-numbers)

- Braided L_∞-algebra: For any $n \geq 2$, choose $\ell_n : E[-1]^{\otimes n} \longrightarrow E[-1]$ as

$$\ell_n(\varphi_1, \dots, \varphi_n) = \varphi_1 \cdots \varphi_n$$

- Interactions: $\lambda\, I = \frac{\lambda^{n-1}}{(n+1)!} \sum_{\{\underline{k}_i\}} I_{\underline{k}_0 \cdots \underline{k}_n}\, e_{\underline{k}_0}^* \cdots e_{\underline{k}_n}^* \in \mathrm{Sym}_R E[1]$

$$I_{\underline{k}_0 \cdots \underline{k}_n} = q^{\sum_{i<j} \underline{k}_i \wedge \underline{k}_j} \langle e_{\underline{k}_0}, \ell_m(e_{\underline{k}_1}, \dots, e_{\underline{k}_n})\rangle = q^{-\sum_{i<j} k_{i1}\, k_{j2}}\, \delta_{\underline{k}_0 + \cdots + \underline{k}_n, \underline{0}}$$

q-symmetric under neighbour swaps: $I_{\cdots \underline{k}_i \underline{k}_{i+1} \cdots} = q^{\underline{k}_i \wedge \underline{k}_{i+1}}\, I_{\cdots \underline{k}_{i+1} \underline{k}_i \cdots}$

- Deformation retract: $H^\bullet(E[1]) = 0$ for $m^2 > 0$:

$$\begin{tikzcd} (0,0) \arrow[r, shift left, "0"] & (E[1], Q_0) \arrow[l, shift left, "0"] \arrow[loop above, "-G"] \end{tikzcd} \qquad G = Q_0^{-1} = (\Delta + m^2)^{-1}$$

Braided Scalar Field Theory on the Fuzzy Torus

- Correlation functions: Only $\Pi(\mathbb{1}) = 1$ is non-zero
- **Example 1:** 4-point function of free braided scalar field ($I = 0$):

$$\widetilde{\Pi}(\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4) \;=\; \Pi\big((\hbar\,\Delta_{\mathrm{BV}}\,\Gamma)^2(\varphi_1\,\varphi_2\,\varphi_3\,\varphi_4)\big)$$
$$=\; \hbar^2\,\big(\langle\varphi_1, G(\varphi_2)\rangle\,\langle\varphi_3, G(\varphi_4)\rangle + \langle\varphi_1, R_\alpha \triangleright G(\varphi_3)\rangle\,\langle R^\alpha \triangleright \varphi_2, G(\varphi_4)\rangle + \langle\varphi_1, G(\varphi_4)\rangle\,\langle\varphi_2, G(\varphi_3)\rangle\big)$$

Braided Wick's Theorem (Oeckl '99)

- **Example 2:** 2-point function at 1-loop in ϕ^4-theory ($n = 3$):

$$\widetilde{\Pi}(\varphi_1\,\varphi_2) \;=\; \Pi\big(\delta\,\Gamma(\varphi_1\,\varphi_2) + (\delta\,\Gamma)^2(\varphi_1\,\varphi_2) + (\delta\,\Gamma)^3(\varphi_1\,\varphi_2)\big)$$
$$=\; -\hbar\,\langle\varphi_1, G(\varphi_2)\rangle - \frac{\lambda^2\,\hbar^2}{2}\sum_{\underline{k},\underline{l}\in\mathbb{Z}_N^2}\frac{\langle e^*_{\underline{k}}, G(\varphi_1)\rangle\,\langle e_{\underline{k}}, G(\varphi_2)\rangle}{[l_1]_q^2 + [l_2]_q^2 + m^2} + O(\lambda^4)$$

No notion of non-planar loop corrections due to *braided* L_∞-structure of I; No UV/IR mixing in continuum? (Oeckl '00; Balachandran *et al.* '06)