Infinite dimensional ambiguity in quantization and Quantum Gravity
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With $>$ 100 years of General Relativity and $>$ 90 years of Quantum Mechanics it is becoming increasingly embarassing the fact that there is not a fully consistent theory of Quantum Gravity.
The candidates to succeed as e.g. String Theory, Loop Quantum Gravity, Causal Dynamical Triangulations, Group Field Theory continue facing conceptual and technical problems.
One of the problems one is faced with and the one we will address today is that of nonuniqueness of quantization of a classical system.

The dream of the founders of quantum mechanics was to have quantization as a well defined process assigning a quantum system to every classical system and satisfying the correspondence principle

$$\text{Quantization Functor (?)} : (M,\omega) \mapsto Q_\hbar(M,\omega) \overset{\hbar\to 0}{\mapsto} (M,\omega)$$

It was soon realized that this can never be the case even for the simplest systems.

Particle in the line (1 dof)

Classical - $(M, \omega) = (\mathbb{R}^2, dp \wedge dq, H = \frac{1}{2}p^2 + V(q))$:

$$f \rightsquigarrow X_f = \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p} \qquad X_H = p\frac{\partial}{\partial q} - V'(q)\frac{\partial}{\partial p}$$

Quantum - $Q_\hbar^{Sch}(\mathbb{R}^2, dp \wedge dq, H)$:

$$
\begin{aligned}
\mathcal{H}^{\mathrm{Q}}_{Sch} &= L^2(\mathbb{R}, dq) \\
q &\mapsto Q^{\mathrm{Sch}}_\hbar(q) = \widehat{q} = q \\
p &\mapsto Q^{\mathrm{Sch}}_\hbar(p) = \widehat{p} = i\hbar\frac{\partial}{\partial q} \\
f(q,p) &\mapsto ?? \\
H = \frac{1}{2}p^2 + V(q) &\mapsto Q^{\mathrm{Sch}}_\hbar(H) = \widehat{H} = -\frac{\hbar^2}{2}\frac{\partial}{\partial q^2} + V(q) \\
\mathcal{H}^{\mathrm{Q}}_{\mathrm{Sch}} &= \mathcal{H}^{\mathrm{Q}}_{q}
\end{aligned}
$$

Groenewold (1946) – van Hove (1951) no go Thm:
It is impossible, even for systems with one degree of freedom, to quantize all observables exactly as Dirac hoped

$$\begin{aligned} Q_\hbar(f) &= \widehat{f} \\ [Q_\hbar(f), Q_\hbar(h)] &= i\hbar\, Q_\hbar(\{f, g\}) \end{aligned}$$

and satisfy natural additional requirements like irreducibility of the quantization.
In order to quantize one needs to add additional data to the classical system. *eg* choose a (sufficiently big but not too big ...) (Lie) subalgebra of the algebra of all observables

$$\mathcal{A} = \mathrm{Span}_{\mathbb{C}}\{1, q, p\}$$

Then we have to study the dependence of the quantum theory on the additional data.

Fortunately Geometric Quantization allows to put an order in the apparent mess of this infinite dimensional family of different quantizations.

1. GQ allows to parametrize the space of quantizations $\mathcal{T}_{(M,\omega)}$ of a system with phase space (M,ω) with (a closure of) a open subset of $C^\infty(M)$:

$$\begin{aligned} \mathcal{T}_{(M,\omega)} &= \overline{U} = \overline{\{k\}} \\ U &= \left\{ k \in C^\infty(M) \, : \, \omega + i\,\partial_I \overline{\partial_I} k > 0 \right\} \subset C^\infty(M), \end{aligned}$$

   where (ω, I) is a Kähler pair.
2. The "group" $\mathcal{H}am_{\mathbb{C}}$ of complex canonical transformations acts transitively in the space of quantizations $\mathcal{T}_{(M,\omega)}$.
3. The action of $\mathcal{H}am_{\mathbb{C}}$ can frequently be lifted to an action on the quantum bundle via Coherent State Transforms.

Geometric quantization is mathematically perhaps the best defined quantization

$$(M,\omega), \quad \frac{1}{2\pi\hbar}[\omega] \in H^2(M,\mathbb{Z})$$

Prequantum data: (L,∇,h), $L \to M, F_\nabla = \frac{\omega}{\hbar}$
Pre-quantum Hilbert space:

$$\mathcal{H}^{\mathrm{prQ}} = \Gamma_{L^2}(M,L) = \overline{\left\{ s \in \Gamma^\infty(M,L) : ||s||^2 = \int_M h(s,s)\ \frac{\omega^n}{n!}\ < \infty \right\}}$$

Pre-quantum observables:

$$\widehat{f} = Q_\hbar^{\mathrm{prQ}}(f) = \widehat{f}^{\mathrm{prQ}} = i\hbar\nabla_{X_f} + f$$

This almost works! But the Hilbert space is too large, the representation is reducible.
We need a smaller Hilbert space: Prequantization $\Rightarrow$ Quantization

Additional Data in Geometric Quantization

Generalizing what is done in the Schrödinger representation, for systems with one degree of freedom, to fix a quantization one chooses (locally) a preferred observable – $F(q,p)$[1] – and then works with wave functions of the form

$$\mathcal{H}^{\mathrm{prQ}} \rightsquigarrow \mathcal{H}^{Q}_{F} = \left\{\Psi \in \mathcal{H}^{\mathrm{prQ}} \,:\, \nabla_{X_F}\Psi = 0, ||\Psi|| < \infty\right\} = \\ = \left\{\Psi(q,p) = \psi(F)\, e^{-k(q,p)},\ ||\Psi|| < \infty\right\} \subset \mathcal{H}^{\mathrm{prQ}}$$

on which the preferred observable F and functions of it $u(F)$ act diagonally

$$Q^{F}_{\hbar}(u(F)) = \widehat{u(F)}^{\mathrm{prQ}}|_{\mathcal{H}^{Q}_{F}} = u(F).$$

[1] for systems with n degrees of freedom one chooses (locally) n independent observables in involution $F_1, \ldots, F_n$, $\{F_j, F_k\} = 0$. The distribution $\mathcal{P} =< X_{F_j}, j = 1, \ldots n >$ is called polarization associated with this choice.

(Non–)Equivalence of different Quantizations

Are all these quantizations (for different choices of F) physically equivalent?
NO!
Consider the observable: $H_\lambda = \frac{p^2}{2} + \frac{q^2}{2} + \lambda \frac{q^4}{4},\ \lambda \geq 0$
and let $Sp^{\mathrm{Sch}}(H_\lambda)$ denote the (discrete) spectrum of H_λ in the Schrödinger quantization, i.e. the spectrum of the operator

$$Q_\hbar^{\mathrm{Sch}}(H_\lambda) = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{q^2}{2} + \lambda \frac{q^4}{4}$$

acting on $\mathcal{H}^Q_{\mathrm{Sch}} = L^2(\mathbb{R}, dq)$.

Now consider the 1–parameter family of quantizations with Hilbert spaces $\mathcal{H}^{\mathrm{Q}}_{H_\lambda}$ for which the role of preferred observable is played by H_λ. Then, one finds that

$$\begin{aligned} \mathcal{H}^{Q}_{H_\lambda} &= \left\{\Psi(q,p) : \nabla_{X_{H_\lambda}} \Psi = 0\right\} = \\ &= \left\{\Psi(q,p) = \psi(H_\lambda)\, e^{i\, G_\lambda(q,p)}\right\} = \\ &= \left\{\sum_{n=0}^{\infty} \psi_n\, \delta(H_\lambda - E_n^\lambda)\, e^{iG_\lambda(q,p)}\right\}, \end{aligned} \tag{1}$$

where E_n^λ are defined by the Bohr-Sommerfeld conditions

$$\oint_{H_\lambda = E_n^\lambda} pdq = \hbar(n + \frac{1}{2}). \tag{2}$$

Since H_λ acts diagonally on this quantization we conclude from (1) that its spectrum in this quantization is given by (2)

$$Sp^{H_\lambda}(H_\lambda) = \{E_n^\lambda, n \in \mathbb{N}_0\}$$

It is known that on one hand $Sp^{\mathrm{Sch}}(H_0) = Sp^{H_0}(H_0)$ but on the other hand $Sp^{\mathrm{Sch}}(H_\lambda) \neq Sp^{H_\lambda}(H_\lambda)$ for all $\lambda > 0$ so that the two quantizations $Q_\hbar^{\mathrm{Sch}}$ and $Q_\hbar^{X_{H_\lambda}}$ are physically inequivalent if $\lambda > 0$!
Wins $Q_\hbar^{\mathrm{Sch}}$!

LQG is facing a similar problem with the Ashtekar–Barbero connection as preferred observable

$$A_\beta = \Gamma(E) + \beta\, K \Rightarrow \Psi_\beta(E, K) = \psi(A_\beta)\, e^{iG_\beta(E,K)}.$$

Are the quantizations based on the choice of connections with different (Immirzi) parameters equivalent? No, because they lead to different spectra of the area operator.
Here it is less obvious which one is the "correct" one. Studies of the black hole entropy formula seemed to indicate the value

$$\beta = \ln(3)/\sqrt{8}\pi\ ??$$

Other, recent studies (e.g. Pranzetti, Sahlmann, Phys Lett 2015, Ben Achour, Livine, arXiv:1705.03772) however seem to point back to $\beta = \sqrt{-1}$. This corresponds to the Ashtekar connection

$$A_{\sqrt{-1}} = \Gamma + \sqrt{-1}K$$

The study of quantizations based on compex valued observables like this has been the focus of most of our recent work.
It turns out that for some choices of complex observables quantization is in fact mathematically better defined then quantization based on real observables and this may help addressing some of the technical issues faced by LQG.

Complex observables and reality conditions: rescued by the power of complex analysis

Let us illustrate the general situation with a one degree of freedom system.
Consider the quantum observable

$$z_f = q + if(p)\,, \qquad dz_f \wedge \overline{dz_f} = -2if'(p)\, dq \wedge dp\,.$$

It turns out that if $f'(p) > 0$ then several remarkable simplifying facts occur:

$$F_f = z_f = q + if(p)$$

1. Complex Structure: There is a unique complex structure J_f on $\mathbb{R}^2$ for which z_f is a global holomorphic coordinate.
2. Kähler Metric: The symplectic form together with the complex structure J_f define on $\mathbb{R}^2$ a Kähler metric

$$\begin{aligned} \gamma_f &= \frac{1}{f'(p)}\, dq^2 + f'(p)\, dp^2 \\ R(\gamma_f) &= -\left(\frac{1}{f'(p)}\right)'' . \end{aligned}$$

3. Quantum Hilbert space much better defined than in the case of quantizations based on real observables:

$$\mathcal{H}^{Q}_{X_{z_f}} = \left\{ \Psi(q,p) = \psi(z_f)\, e^{-k_f(p)},\ ||\Psi|| < \infty \right\}$$

where ψ is a J_f–holomorphic function and $k_f(p) = pf(p) - \int f(p)dp$ is a Kähler potential.

4. The inner product is not ambiguous and it fixes the reality conditions:

$$< \Psi_1, \Psi_2 >= \int_{\mathbb{R}^2} \overline{\psi_1(z_f)}\psi_2(z_f)\, e^{-2k_f(p)}\, dqdp\,.$$

The main applications so far:

1. Donaldson–Tian theory of stability of Kähler manifolds
   Extend Kempf–Ness to the "action" of $Ham_{\mathbb{C}}(M, \omega)$ on $\mathcal{T}$.
2. Quantization
3. Geometry dependence of Fractional Quantum Hall trial states
4. Representation theory

We will concentrate on the application 2 to Quantum Theory.

Let $\mathcal{T}$ be the space of polarizations. In $\mathcal{T}$ we have the space of Kähler polarizations – $\mathcal{T}_{\mathrm{Kah}}$ – and in its boundary real and mixed polarizations.
Geometric quantization gives us the quantum Hilbert bundle

$$\mathcal{H}^Q \longrightarrow \mathcal{T} \supset \mathcal{T}_{\mathrm{Kah}} = \{k\}$$

and the tools to study the dependence of quantization on the choice of the complex structure or, more generaly, on the choice of polarization.

Integral transforms relating different quantizations

Step 1 Given two polarizations $\mathcal{P}^1$ and $\mathcal{P}^2$ we can hope to link them with a geodesic on $\mathcal{T}$, i.e. that there exists an Hamiltonian $H \in C^\omega(M)$ such that

$$\begin{aligned} \mathcal{P}^2 &= e^{it\,\mathcal{L}_{X_H}}\big|_{t=1}\,\mathcal{P}^1 = e^{it\,\mathcal{L}_{X_H}}\big|_{t=1} < X_{F_1}, \dots, X_{F_n} >= \\ &= < X_{e^{iX_H}(F_1)}, \dots, X_{e^{iX_H}(F_n)} > \end{aligned} \tag{3}$$

Step 2 Then geometric quantization gives us a way of lifting the geodesics to the quantum bundle and thus construct construct an integral transform

$$C^{iH}_{\mathcal{P}^1\mathcal{P}^2} \,:\, \mathcal{H}^{\mathrm{Q}}_{\mathcal{P}^1} \longrightarrow \mathcal{H}^{\mathrm{Q}}_{\mathcal{P}^2}$$

Imaginary time: why??

It is precisely to study the dependence of $Q_\hbar$ on the choice of preferred complex observables that evolution in imaginary time enters the scene.

$$H = \int f(p)dp \rightsquigarrow X_H = f(p)\frac{\partial}{\partial q} \; : \; q \mapsto q + t\, f(p) \overset{t \rightsquigarrow \sqrt{-1}s}{\mapsto} q + \sqrt{-1}s\, f(p)$$

Imaginary time evolution is not new in quantum mechanics. Many amplitudes can be obtained by making the famous (but misterious) Wick rotation: $t \rightsquigarrow is$ – e.g. semiclassical probabilites of tunneling given by imaginary time evolution.

What we are studying is a new way of looking at imaginary (or complex) time evolution in (some situations in) quantum mechanics and in geometry.

In Kähler geometry imaginary time evolution leads to geodesics in the (infinite dimensional) space of Kähler potentials ($\subset$ quantizations) in a given cohomology class, and is used to study the stability of polarized varieties [Semmes, Donaldson, Tian].
In loop quantum gravity complex time Hamiltonian evolution was proposed by Thiemann in '96 in order to transform the spin connection into the Ashtekar connection.

$$\Gamma \mapsto A_i = \Gamma + iK.$$

On the way we will see how geometric quantization explains the misterious factors in the Segal–Bargmann-Hall coherent state transforms.

In 1994 Brian Hall constructed an unitary transform for Lie groups of compact type G

$$
\begin{aligned}
U \,:\, L^2(G, dx) \quad &\longrightarrow \quad \mathcal{H}L^2(G_{\mathbb{C}}, d\nu(g)) \\
U \quad &= \quad \mathcal{C} \circ e^{\frac{\Delta}{2}}
\end{aligned}
$$

where $G_{\mathbb{C}}$ is the unique complexification of G, $\mathcal{H}L^2$ means holomorphic L^2 functions and ν is the averaged heat kernel measure on $G_{\mathbb{C}}$.

Let us show how geometric quantization reveals the intimate relation of the two factors in the rhs of (4).
For simplicity we restrict ourselves to the case $G = \mathbb{R}$, $G_{\mathbb{C}} = \mathbb{C}$ but the argument is valid for any Lie group of compact type.
Then (4) reads

$$\begin{aligned} U \,:\, L^2(\mathbb{R}, dq) &\longrightarrow \mathcal{H}L^2(\mathbb{C}, e^{-p^2} dpdq) \\ U &= \mathcal{C} \circ e^{\frac{\Delta}{2}} \\ \psi(q) &\mapsto (e^{\frac{\Delta}{2}}\psi)(q) \mapsto (e^{\frac{\Delta}{2}}\psi)(q + \sqrt{-1}p)\,. \end{aligned}$$

Notice that, for $H = \frac{p^2}{2}$, $X_H = p\frac{\partial}{\partial q}$ and therefore

$$e^{\tau X_H}(q)|_{\tau=i} = (q + \tau p)|_{\tau=i} = q + ip = z$$

We see therefore that, for $H = \frac{p^2}{2}$,

$$\mathcal{C} = e^{iX_H}$$

and since $\widehat{H}^{\mathrm{prQ}} = iX_H - \frac{p^2}{2}$, we conclude that

$$e^{-i\tau \widehat{H}^{\mathrm{prQ}}}|_{\tau=i} = e^{\widehat{H}^{\mathrm{prQ}}} = \mathcal{C} \circ e^{-\frac{p^2}{2}} \, .$$

On the other hand, since, $\widehat{p}^{\mathrm{Sch}} = -i\frac{\partial}{\partial q}$, we have also

$$e^{\frac{\Delta}{2}} = e^{-\widehat{H}^{\mathrm{Sch}}} = e^{-i\tau \widehat{H}^{\mathrm{Sch}}}|_{\tau=-i},$$

We see therefore that the Hall CST transform in (4) is equivalent to the follwing transform lifting the complex canonical transformation, $e^{\tau X_H}|_{\tau=i} = e^{ip\frac{\partial}{\partial q}}$:

$$\begin{array}{rcl} \mathcal{H}^{Q}_{\mathrm{Sch}} = \mathcal{H}^{\mathrm{Q}}_{q} & \xrightarrow{V_i^H} & \mathcal{H}^{Q}_{z} = \mathcal{H}^{\mathrm{Q}}_{\mathrm{Fock}} \\ V_i^H & = & e^{-i\tau \widehat{H}^{\mathrm{prQ}}}|_{\tau=i} \circ e^{-i\tau \widehat{H}^{\mathrm{Sch}}}|_{\tau=-i} = \\ & = & e^{+\widehat{H}^{\mathrm{prQ}}} \circ e^{-\widehat{H}^{\mathrm{Sch}}} \end{array} \tag{4}$$

with the (extra bonus of the) averaged heat kernel measure being absorbed into the prequantization of the complexified canonical transformation.

Representation Theoretic meaning of the factors in the CST

Notice that the prequantization of the observables q, p preserve both Hilbert spaces $\mathcal{H}^Q_{\mathrm{Sch}}$ and $\mathcal{H}^Q_{\mathrm{Fock}}$ so that there is a $*$–representation of the complexified Heisenberg algebra on both.

One can check that the first factor to act in (4) maps the self-adjoint $\widehat{q}^{\mathrm{Sch}}$ to the non self-adjoint $\widehat{q - ip}^{\mathrm{Sch}}$ and the second factor to act maps $\widehat{q}^{\mathrm{Sch}}$ to $\widehat{q + ip}^{\mathrm{Fock}}$ and therefore V_i^H maps $\widehat{q}^{\mathrm{Sch}}$ to $\widehat{q}^{\mathrm{Fock}}$.

Then V_i^H intertwines $\widehat{q}^{\mathrm{Sch}}$ and $\widehat{p}^{\mathrm{Sch}}$ with $\widehat{q}^{\mathrm{Fock}}$ and $\widehat{p}^{\mathrm{Fock}}$ respectively which makes its projective unitarity a consequence of Schur's lemma.

Some of our works on this topic

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Thank you!