Multi-matrix Trace relations and Hagedorn Transitions

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The Cayley-Hamilton Theorem

Every matrix obeys its own characteristic polynomial.

$$p_X(\lambda) = \det(\lambda I_N - X) = \lambda^N + c_{N-1}\lambda^{N-1} + \cdots + c_1\lambda + c_0$$

$$p_X(X) = X^N + c_{N-1}X^{N-1} + \cdots + \det(-X) = 0$$

$$c_{n-1} = -\operatorname{tr}(X), \quad c_{n-2} = \frac{1}{2}\left((\operatorname{tr}(X))^2 - \operatorname{tr}(X^2)\right), \quad \text{etc.}$$

$$\mathrm{Tr}(p_X) = 0 \implies \det(X) \quad \text{expressed in terms of traces of } X^k$$

$$\mathrm{Tr}(X\, p_X(X)) = 0 \implies \mathrm{Tr}(X^{n+1}) = \text{sum of products of traces.}$$

Two or more matrices

$$p_{X_1}(X_1) = 0, \quad p_{X_2}(X_2) = 0 \quad \text{and} \quad p_{X_1+X_2}(X_1 + X_2) = 0$$

The only independent traces are

$$\mathrm{Tr}(X_1), \mathrm{Tr}(X_2), \mathrm{Tr}(X_1^2), \mathrm{Tr}(X_2^2) \quad \text{and } \mathrm{Tr}(X_1 X_2)$$

For three 2×2 matrices one needs in addition

$$\mathrm{Tr}(X_1 X_2 X_3)$$

and generically such products up to triples is sufficient for any number of 2×2 matrices.

It quickly gets complicated!

Two 3×3 matrices

The independent traces for $N = 3$ are:

$$\mathrm{Tr}(X_1), \mathrm{Tr}(X_2), \mathrm{Tr}(X_1^2), \mathrm{Tr}(X_1\,X_2), \mathrm{Tr}(X_2^2),$$

$$\mathrm{Tr}(X_1^3), \mathrm{Tr}(X_1^2 X_2), \mathrm{Tr}(X_1\,X_2^2), \mathrm{Tr}(X_2^3),$$

$$\mathrm{Tr}(X_1^2\,X_2^2), \mathrm{Tr}(X_1^2\,X_2^2\,X_1\,X_2)$$

We have a 6th order term!

For a nice review of some of the mathematics background see V. Drensky, *Computing with Matrix Invariants*, arXiv:math/0506614.

Matrices of oscillators

Consider matrices of creation oscellators $a^\dagger_{ij}$ and $b^\dagger_{ij}$ we can ask about the allowed states $U(N)$ invariant states e.g. states such as:

$$\mathrm{Tr}(a^\dagger)|0\rangle, \quad \mathrm{Tr}(b^\dagger)|0\rangle, \mathrm{Tr}((a^\dagger)^2)|0\rangle, \quad \mathrm{Tr}(a^\dagger)\,\mathrm{Tr}(a^\dagger)|0\rangle \ldots$$

$$\mathrm{Tr}(a^\dagger)^2|0\rangle \ldots \quad \mathrm{Tr}((a^\dagger)^{n_1}(b^\dagger)^{n_2})\cdots(b^\dagger)^{n_k}|0\rangle$$

How do we avoid double counting?

A gauge Gaussian model

$$S_{GG}[X] = \int dt \sum_{a=1}^{D} \frac{1}{2} \operatorname{Tr} \left[D_t X^a D_t X^a - m^2 X^a X^a \right]$$

$D_t = \partial_t - i[A, \ ]$.

The Hamiltonian formulation involves a system of harmonic oscillators with a Gauss law constraint which implies the states must be $U(N)$ singlets.

In a thermal bath: for small N

A single oscillator

$$Tr(\mathrm{e}^{-\beta H}) = \sum_{n=0}^{\infty} \mathrm{e}^{-\beta m(n+\frac{1}{2})} = \frac{\mathrm{e}^{-\beta \frac{m}{2}}}{1-\mathrm{e}^{-\beta m}}$$

Without zero point energy

$$\mathbb{S}(t) = Tr(t^{\hat{a}^\dagger a}) = \frac{1}{1-t}, \qquad t = \mathrm{e}^{-\beta m}.$$

For k oscillators

$$\mathbb{S}(t,k) = \frac{1}{(1-t)^k} = 1 + kt + \frac{k(k+1)}{2}t^2 + \frac{k(k+1)(k+2)}{3!}t^3 + \cdots$$

The coefficient of t^n is the dimension of the level n vector space.
$\mathbb{S}(t,k)$ is called a Hilbert (or Poincaré) series.

A single matrix Gauge Gaussian model

$$S_{GG}[X] = N \int_0^\beta d\tau \, \frac{1}{2} \operatorname{Tr} \left[(D_\tau X)^2 + m^2 X^2 \right]$$

The allowed states

$$|0\rangle$$
$$\operatorname{Tr}(a^\dagger)|0\rangle$$
$$\operatorname{Tr}(a^\dagger a^\dagger)|0, \operatorname{Tr}(a^\dagger)\operatorname{Tr}(a^\dagger)|0\rangle$$
$$\cdots$$
$$\operatorname{Tr}(a^\dagger)^n|0\rangle, \operatorname{Tr}(a^\dagger)^n \operatorname{Tr}(a^\dagger)|0\rangle \cdots (\operatorname{Tr} a^\dagger)^n$$
$$\cdots$$

The Hilbert Series in the $N \to \infty$ limit is then

$$\mathbb{S}(t) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n} = \frac{1}{\Phi(t)}$$

with $\Phi(t)$ the Euler function.

The Multi-matrix case

On integrating out X^a and suppressing the zero-point energy, the effective action for θ_i, the eigenvalues of βA is

$$S_{GG}(\theta) = \frac{D}{2}\sum_{i,j=1}^{N} \ln\left|1 - \mathrm{e}^{-\beta m + i(\theta_i - \theta_j)}\right|^2$$

$$- D\ln(1 - \mathrm{e}^{-\beta m}) - \frac{1}{2}\sum_{i\neq j=1}^{N} \ln\left|1 - \mathrm{e}^{i(\theta_i-\theta_j)}\right|^2.$$

Hilbert-Poincaré series from gauge Gaussian Matrix Model

$$\mathbb{S}(t, D) = \frac{1}{N!}\int \prod_{i=1}^{N} \frac{d\theta_i}{2\pi}\mathrm{e}^{-S_{GG}(\theta)}$$

Equivalently $\mathbb{S}$ is the sentence generating function for words formed from traces of creation operators.
See Furuuchi et al arXiv:0310286.

Hilbert-Poincaré series as Molien-Weyl formula.

The integrations can be case as contours integral giving

$$\mathbb{S}(t,D) = \frac{1}{(1-t)^{D(N-1)}} \frac{1}{N!} \int \prod_{i=1}^{N} \frac{dz_i}{2\pi i z_i} \frac{\Delta(\{z\})\Delta(\{1/z\})}{(\Delta(t,\{z\})\Delta(t,\{1/z\}))^D}$$

where $\Delta(t,\{z\}) = \prod_{1\leq i<j\leq N}(tz_i - z_j)$ and $\Delta(\{z\}) = \Delta(1,\{z\})$ is the Vandermonde determinant.

This expression is the Molien-Weyl formula for the Hilbert-Poincaré series. (See F. Dolan arXiv:0704.1038 and Kristensson et al arXiv:2005.06480).

Caley-Hamilton

But for finite N Caley-Hamilton tells us that $\mathrm{Tr}((a^\dagger)^{N+1})$ is related to lower traces.
The finite N result for one matrix is in fact

$$\mathbb{S}(t) = \prod_{n=1}^{N} \frac{1}{1-t^n} = P_N(t)$$

and $P_N(t)$ is the generating function for $p_N(n)$ the number of partitions of n into no more than N parts and

$$P_N(t) = \sum_{n=0}^{\infty} p_N(n) t^n$$

which can be seen directly from the Fock basis.

Example D=2

With different masses for the two matrices

$$\mathbb{S}^{U(2)}_{GG}(t_1, t_2, 2) = \frac{1}{(1-t_1)(1-t_2)(1-t_1^2)(1-t_2^2)(1-t_1 t_2)}$$

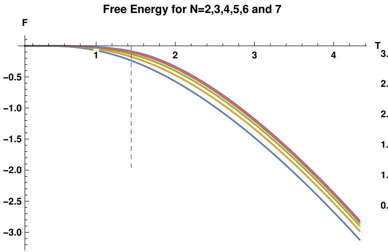
For $t_1 = t_2$ and $SU(2)$ it becomes:

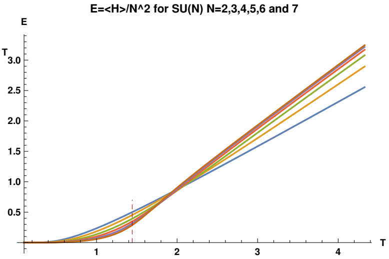
$$\mathbb{S}^{SU(2)}_{GG}(t, 2) = \frac{1}{(1-t^2)^3}$$

$$\mathbb{S}^{SU(3)}_{GG}(t, 2) = \frac{1}{(1-t)^4} \frac{(1-t^2+t^4)}{(1-t^2)^4(1+t+t^2)^4} = 1+3t^2+4t^3+7t^4+\cdots$$

The results are know up to $SU(7)$ with $SU(7)$ computed in Kristensson et al arXiv:2005.06480.

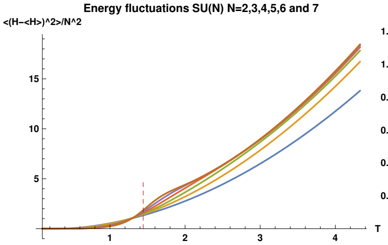
Observables for small N.

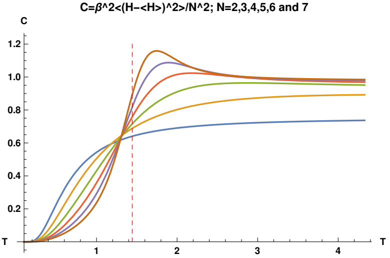




The free Energy and Internal Energy for gauged Gaussian matrix models.

Fluctuation Observables





The Standard Deviation of the Energy and the Heat Capacity for gauged Gaussian matrix models.

Dimensional Reduction of Yang-Mills

The Matrix models of interest are the zero volume limits of Yang-Mills compactified on a torus.
On $\mathbb{R}^{3+1}$ we have the Yang-Mills action:

$$S_{YM} = \frac{1}{4g^2} \int dt d^3x F_{\mu\nu} F^{\mu\nu}$$

Dimensional reduction on $\mathbb{T}^3$ gives 3-matrix model with a gauge-field $A_0 \to A$

Path Integral Quantization in a Thermal Bath

$$Z = \int [dX][dA] \mathrm{e}^{-\int_0^\beta d\tau \, \mathrm{Tr}(\frac{1}{2}(D_\tau X^a)^2 - \frac{1}{4}[X^a, X^b]^2)}$$

One can evaluate observables with the path integral by standard techniques.

Hamiltonian Quantization

The residual gauge field A is not dynamical and appears only in

$$D_\tau X^a = \partial_\tau X^a - i[A, X^a].$$

It leads to a constraint on the dynamics.

Gauss law constraint

A is a Lagrange multiplier field that leads to the Gauss Law constraint and implies a projection onto physical states.

From the action we can obtain the Hamiltonian and once we have the Hamiltonian H we can equally consider thermal ensembles whose partition function is given by

$$Z = \mathrm{Tr}_{Phys}(\mathrm{e}^{-\beta H})$$

where the physical constraint means the states are $SU(N)$ invariant.

Introducing an Auxiliary Field

Expanding the reduced action gives

$$S[X,A] = \int_0^\beta d\tau \, \mathrm{Tr}\left(\frac{1}{2}(D_\tau X^a)^2\right) - \frac{1}{4}\lambda^{ABCD}\int_0^\beta dt \, X_A^a X_B^a \, X_C^b X_C^b \, .$$

Next (with μ_{ABCD} the inverse of λ^{ABCD}) add to the action

$$\Delta S = \frac{1}{4}\mu_{ABCD}\int_0^\beta dt \left(k^{AB} + \lambda^{ABEF}X_E^a X_F^a\right)\left(k^{CD} + \lambda^{CDGH}X_G^a X_H^a\right) \, ,$$

the action $S[X,A,k] = S[X,A] + \Delta S[X,k]$ then becomes:

$$S[X,A,k] = \frac{1}{2}\int_0^\beta d\tau \left\{\mathrm{Tr}(D_\tau X^a)^2 + k^{AB}X_A^a X_B^a + \frac{1}{4}\mu_{ABCD}k^{AB}k^{CD}\right\} \, .$$

This addition is trivial since $\int [dk] \, e^{-\Delta S} = \mathrm{const}.$

Gauge Gaussian Model

The auxiliary field $k^{AB} \simeq m^2 \delta^{AB}$ so

$$S[X, A, k] = \frac{1}{2} \int_0^\beta d\tau \left\{ \mathrm{Tr}(D_\tau X^a)^2 + k^{AB} X_A^a X_B^a + \frac{1}{4} \mu_{ABCD} k^{AB} k^{CD} \right\} .$$

becomes:

A Gauge Gaussian Model

$$S_{GG}[X, A] = \frac{1}{2} \int_0^\beta d\tau \, \mathrm{Tr} \left\{ (D_\tau X^a)^2 + m^2 X^a X^a \right\}$$

with m^2 approximately the lowest glueball mass times the toroidal volume.

Similar models e.g. the large μ BMN model

The massive deformation of the BFSS model gives the BMN model

$$S[X,\psi] = \int_0^\beta d\tau \; \mathrm{Tr}\left[\frac{1}{2}D_\tau X^i D_\tau X^i - \frac{1}{4}\left([X^r,X^s] + \frac{i\mu}{3}\varepsilon^{rst}X_t\right)^2 - \frac{1}{2}[X^r,X^m]^2 - \frac{1}{4}[X^m,X^n]^2 + \frac{1}{2}\left(\frac{\mu}{6}\right)^2 X_m^2 + \frac{1}{2}\psi^T\mathcal{C}\left(D_\tau - \frac{i\mu}{4}\gamma^{567}\right)\psi - \frac{1}{2}\psi^T\mathcal{C}\gamma^i[X^i,\psi]\right]$$

Taking μ to infinity gives a supersymmetric gauge Gaussian model.

Connection to other models

These matrix models are dimensional reductions of higher dimensional Yang-Mills models.

- The bosonic membrane in flat $3+1$ spacetime is a toroidal compactification of SU(N) Yang-Mills.
- The Bosonic model on a pp-wave background is is a reduction of Yang-Mills on S^3— also studied in

A. P. Balachandran, S. Vaidya and A. R. de Queiroz, Mod. Phys. Lett. A 30 (2015) no.16, 1550080[] and
Nirmalendu Acharyya, A. P. Balachandran, Mahul Pandey, Sambuddha Sanyal, Sachindeo Vaidya, IJMP A Vol. 33, No. 13 (2018) 1850073 [arXiv:1606.08711].

- The BFSS model is a toroidal compactification of $9+1$ dim $\mathcal{N}=1$ Susy on a 9-torus or $3+1$ $\mathcal{N}=4$ susy on a 3-torus.
- The BFSS model is a compactification of $\mathcal{N}=4$ susy on S^3.
- The BD model is a compactification of $5+1$ $\mathcal{N}=1$ on a 5-torus or $3+1$ $\mathcal{N}=2$ on a 3-torus.

One could continue!

These compactifications give hints of the physics to expect.

Analysing the gauge Gaussian model.

Gauge Gaussian models e.g.

$$S_{GG}[X] = N \int_0^\beta d\tau \sum_{a=1}^{D} \frac{1}{2} \mathrm{Tr} \left[D_\tau X^a D_\tau X^a + m^2 X^a X^a \right]$$

are the simplest model of this type we can study and can be analysed in great detail.

The Hamiltonian formulation involves a system of harmonic oscillators with a Gauss law constraint which insists on $SU(N)$ singlets.

Analysing gauge Gaussian models

Integrating out the X^a gives the effective action

$$S_{GG}(\theta) = \frac{D(N^2-1)}{2}\beta m + \frac{D}{2}\sum_{i,j=1}^{N} \ln\left|1-\mathrm{e}^{-\beta m + i(\theta_i-\theta_j)}\right|^2$$

$$-D\ln(1-\mathrm{e}^{-\beta m}) - \frac{1}{2}\sum_{i\neq j=1}^{N} \ln\left|1-\mathrm{e}^{i(\theta_i-\theta_j)}\right|.$$

The θ_i are eigenvalues of βA in static gauge.
Expanding the logarithms and with $u_n = \frac{1}{N}\sum_{i=1}^{n} \mathrm{e}^{in\theta_i}$ gives

$$S_{GG}(\theta) = \frac{D(N^2-1)}{2}\beta m + N^2\sum_{n=1}^{\infty}\left\{\frac{1-D\mathrm{e}^{-nm\beta}}{n}|u_n|^2 - \frac{1}{nN}\right\}$$

In large N the u_n are moments of the eigenvalue distribution $\rho(\theta)$.

The Hagedorn transition

Examining

$$S_{GG}(\theta) = \frac{D(N^2-1)}{2}\beta m + N^2 \sum_{n=1}^{\infty}\{\frac{1-D\mathrm{e}^{-nm\beta}}{n}|u_n|^2 - \frac{1}{nN}\}$$

At low temperature (large β) all u_n have a minimum at 0 and the free energy is given by the zero point energy term. As the temperature is increased u_1 becomes unstable first.

The Hagedorn temperature

For $D > 1$ there is a large N phase transition at:

$$\beta_H = \frac{\ln D}{m}$$

A Hagedorn Transition

A Hagedorn transition is characterised by exponential growth in the degeneracy of energy levels at large energy .

Degeneracy growth

$$\Omega(E) \sim \mathrm{e}^{\beta_H E}$$

as the energy increases.

A transition occurs at $\beta = \beta_H$

Such exponential growth seems unphysical so we expect a phase transition from a low temperature to a high temperature phase.

The Hagedorn/Deconfinement Phase Transition in Weakly Coupled Large N Gauge Theories — Aharony, Marsano, Minwalla, Papadodimas and Van Raamsdonk, arXiv:hep-th/0310285 argued that weakly coupled gauge theories on spheres typically undergo such a phase transition.

The large N limit

In the large N limit the free energy per matrix element is

$$\beta F(\rho) = \frac{Dm\beta}{2} + \frac{D}{2}\int \rho(\alpha)\int \rho(\alpha\prime)\ln|1-\mathrm{e}^{-\beta m + i(\alpha-\alpha\prime)}|^2 d\alpha d\beta$$
$$-\frac{1}{2}P\int \rho(\alpha)\rho(\alpha\prime)\ln|1-\mathrm{e}^{i(\alpha-\alpha\prime)}|d\alpha d\alpha\prime\,.$$

For low temperatures including the transition expanding in $\mathrm{e}^{-m\beta}$ and only retaining the leading exponential is sufficient and equivalent to solving the model

$$Z_{a1} = \int [dU]\mathrm{e}^{a_1 \mathrm{Tr}(U)\mathrm{Tr}(U^{-1})} \quad \text{with } a_1 = D\mathrm{e}^{-m\beta}$$

resulting in

$$\beta F_{a1} = -a_1|u_1|^2 - \frac{1}{2}P\int \rho(\alpha)\rho(\alpha\prime)\ln|1-\mathrm{e}^{i(\alpha-\alpha\prime)}|d\alpha d\alpha\prime\,.$$

The eigenvalue distribution is given by (see Aharony et al [hep-th/0310285])

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for} \quad \beta > \beta_H \\ \frac{1}{\pi s^2}\sqrt{s^2 - \sin^2(\frac{\theta}{2})}\cos(\frac{\theta}{2}) & \text{for} \quad \beta < \beta_H \end{cases}$$

with

$$s^2 \equiv \sin^2(\frac{\theta_0}{2}) = 1 - \sqrt{1 - \frac{1}{a_1}} = 1 - \sqrt{1 - \mathrm{e}^{-m(\beta - \beta_H)}}.$$

and

$$\beta F_{GG} = \begin{cases} \frac{Dm\beta}{2} & \text{for} \quad \beta > \beta_H \\ \frac{Dm\beta}{2} + \frac{1}{2} - \frac{1}{2s^2} - \frac{1}{2}\ln s^2 & \text{for} \quad \beta < \beta_H \end{cases}$$

Near the transition we have

$$\beta F_{GG} = \begin{cases} \frac{Dm\beta}{2} & \text{for} \quad \beta > \beta_H \\ \frac{Dm\beta}{2} + \frac{m(\beta - \beta_H)}{4} - \frac{m^{3/2}}{3}(\beta_H - \beta)^{3/2} + \cdots & \text{for} \quad \beta < \beta_H \end{cases}$$

The eigenvalue distribution is given by (see Aharony et al [hep-th/0310285])

$$\rho(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for} \quad \beta > \beta_H \\ \frac{1}{\pi s^2}\sqrt{s^2 - \sin^2(\frac{\theta}{2})}\cos(\frac{\theta}{2}) & \text{for} \quad \beta < \beta_H \end{cases}$$

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The Energy and Specific Heat

The energy near the Hagedorn temperature is

$$E = \frac{\partial(\beta F)}{\partial \beta} = \begin{cases} \frac{Dm}{2} & \text{for} \quad \beta > \beta_H \\ \frac{Dm}{2} + \frac{m}{4} + \frac{m^{3/2}}{2}\sqrt{\beta_H - \beta} + \cdots & \text{for} \quad \beta < \beta_H \,. \end{cases}$$

We can furthermore obtain that the specific heat

$$C_{\rm v} = -\beta^2 \frac{\partial^2(\beta F)}{\partial \beta^2} = \begin{cases} 0 & \text{for} \quad \beta > \beta_H \\ \beta_H^2 \frac{m^{3/2}}{4\sqrt{\beta_H - \beta}} + \cdots & \text{for} \quad \beta < \beta_H \,. \end{cases}$$

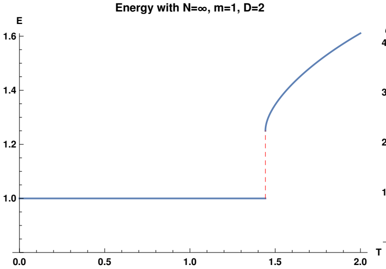
Divergent fluctuations

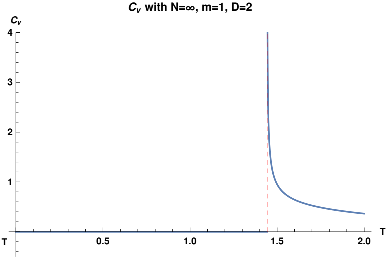
The specific heat of the gauge Gaussian model is predicted to diverge with a square root singularity as the Hagedorn temperature is approached from the deconfined high temperature side of the transition!

3/2-order phase transition

The transition is NOT to in fact first order.

The transition has a divergent specific heat on either side of the transition. The stronger divergence appears to be on the low temperature side, but this is coming from subdominant contributions as the limit is approached.





For other examples of 3/2 order phase-transitions see:
Bhattacharjee, Nagle, Huse and Fisher J.Stat.Phys. 32 (1983) 361.
Nash and O'Connor J.Phys. A 42 (2009) 012002 [arXiv:0809.2960]

Divergent $1/N$ corrections

When the leading $1/N$ corrections in the large N limit are taken into account the partition function in the confined phase becomes

$$Z_{GG}^{Conf} = \mathrm{e}^{-D(N^2-1)m\beta/2} \prod_{n=1}^{\infty} \frac{1}{1-\mathrm{e}^{-nm(\beta-\beta_H)}}$$

which is well approximated by the $n=1$ term i.e.

$$Z_{GG}^{Conf} \simeq \mathrm{e}^{-D(N^2-1)m\beta/2} \frac{1}{1-\mathrm{e}^{-m(\beta-\beta_H)}}$$

Near the Hagedorn temperature

$$\beta F = \frac{Dm\beta}{2} + \frac{1}{N^2} \ln\left(m(\beta - \beta_H)\right) + \cdots$$
$$E = \frac{Dm}{2} + \frac{1}{N^2} \frac{1}{\beta - \beta_H} + \cdots \qquad (1)$$

The $1/N^2$ corrections diverge as the Hagedorn temperature is approached. For $T \simeq T_H - \frac{2T_H^2}{N^2 mD}$ the $1/N^2$ corrections can compete with the leading ground state energy contribution.

N.B. Fluctuations are large!

One needs more care in taking the limit in the vicinity of the transition.

A micorcanonical analysis

The entropy (over N^2) in a microcanonical ensemble is given by

$$\mathcal{S} = \frac{\ln \Omega}{N^2}$$

Low temperatures

$$\mathbb{S}(t, D) = \prod_{k=1}^{\infty} \frac{1}{1 - Dt^k} \sim \sum_{n=1}^{\infty} \mathrm{e}^{n \ln D - n\beta m}$$

We can read off the entropy

$$\mathcal{S}_{-} = \frac{n}{N^2} \ln D = \frac{E}{m} \ln D$$

High temperature

We can solve $E(\beta)$ for $\beta(E)$ then note:

$$\frac{d\mathcal{S}}{dE} = \beta(E)$$

Integrating back and supplying a boundary condition gives $\mathcal{S}(E)$.

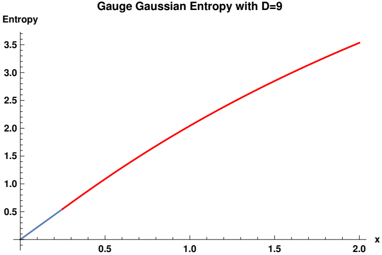
We should match β_H where $\frac{E}{m} = \frac{1}{4}$.

The Microcanonical Boltzmann Entropy

$$\mathcal{S}_- = x \ln D \qquad x \leq \frac{1}{4}$$

$$\mathcal{S}_+ = -\frac{1}{4} + x + x \ln(\frac{16Dx}{(1+4x)^2}) + \frac{1}{2}\ln(\frac{2}{(1+4x)}) \quad x \geq \frac{1}{4}$$

$x = \frac{E}{m} = \frac{n}{N^2}$.



Conclusions

- We have seen gauge Gaussian models have Hagedorn transitions and potentially divergent $1/N$ corrections.
- The transitions are $3/2$ order rather than 1st order.
- Analytic studies of small N are instructive.
- For large N trace relations become dominant for strings of length $\frac{N^2}{4}$.

Thank you for your attention