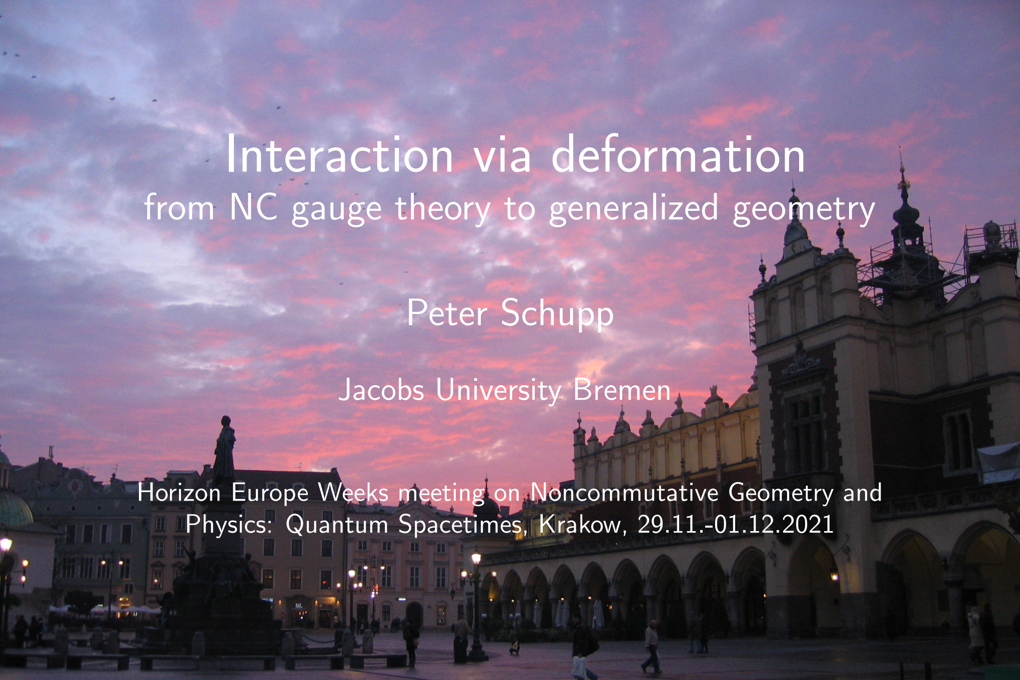


Interaction via deformation

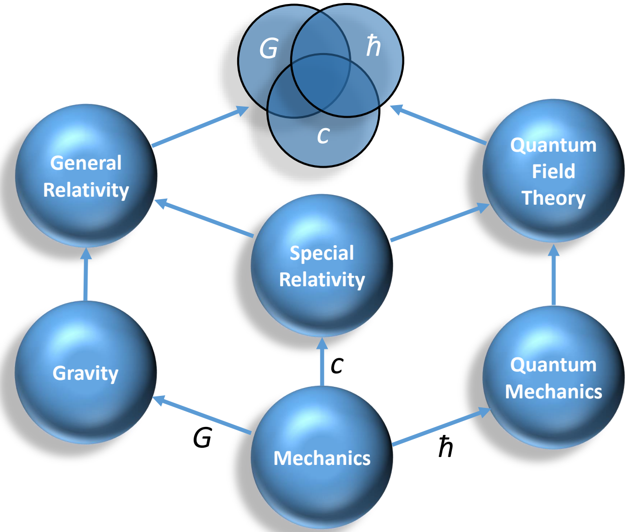
from NC gauge theory to generalized geometry

Peter Schupp

Jacobs University Bremen

Horizon Europe Weeks meeting on Noncommutative Geometry and Physics: Quantum Spacetimes, Krakow, 29.11.-01.12.2021

"Geometry" $\longrightarrow$ Noncommutative/Generalized Geometry $\longleftarrow$ "Algebra"



deformation and unification $G, c, \hbar$ – plus Boltzmann's k

Interaction via deformation

- gravity = free fall in curved spacetime
  $\rightarrow$ extend this idea to all forces!

- $\rightarrow$ free Hamiltonian, interaction via deformation:
  deformed symplectic structure (or operator algebra)

[, ] $\neq 0$

- gauge theory recovered via Moser's lemma:
  deformation maps are not unique $\Rightarrow$ gauge symmetry
- works also in a graded setting, allows magnetic sources

Outline

- Noncommutative gauge theory (general, review)
- Effective actions for open strings and branes
- Graded/generalized geometry and gravity actions

Noncommutativity in electrodynamics and string theory

- electron in constant magnetic field $\vec{B} = B\hat{e}_z$:

$$\mathcal{L} = \frac{m}{2}\dot{\vec{x}}^2 - e\dot{\vec{x}} \cdot \vec{A} \qquad \text{with} \qquad A_i = -\frac{B}{2}\epsilon_{ij}x^j$$

$$\lim_{m\to 0} \mathcal{L} = e\frac{B}{2}\dot{x}^i\epsilon_{ij}x^j \qquad \Rightarrow \qquad [\hat{x}^i, \hat{x}^j] = \frac{2i}{eB}\epsilon^{ij}$$

- bosonic open strings in constant B-field

$$S_\Sigma = \frac{1}{4\pi\alpha'}\int_\Sigma \left(g_{ij}\partial_a x^i \partial^a x^j - 2\pi i\alpha' B_{ij}\epsilon^{ab}\partial_a x^i \partial_b x^j\right)$$

in low energy limit $g_{ij} \sim (\alpha')^2 \to 0$:

$$S_{\partial\Sigma} = -\frac{i}{2}\int_{\partial\Sigma} B_{ij}x^i\dot{x}^j \qquad \Rightarrow \qquad [\hat{x}^i, \hat{x}^j] = \left(\frac{i}{B}\right)^{ij}$$

C-S Chu, P-M Ho (1998); V Schomerus (1999); Seiberg, Witten

Open strings on D-branes in B-field background

$$\langle [x^i(\tau), x^j(\tau')] \rangle = i\theta^{ij}$$

non-commutative string endpoints with $\star$-product depending on θ via

$$\frac{1}{g+B} = \frac{1}{G+\Phi} + \theta \qquad \text{(closed-open string relations)}$$

add fluctuations $B \rightsquigarrow B + F$; depending on regularization scheme:

$$\rightarrow \quad \begin{cases} \text{ordinary gauge theory} & \text{(e.g. Pauli-Villars)} \\ \text{non-commutative gauge theory} & \text{(e.g. point-splitting)} \end{cases}$$

$\Rightarrow$ SW map: commutative $\leftrightarrow$ noncommutative theory (duality)

Star products

Deformation quantization of the point-wise product in the direction of a Poisson bracket $\{f,g\} = \theta^{ij}\partial_i f \cdot \partial_j g$:

$$f \star g = fg + \frac{i\hbar}{2}\{f,g\} + \hbar^2 B_2(f,g) + \hbar^3 B_3(f,g) + \dots\ ,$$

with suitable bi-differential operators B_n such that $\star$ is associative.

There is a natural gauge symmetry: "equivalent star products"

$$\star \mapsto \star'\ , \quad Df \star Dg = D(f \star' g)\ ,$$

will yield a new associative star product for any (invertible) differential operator $Df = f + \hbar D_1 f + \hbar^2 D_2 f + \dots$

Kontsevich formality and star product

U_n maps n k_i-multivector fields to a $(2-2n+\sum k_i)$-differential operator

$$U_n(\mathcal{X}_1,\ldots,\mathcal{X}_n) = \sum_{\Gamma\in G_n} w_\Gamma\, D_\Gamma(\mathcal{X}_1,\ldots,\mathcal{X}_n)\ ,$$

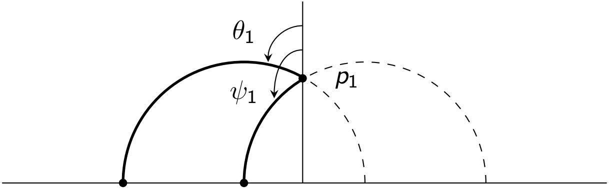
where the sum is over all possible diagrams with weight

$$w_\Gamma = \frac{1}{(2\pi)^{\sum k_i}} \int_{\mathbb{H}_n} \bigwedge_{i=1}^{n} \left(\mathrm{d}\phi^h_{e_i^1} \wedge \cdots \wedge \mathrm{d}\phi^h_{e_i^{k_i}}\right)\ .$$

The star product for a given bivector θ is:

$$f\star g = \Phi(\theta)(f,g) := \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\hbar)^n}{n!}\, U_n(\theta,\ldots,\theta)(f,g)$$

Example constant θ:
The graphs and hence the integrals factorize. The basic graph



yields the weight

$$w_{\Gamma_1} = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\psi_1 \int_0^{\psi_1} d\phi_1 = \frac{1}{(2\pi)^2} \left[\frac{1}{2} (\psi_1)^2 \right]_0^{2\pi} = \frac{1}{2}$$

and the star product turns out to be the Moyal-Weyl one:

$$f \star g = \sum \frac{(i\hbar)^n}{n!} \left(\frac{1}{2} \right)^n \theta^{\mu_1 \nu_1} \ldots \theta^{\mu_n \nu_n} (\partial_{\mu_1} \ldots \partial_{\mu_n} f)(\partial_{\nu_1} \ldots \partial_{\nu_n} g)$$

Formality condition

The U_n define an L_∞ quasi-isomorphisms of DGL algebras and satisfy

$$\mathrm{d}.\, U_n(\mathcal{X}_1, \ldots, \mathcal{X}_n) + \frac{1}{2} \sum_{\substack{\mathcal{I}\sqcup\mathcal{J}=(1,\ldots,n)\\ \mathcal{I},\mathcal{J}\neq\emptyset}} \varepsilon_{\mathcal{X}}(\mathcal{I},\mathcal{J}) \left[U_{|\mathcal{I}|}(\mathcal{X}_{\mathcal{I}})\,,\, U_{|\mathcal{J}|}(\mathcal{X}_{\mathcal{J}})\right]_{\mathrm{G}}$$
$$= \sum_{i<j} (-1)^{\alpha_{ij}}\, U_{n-1}\big([\mathcal{X}_i, \mathcal{X}_j]_{\mathrm{S}}, \mathcal{X}_1, \ldots, \widehat{\mathcal{X}}_i, \ldots, \widehat{\mathcal{X}}_j, \ldots, \mathcal{X}_n\big)\ ,$$

relating Schouten brackets to Gerstenhaber brackets.

Kontsevich (1997)

For $\mathrm{d}_\star = -[-,\star]_G$ and $\mathrm{d}_\theta = -[-,\theta]_S$, FC implies $\mathrm{d}_\star \Phi(\theta) = \mathrm{i}\,\hbar\,\Phi(\mathrm{d}_\theta\theta)$, i.e.

$$\star \text{ associative} \quad \Leftrightarrow \quad \theta \text{ Poisson}$$

Up to gauge equivalence

$$f \star g = f \cdot g + \frac{i}{2} \sum \theta^{ij}\, \partial_i f \cdot \partial_j g - \frac{\hbar^2}{4} \sum \theta^{ij}\theta^{kl}\, \partial_i\partial_k f \cdot \partial_j\partial_l g$$
$$- \frac{\hbar^2}{6} \Big(\sum \theta^{ij} \partial_j \theta^{kl}\, (\partial_i\partial_k f \cdot \partial_l g - \partial_k f \cdot \partial_i\partial_l g) \Big) + \dots \ ,$$

where $\theta = \theta^{ij}\partial_i \otimes \partial_j$ is a Poisson bi-vector.

Global vs local symmetry: The noncommutative theory has global symmetries, like the commutative theory (if θ is transformed as well). But partial derivatives in the star product cause problems with local symmetries.

Could introduce covariant derivatives in $\star$ (as in gauge theories and GR), but this will i.g. break associativity.

$\longrightarrow$ symplectomorphisms, twisted symmetries, or NC gauge transformations

Formality: vector field $\mapsto$ differential operator:

$$\xi = \xi^i(x)\partial_i \quad \mapsto \quad \Xi = \Phi(\xi) := \sum \frac{(i\hbar)^n}{n!} U_{n+1}(\xi, \theta, \ldots, \theta)$$

$$\Xi(f \star g) = \Xi f \star g + g \star \Xi g + f[\mathcal{L}_\xi \star]g$$

The differential operator Ξ_t generates deformed diffeomorphisms.
If ξ is a symplectomorphism, then $\mathcal{L}_\xi \star = 0$.

Twisted symmetry & coproduct $\Rightarrow$ ξ does not "see" the $\star$-product

$$\delta_\xi(f \star g) = \sum \xi_{(1)} f \star \xi_{(2)} g \qquad \delta\xi \equiv \sum \xi_{(1)} \otimes \xi_{(2)}$$

NC gauge transformations are automatically $\star$-derivations

$$\delta F = i(\Lambda \star F - F \star \Lambda) \qquad \delta\Psi = i\Lambda \star \Psi$$

note: $\delta(f \star \Psi) = f \star i\Lambda \star \Psi \neq i\Lambda \star f \star \Psi$

$\Rightarrow$ introduce *covariant coordinates* $\mathcal{D}x^\mu = x^\mu + \hat{A}^\mu$ and *covariant functions*

$$\mathcal{D}f = f + f_A \qquad \text{with} \qquad \delta\mathcal{D}f = i[\Lambda \stackrel{\star}{,} \mathcal{D}f]$$

$\Rightarrow$ equivalent star products

$$\mathcal{D}f \star \mathcal{D}g =: \mathcal{D}(f \star' g)$$

$\Rightarrow$ deformation of $\star$ by covariantizing change of coordinates (SW map)

$$\begin{array}{ccccc} B: & & \Theta & \xrightarrow{\text{quantization}} & \star \\ \text{Moser} \downarrow \rho & & \downarrow \rho & & \downarrow \mathcal{D} \\ B+F: & & \Theta' & \xrightarrow{\text{quantization}} & \star' \end{array}$$

Jurčo, PS, Wess (2001)

Moser's lemma on "nearby symplectic structures"

B: closed ($dB = 0$), non-degenerate ($\theta := B^{-1}$) 2-form
$B' = B + F$, F exact ($F = dA$)

$B_t = B + tF$, non-degenerate, $t \in [0,1]$.

$\Rightarrow B'$ is obtained from B by a change of coordinates.

Proof: Let $\xi_t = \theta_t^{ij} A_j \partial_i$, i.e $i_{\xi_t} B_t = -A$.

$$\Rightarrow \mathcal{L}_{\xi_t} B_t = i_{\xi_t} dB + d i_{\xi_t} B = 0 - dA = -F = -\partial_t B_t \ .$$

integrate the flow generated by $\mathcal{L}_{\xi_t}$ from $t = 0$ to $t = 1$ to obtain a map ρ that depends on A and relates B' to B.

B' is gauge invariant, but the map ρ transforms by a canonical transformation $=$ (semi-classical) NC gauge transformation.

Semi-classical "Poisson" Moser

θ: Poisson bi-vector (can be degenerate) ; $F = dA$
$\theta' = \theta - \theta \cdot F \cdot \theta + \theta \cdot F \cdot \theta \cdot F \cdot \theta - + \ldots$

$\theta_t = \theta \cdot (1 + tF \cdot \theta)^{-1}$, Poisson, $t \in [0,1]$; $\xi_t = -A \cdot \theta_t \cdot \partial$.

$$\Rightarrow \partial_t \theta_t = -\mathcal{L}_{\xi_t} \theta_t = -[\xi_t, \theta_t]_S$$

$$\rho^*(\theta') = \theta \ , \text{ with } \rho^* = \exp(\mathcal{L}_{\xi_t} + \partial_t) \exp(-\partial_t)\big|_{t=0}$$

Gauge transformation: $\delta A = d\lambda$ implies $\delta\rho^*(f) = \{\rho^*(f), \tilde{\lambda}\}$,
where $\tilde{\lambda} = \sum \frac{1}{n!}(\xi_t + \partial_t)^{n+1}(\lambda)\big|_{t=0}$.

$\rho^*(x^\mu) =: x^\mu + \tilde{a}^\mu, \quad \delta\tilde{a}^\mu = \theta^{\mu\nu}\partial_\nu\tilde{\lambda} + \{\tilde{a}^\mu, \tilde{\lambda}\}$, etc.

"Poisson gauge theory"

Quantum Moser (= Seiberg-Witten map)

Start again with the Moser vector field ξ_t. The differential operator

$$\Xi_t = \Phi(\xi_t) := \sum \frac{(i\hbar)^n}{n!} U_{n+1}(\xi_t, \theta, \ldots, \theta)$$

generates deformed diffeomorphisms that can be integrated to a flow $\mathcal{D}$, which is the SW map (exact, to all orders):

Let $\star_t = \sum \frac{i\hbar}{n!} U_n(\theta_t, \ldots \theta_t), \quad \star' = \star_1,$

$$\Rightarrow \partial_t(\star_t) = -[\Xi_t, \star_t]_G$$

$$\mathcal{D}(\star') = \star \ , \text{ with } \mathcal{D} = \exp(\Xi_t + \partial_t) \exp(-\partial_t)\big|_{t=0}$$

Let $\Lambda = \Phi(\lambda) = \sum \frac{(i\hbar)^n}{n!} U_{n+1}(\lambda, \theta, \ldots, \theta)$ and $\tilde{\Lambda} = \sum \frac{1}{n!}(\xi_t + \partial_t)^{n+1}(\Lambda)\big|_{t=0}$.
Gauge transformation $\delta A = d\lambda$ implies $\delta \mathcal{D} f = i[\tilde{\Lambda} \stackrel{\star}{,} \mathcal{D} f]$

NC gauge theory and equivalent star products

NC gauge theory = gauge theory of noncommutativity:

$$\mathcal{D}_{[a]}(f \star' g) = \mathcal{D}_{[a]} f \star \mathcal{D}_{[a]} g$$

Star products $\star$, $\star'$: locally equivalent, globally Morita equivalent.

Finite gauge transformations

classical gauge transformation: $\psi \mapsto \psi_g = g\psi$ and $a \mapsto a_g = a + g d g^{-1}$
gauge equivalence $\Rightarrow$

$$\Psi_{[\psi_g, a_g]} = G_{[g,a]} \star \Psi_{[\psi,a]} \ , \quad \mathcal{D}_{[a_g]}(f) = G_{[g,a]} \star \mathcal{D}_{[a]}(f) \star (G_{[g,a]})^{-1}$$

$$G_{[g_1, a_{g_2}]} \star G_{[g_2,a]} = G_{[g_1 \cdot g_2, a]} \quad \text{(noncommutative group law)}$$

can be used to patch and globalize construction $\Rightarrow$ NC line bundles

It is consistent to use an abbreviated notation

$$G_{jk} \equiv G_{g_{jk}}[a_k], \qquad \mathcal{D}_k \equiv \mathcal{D}_{[a_k]}.$$

The fundamental relations on $U_i \cap U_j \cap U_k$ are

$$G_{ij} \star G_{jk} = G_{ik}, \qquad G_{kj} \star G_{jk} = 1, \qquad \mathcal{D}_j \star G_{jk} = G_{jk} \star \mathcal{D}_k.$$

(There is no summation over j or k in these formulas.)
The G_{jk} play the role of noncommutative transition functions.

Sections: $\Psi = (\Psi_k)$ with $\Psi_j = G_{jk} \star \Psi_k$
bimodule structure: $f.\Psi = (\mathcal{D}_k(f) \star \Psi_k), \quad \Psi.f = (\Psi_k \star f)$
$f.(g.\Psi) = (f \star' g).\Psi$ while $(\Psi.f).g = \Psi.(f \star g)$ etc. ...

Jurco, PS, Wess, *Noncommutative line bundle and Morita equivalence*,
Lett.Math.Phys. 61 (2002) 171-186

Open string effective action

$$\mathcal{S}_{\text{DBI}} = \int d^n x \frac{1}{g_s} \det{}^{\frac{1}{2}}(g + B + F) = \int d^n x \frac{1}{\hat{G}_s} \det{}^{\frac{1}{2}}(\hat{G} + \hat{\Phi} + \hat{F})$$

commutative $\leftrightarrow$ non-commutative duality

Expand to second order, ignore (cosmological) constants $\Rightarrow$

$$\mathcal{S}_{\text{DBI}} = \int d^n x \frac{|-g|^{\frac{1}{2}}}{4 g_s} g^{ij} g^{kl} (B+F)_{ik} (B+F)_{jl} \qquad \text{(Maxwell/Yang-Mills)}$$

$$\mathcal{S}_{\text{DBI}}^{NC} = \int d^n x \frac{|\theta|^{-\frac{1}{2}}}{4 \hat{g}_s} \hat{g}_{ij} \hat{g}_{kl} \{\hat{X}^i, \hat{X}^k\}\{\hat{X}^j, \hat{X}^l\} \qquad \text{(Matrix Model)}$$

Covariant coordinates: $\hat{X}^i = x^i + \hat{A}^i$

Commutative $\leftrightarrow$ non-commutative duality fixes form of action

higher Born Infeld action

open p-brane ($p > 1$) effective action?

$\det[g + B]$ makes no sense for B a $p+1$-form ($p > 1$) ,
but $\det[g + B\tilde{g}^{-1}B^T]$ does, where $\tilde{g}$ is antisymmetrized $g^{\otimes p}$
(B: $(p+1)$-form written as rectangular matrix B_{iJ} with multiindex J)

$$S_{DBI} = \frac{1}{g_m}\int d^{p'+1}x \det{}^{x}[g] \det{}^{y}\big[\underbrace{g + B\tilde{g}^{-1}B^T}_{G}\big]$$

$$= \frac{1}{g_m}\int d^{p'+1}x \det{}^{\frac{1}{2}}[g] \det{}^{y}\big[1 + g^{-1}B\tilde{g}^{-1}B^T\big]$$

$y = ?$
noncommutative “Nambuian” version of this?
Π: Nambu-Poisson $p+1$ multi-vector field

Effective open p-brane action

Miraculous identity

$$\det[g+(B+F)\tilde{g}^{-1}(B+F)^T] = \det{}^2[1-F\Pi^T]\cdot\det[G+(\Phi+F')\tilde{G}^{-1}(\Phi+F')^T]$$

where $F' = (I - F\Pi^T)^{-1}F$, holds for all p.

The Jacobian of the Nambu-Poisson map fixes the appropriate power:

$\Rightarrow$ Effective action (conjecture)

$$S_{p\text{-DBI}} = \int d^{p'+1}x \, \frac{1}{g_m} \det{}^{\mathrm{x}}(g) \cdot \det{}^{\mathrm{y}}\left[g + (C+F)\tilde{g}^{-1}(C+F)^T\right]$$

with $\mathrm{x} = \frac{p}{2(p+1)}$, $\mathrm{y} = \frac{1}{2(p+1)}$

Effective open p-brane action

NC Dual

$$S_{p\text{-NCDBI}} = \int d^{p'+1}x \, \frac{1}{\widehat{G}_m} \frac{\widehat{|\Pi|}^{\frac{1}{p+1}}}{|\Pi|^{\frac{1}{p+1}}} \det{}^{\mathrm{x}}(\widehat{G}) \cdot \det{}^{\mathrm{y}}\big[\widehat{G} + (\widehat{\Phi} + \widehat{F}')\widehat{\tilde{G}}^{-1}(\widehat{\Phi} + \widehat{F}')^{T}\big]$$

$\widehat{}$ denotes objects evaluated at covariant coordinates

$\widehat{F}'$ is the Nambu (NC) field strength

open-closed membrane coupling constants

$$G_m = g_m \left(\frac{\det G}{\det g} \right)^{\frac{p}{2(p+1)}}$$

Expansion of action

ignore a cosmological constant term and let $\mathcal{F} = C + F$

$$\mathcal{S}_{p\text{-DBI}} = \frac{1}{2(p+1)g_m} \det{}^{\frac{1}{2}}(g)\,\mathrm{tr}\left[g^{-1}\mathcal{F}\tilde{g}^{-1}\mathcal{F}^T\right] + \dots$$

the coupling constant g_m is dimensionless for:

- strings on D3 with 2-form field strength (Maxwell/Yang-Mills)
- 2-brane on 5-brane with 3-form field strength ($\rightsquigarrow$ M2-M5 system)
- p-brane on $2(p+1)$-brane with $p+1$ form field strength

consider $p = 2$, $p' = 5$ and expand further ($k = \mathcal{F}_i^{kl}\mathcal{F}_{jkl}$):

$$det^{\frac{1}{6}}(1+k) = \sqrt{1 + \frac{1}{3}\mathrm{tr}k - \frac{1}{6}\mathrm{tr}k^2 + \frac{1}{36}(\mathrm{tr}\,k)^2 + \dots}.$$

$\Rightarrow$ exact match with κ-symmetry computation of
Cederwall, Nilsson, Sundell, "An Action for the superfive-brane" (1998)

Effective open p-brane action

From higher gauge theory to matrix model...

Expanding to lowest order (ignoring a non-cosmological constant) $\Rightarrow$

semi-classical/infinite-dimensional version of a matrix model

$$\int d^{p+1}x \, \frac{1}{|\Pi|^{\frac{1}{p+1}}} \frac{1}{2(p+1)\widehat{g}_m} \cdot \hat{g}_{i_0 j_0} \cdots \hat{g}_{i_p j_p} \{\widehat{X}^{j_0}, \ldots, \widehat{X}^{j_p}\}\{\widehat{X}^{i_0}, \ldots, \widehat{X}^{i_p}\}$$

quantize:

$$\rightsquigarrow \frac{1}{2(p+1)\widehat{g}_m} \operatorname{Tr}\left(\hat{g}_{i_0 j_0} \cdots \hat{g}_{i_p j_p} \left[\widehat{X}^{j_0}, \ldots, \widehat{X}^{j_p}\right] \left[\widehat{X}^{i_0}, \ldots, \widehat{X}^{i_p}\right]\right)$$

Closed string effective action, gravity

Massless bosonic modes

- open strings: A_μ, ϕ^i $\rightarrow$ gauge and scalar fields
- closed strings: $g_{\mu\nu}$, $B_{\mu\nu}$, Φ $\rightarrow$ background geometry, gravity

Closed string effective action

Weyl invariance (at 1 loop) requires vanishing beta functions:

$$\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$$
$$\Downarrow$$
equations of motion for $g_{\mu\nu}$, $B_{\mu\nu}$, Φ
$$\Uparrow$$
closed string effective action

$$\int d^D x |-g|^{\frac{1}{2}} \left(R - \frac{1}{12} e^{-\Phi/3} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{6} \partial_\mu \Phi \partial^\mu \Phi + \dots \right)$$

Noncommutative version of this?

graded Poisson structure

Now try to do the same for gravity! Since the metric $g_{\mu\nu}$ is symmetric (unlike $F^{\mu\nu}$) we need odd variables to deform Poisson brackets with it.

Graded Poisson algebra on $T^*[2] \oplus T[1]M$, deformed by metric:

$$\{\underset{1}{\theta^\mu}, \underset{1}{\theta^\nu}\} = 2\underset{0}{g^{\mu\nu}}(x) \qquad \{\underset{2}{p_\mu}, \underset{0}{x^\nu}\} = \underset{0}{\delta^\nu_\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

Jacobi identity (i.e. associativity) $\Leftrightarrow$ metric connection

$$\{\underset{2}{p_\mu}, \underset{1}{\theta^\alpha}\} = \Gamma^\alpha_{\mu\beta}\underset{1}{\theta^\beta} =: \nabla_\mu \theta^\alpha$$

$$\{p_\mu, \{\theta^\alpha, \theta^\beta\}\} = 2\partial_\mu g^{\alpha\beta} = \{\{p_\mu, \theta^\alpha\}, \theta^\beta\} + \{\theta^\alpha, \{p_\mu, \theta^\beta\}\}$$

and curvature

$$\{\{p_\mu, p_\nu\}, \theta^\alpha\} = [\nabla_\mu, \nabla_\nu]\theta^\alpha = \theta^\beta R_\beta{}^\alpha{}_{\mu\nu}$$

$$\Rightarrow \quad \{\underset{2}{p_\mu}, \underset{2}{p_\nu}\} = \tfrac{1}{4}\underset{1}{\theta^\beta}\underset{1}{\theta^\alpha} R_{\beta\alpha\mu\nu}$$

deformations

our initial example:

deformation by a gauge field A

$$\Omega' = dx^i \wedge dp_i + \tfrac{1}{2} F_{ij}(x) dx^i \wedge dx^j, \; dF = 0, \text{ locally } F = dA$$

$$\boxed{\Omega_t = \Omega + t\, dA}, \quad A = A_i(x) dx^i$$

$$V_t = A_i(x) \frac{\partial}{\partial p_i}, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[A]}(p) = p + A$$

$$\{p_i, x^j\}_t = \delta_i^j$$

$$\{p_i, p_j\}_t = t\, F_{ij}(x)$$

gauge transformation $\delta A = d\lambda \leftrightarrow \delta\rho_{[A]}$: canonical transformation

non-abelian versions: $A_i^\alpha(x) \ell_\alpha dx^i$ and $A_{ia}^b(x) \theta^a \chi_b dx^i$

$\rightsquigarrow$ Abelian and non-abelian gauge theory

deformations

deformation by a spin connection ω

$$\Omega = dx^i \wedge dp_i + \tfrac{1}{2}\eta_{ab} d\theta^a \wedge d\theta^b \qquad \theta^a = e^a_i \theta^i \,, \quad g_{ij} = e^a_i e^b_j \eta_{ab}$$

$$\boxed{\Omega_t = \Omega + t\, d\omega}, \quad \omega = \omega_i(x,\theta) dx^i = \tfrac{1}{2}\omega_{iab}(x)\theta^a\theta^b dx^i$$

$$V_t = \omega_i \partial_{p_i} \,, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[\omega]}(p) = p + \omega$$

$$\{p_i, x^j\}_t = \delta^j_i \qquad \{\theta^a, \theta^b\}_t = \eta^{ab}$$

$$\{p_i, \theta^a\} = t\, \eta^{ab} \omega_{ibc}(x)\, \theta^c \qquad \omega_{ibc} = -\omega_{icb}$$

$$\{p_i, p_j\}_t = t\, R_{ij} \qquad R = d\omega + t\omega \wedge \omega$$

gauge transformation $\delta\omega = d\lambda \leftrightarrow \delta\rho_{[\omega]}$: canonical transformation

$\rightsquigarrow$ Einstein-Cartan gravity

deformations

deformation by a general connection Γ

$$\Omega = dx^i \wedge dp_i + d\theta^i \wedge d\chi_i$$

$$\boxed{\Omega_t = \Omega + t\, d\Gamma}\,, \quad \Gamma = \Gamma_i dx^i = \Gamma_{ij}{}^k(x)\theta^j \chi_k dx^i$$

$$V_t = \Gamma_i \partial_{p_i}\,, \quad \mathcal{L}_{V_t} \rightsquigarrow \rho_{[\Gamma]}(p) = p + \Gamma$$

$$\{p_i, x^j\}_t = \delta_i^j \qquad \{\chi_i, \theta^j\}_t = \delta_i^j$$

$$\{p_i, \theta^j\} = t\, \Gamma^j_{ik}\theta^k \qquad \{p_i, \chi_j\} = -t\, \Gamma^k_{ij}\chi_k$$

$$\{p_i, p_j\}_t = t\, R_k{}^l{}_{ij}\theta^k \chi_l \qquad R_k{}^l{}_{ij} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{ik}\Gamma^l_{jm} - \Gamma^m_{jk}\Gamma^l_{im}$$

gauge transformation $\delta\Gamma = d\Lambda \leftrightarrow \delta\rho_{[\Gamma]}$: canonical transformation

$\rightsquigarrow$ General relativity and alternative gravity theories

Graded geometry

Graded Poisson manifold $T^*[2]T[1]M$

- degree 0: x^i "coordinates"
- degree 1: $\xi^\alpha = (\theta^i, \chi_i)$
- degree 2: p_i "momenta"

symplectic 2-form

$$\omega = dp_i \wedge dx^i + \frac{1}{2} G_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta = dp_i \wedge dx^i + d\chi_i \wedge d\theta^i$$

even (degree -2) Poisson bracket on functions $f(x, \xi, p)$

$$\{x^i, x^j\} = 0, \quad \{p_i, x^j\} = \delta_i^j, \quad \{\xi^\alpha, \xi^\beta\} = G^{\alpha\beta}$$

metric $G^{\alpha\beta}$: natural pairing of TM, T^*M:

$$\{\chi_i, \theta^j\} = \delta_i^j\ , \quad \{\chi_i, \chi_j\} = 0\ , \quad \{\theta^i, \theta^j\} = 0$$

Graded geometry

degree-preserving canonical transformations

- infinitesimal, generators of degree 2:

$$v^\alpha(x)p_\alpha + \tfrac{1}{2}M^{\alpha\beta}(x)\xi_\alpha\xi_\beta \quad \rightsquigarrow \quad \text{diffeos and } o(d,d)$$

- finite, idempotent ("coordinate flip"): $(\tilde{\chi}, \tilde{\theta}) = \tau(\chi, \theta)$ with $\tau^2 = \mathrm{id}$

  $\rightsquigarrow$ generating function F of type 1 with $F(\theta, \tilde{\theta}) = -F(\tilde{\theta}, \theta)$:

$$F = \theta \cdot g \cdot \tilde{\theta} - \tfrac{1}{2}\,\theta \cdot B \cdot \theta + \tfrac{1}{2}\,\tilde{\theta} \cdot B \cdot \tilde{\theta}$$

$$\chi = \frac{\partial F}{\partial \theta} = \tilde{\theta} \cdot g + \theta \cdot B\ , \qquad \tilde{\chi} = -\frac{\partial F}{\partial \tilde{\theta}} = \theta \cdot g + \tilde{\theta} \cdot B$$

$$\Rightarrow \quad \tau(\chi, \theta) = (\chi, \theta) \cdot \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix}$$

  $\rightsquigarrow$ generalized metric

Generalized geometry as a derived structure

degree 3 “Hamiltonian”: Dirac operator

$$\Theta = \xi^\alpha h^i_\alpha(x) p_i \;+\; \underbrace{\tfrac{1}{6} C_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma}_{\text{twisting/flux terms}}$$

For $e = e_\alpha(x)\xi^\alpha \in \Gamma(TM \oplus T^*M)$ (degree 1, odd):

- pairing: $\langle e, e' \rangle = \{e, e'\}$
- anchor: $h(e)f = \{\{e, \Theta\}, f\}$
- bracket: $[e, e']_D = \{\{e, \Theta\}, e'\}$

Generalized geometry as a derived structure

Courant algebroid axioms from associativity and $\{\Theta, \Theta\} = 0$:

$$\begin{aligned} h(\xi_1)\left\langle \xi_2, \xi_2 \right\rangle &= \{\{\Theta, \xi_1\}, \{\xi_2, \xi_2\}\} \\ &= 2\{\{\{\Theta, \xi_1\}, \xi_2\}, \xi_2\} = 2\left\langle [\xi_1, \xi_2], \xi_2 \right\rangle \qquad \text{(axiom 1)} \\ &= 2\{\xi_1, \{\{\Theta, \xi_2\}, \xi_2\}\} = 2\left\langle \xi_1, [\xi_2, \xi_2] \right\rangle \qquad \text{(axiom 2)} \end{aligned}$$

$$\begin{aligned} [\xi_1, [\xi_2, \xi_3]] &= \{\{\Theta, \xi_1\}, \{\{\Theta, \xi_2\}, \xi_3\}\} \\ &= [[\xi_1, \xi_2], \xi_3] + [\xi_2, [\xi_1, \xi_3]] + \frac{1}{2}\{\{\{\{\Theta, \Theta\}, \xi_1\}, \xi_2\}, \xi_3\}. \end{aligned}$$

$\{\Theta, \Theta\} = 0 \quad \Leftrightarrow \quad [\,,]$-Jacobi identity (in 1st slot) (axiom 3)

Deformation

general (deformed) Poisson structure

$$\{v, f\} = v.f$$
$$\{V, W\} = G(V, W) \equiv \langle V, W \rangle$$
$$\{v, V\} = \nabla_v V \qquad \leftarrow \text{connection metric wrt. } G$$
$$\{v, w\} = [v.w]_{\text{Lie}} + R(v, w) \qquad \leftarrow \text{curvature of } \nabla$$

with

- degree 0: $f(x)$
- degree 1: $V = V^\alpha(x)\xi_\alpha$ "generalized vectors"
- degree 2: $v = v^i(x)p_i$ "vector fields"

general Hamiltonian

$$\Theta = \tilde{\xi}^\alpha h(\xi_\alpha) + \frac{1}{6} C_{\alpha\beta\gamma}\tilde{\xi}^\alpha\tilde{\xi}^\beta\tilde{\xi}^\gamma \qquad \leftarrow \text{general flux (H,f,Q,R)}$$

Deformation

derived bracket

$$\{\{\{V,\Theta\},W\},X\} = \langle\nabla_V W, X\rangle - \langle\nabla_W V, X\rangle + \langle\nabla_X V, W\rangle + C(V,W,X)$$

$$\{\{\{\xi_\alpha,\Theta\},\xi_\beta\},\xi_\gamma\} = \underbrace{\Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma}}_{\text{torsion}} + \Gamma_{\gamma\alpha\beta} + C_{\alpha\beta\gamma} =: \Gamma^{\text{new}}_{\gamma\alpha\beta}$$

"mother of all brackets"

$$\begin{aligned}[V,W] &= \nabla_V W - \nabla_W V + \langle\nabla V, W\rangle + C(V,W,-)\\ &= [[V,W]] + T(V,W) + \langle\nabla V, W\rangle + C(V,W,-)\end{aligned}$$

In order to obtain a regular Courant algebroid, impose

$$\{\Theta,\Theta\} = 0 \quad \Leftrightarrow \quad \nabla C + \frac{1}{2}\{C,C\} = 0\,, \quad G^{-1}|_h = 0\,, \ldots$$

Generalized differential geometry

generalized Lie-bracket (involves anchor $h : E \to TM$)

$$[[V,W]] = -[[W,V]]\,, \quad [[V,fW]] = (h(V)f)W + f\,[[V,W]]$$

generalized connection "type I" and miraculous triple identity

$$\Gamma(V;fW,U) = (h(V)f)\langle W,U\rangle + f\Gamma(V;W,U)\,,$$

$$\boxed{\langle V,[W,Z]\rangle = \langle V,[[W,Z]]\rangle + \Gamma(V;W,Z)}$$

$$\langle \nabla_V W,U\rangle := \Gamma(V;W,U)$$

generalized curvature and torsion

$$R(V,W) = \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[[V,W]]}$$

$$T(V,W) = \nabla_V W - \nabla_W V - [[V,W]]$$

Boffo, PS (2019/2020)

Graded/generalized geometry and gravity

cookbook recipe

- deform graded Poisson structure
- pick Hamiltonian Θ (e.g. canonical), compute derived brackets
- choose generalized Lie bracket $[[\,,\,]]$ (e.g. canonical)
- determine connection Γ from triple identity
- project (or rather embed) via non-isotropic splitting (e.g. canonical)

$$s : \Gamma(TM) \to \Gamma(E) \qquad \rho \circ s = \mathrm{id} \qquad \langle X, Y \rangle_{TM} := \langle s(X), s(Y) \rangle$$

$$\langle \nabla_Z X, Y \rangle_{TM} := \Gamma(s(Z); s(X), s(Y))$$

- compute Riemann and Ricci tensors, take trace with $g + B$, write action in terms of resulting Ricci scalar

deformation by generalized vielbein E

$$\Omega = dx^i \wedge dp_i + d\theta^i \wedge d\chi_i$$

deformation by change of coordinates in the odd (degree 1) sector
two choices:

$$\begin{pmatrix}\theta\\ \chi\end{pmatrix} \mapsto \begin{pmatrix}1 & 0\\ g+B & 1\end{pmatrix} \cdot \begin{pmatrix}\theta\\ \chi\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1 & \Pi+G\\ -g+B & 1\end{pmatrix} \cdot \begin{pmatrix}\theta\\ \chi\end{pmatrix}$$

Boffo, PS

now crank the "machine" (deformed derived bracket, connection, project, Riemann, Ricci) $\rightsquigarrow$ (effective) gravity actions ...

Graded/generalized geometry and gravity

generalized Koszul formula for nonsymmetric $\mathcal{G} = g + B$

$$\begin{aligned} 2g(\nabla_Z X, Y) &= \langle Z, [X, Y]' \rangle' \\ &= X\mathcal{G}(Y, Z) - Y\mathcal{G}(X, Z) + Z\mathcal{G}(X, Y) \\ &\quad -\mathcal{G}(Y, [X, Z]_{\text{Lie}}) - \mathcal{G}([X, Y]_{\text{Lie}}, Z) + \mathcal{G}(X, [Y, Z]_{\text{Lie}}) \\ &= 2g(\nabla^{LC}_X Y, Z) + H(X, Y, Z) \end{aligned}$$

$\Rightarrow$ non-symmetric Ricci tensor

$$R_{jl} = R^{LC}_{jl} - \frac{1}{2}\nabla^{LC}_i H_{jl}{}^i - \frac{1}{4} H_{lm}{}^i H_{ij}{}^m \qquad R = \mathcal{G}_{ij} g^{ik} g^{jl} R_{kl}$$

$\Rightarrow$ gravity action (closed string effective action) after partial integration:

$$S_{\mathcal{G}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(R^{LC} - \frac{1}{12} H_{ijk} H^{ijk} \right)$$

Khoo, Vysoky, Jurco, Boffo, PS

Graded/generalized geometry and gravity

This formulation consistently combines all approaches of Einstein: Non-symmetric metric, Weitzenböck and Levi-Civita connections, without any of the usual drawbacks.

The dilaton $\phi(x)$ rescales the generalized tangent bundle. The deformation can be formulated in terms of vielbeins

$$E = e^{-\frac{\phi}{3}} \begin{pmatrix} 1 & 0 \\ g+B & 1 \end{pmatrix} \qquad E^{-1}\partial_i E = \begin{pmatrix} -\frac{1}{3}\partial_i\phi & 0 \\ \partial_i(g+B) & -\frac{1}{3}\partial_i\phi \end{pmatrix}$$

Going through the same steps as before we find in $d = 10$

$$S = \frac{1}{2\kappa}\int d^{10}x\, e^{-2\phi}\sqrt{-g}\Big(R^{\mathrm{LC}} - \tfrac{1}{12}H^2 + 4(\nabla\phi)^2\Big)$$

Boffo, PS (JHEP 2020)

Graded/generalized geometry and gravity

new approach, symmetric in open-closed string relations

$$(g - B)^{-1} =: G^{-1} - \Pi$$

deformation via

$$\mathcal{E} = \begin{pmatrix} 1 & (g-B)^{-1} \\ -(g+B) & 1 \end{pmatrix} \quad \rightsquigarrow \quad \mathcal{G} = \begin{pmatrix} -2g & 0 \\ 0 & 2G^{-1} \end{pmatrix}$$

$\rightsquigarrow$ low energy effective action for non-geometric closed strings

$$S[G^{-1}, \Pi] = \int_M \mathrm{d}^d x \sqrt{\det G^{-1}} \Big[\mathrm{R}_G - \frac{1}{12} R^2 - \frac{1}{2} R^{lmi} Q^{jk}{}_i G_{lj} G_{mk} - \frac{1}{4} Q^{jl}{}_m Q^{kn}{}_i G_{jk} G_{ln} G^{mi} - \frac{1}{2} Q^{lj}{}_k Q^{km}{}_l G_{jm} \Big]$$

where locally $R^{ijk} = 3\Pi^{[i|l}\partial_l \Pi^{|jk]}$ and $Q^{ij}{}_k = \partial_k \Pi^{ij}$

Boffo, PS (2021)

Conclusion

- interaction via deformation: “forces = free fall in deformed phase space”
- powerful approach to NC gauge theory: allows to find SW to all orders
- effective actions via commutative–noncommutative duality
- graded/generalized geometry provides a perfect setting for the formulation of low energy effective actions and theories of gravity
- approach is based on deformed graded geometry is algebraic in nature: everything follows from associativity as unifying principle

Thanks for listening!