On Hopf and L_∞-algebras

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work in progress with

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Personal perspective

Noncommutative geometry:

- replace the space by generalised structure living on that would-be space (e.g. noncommutative algebra of functions over a manifold).
- analyse the consequences in field-theoretical models, both kinematical and dynamical

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- noncommutative star-gauge symmetry, twisted-gauge symmetry
- string theory dualities

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$\rightsquigarrow$ higher homotopy structures A_∞, L_∞, ... Stasheff '63, Stasheff, Schlesinger '77

Motivation

L_∞-algebra $\rightsquigarrow$ useful for understanding quantization of field theory and gravity.

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- Quantization
  - BV formalism $\sim$ L_∞-algebra Zwiebach '92, cf. Richard's talk
  - Deformation quantization: formality thm $\sim$ L_∞ quasi-isomorphism Kontsevich '97;
    Poisson sigma model quantization Cattaneo, Felder, '99

Motivation

L_∞-algebra $\rightsquigarrow$ useful for understanding quantization of field theory and gravity.

- Quantization $\rightsquigarrow$ BV-BRST, deformation quantization
- Geometry
  - Graded geometry: L_∞-algebra (cyclic) $\equiv$ Q(P) manifolds AKSZ '95, cf. Peter's talk
  - Generalized geometry of Courant, double field theory and exceptional algebroids
    Roytenberg, Weinstein '98; Deser, Saemann '16, LJ, Grewcoe '20; Cederwall, Palmkvist '18

Motivation

L_∞-algebra $\rightsquigarrow$ useful for understanding quantization of field theory and gravity.

- Quantization $\rightsquigarrow$ BV-BRST, deformation quantization
- Graded and generalized geometry
- NC/NA field theory and gravity
  - $\star$-product: bootstraping nc gauge theories using L_∞ Blumenhagen et al '18, cf. Patrizia's talk
  - Drinfel'd twist and braided L_∞ Dimitrijević Ćirić et al '21, Nguyen, Schenkel, Szabo '21
  - HS in unfolded formalism Vasiliev, cf. Harold's talk

In this talk

GOAL

Argue that (curved) L_∞-algebra is (graded) Hopf algebra with codifferential.

PLAN

- L_∞-algebra - coalgebra formulation
- Hopf algebra in brief
- Drinfel'd twist of L_∞-algebra
- Outlook

L_∞ - coalgebra formulation

There is one-to-one correspondence between an L_∞ structure on a differential graded vector space $X = \bigoplus_{d\in\mathbb{Z}} X_d$ and a degree 1 coderivation on the coalgebra generated by the suspension of X. Lada, Stasheff '92, Lada, Markl '94

Suspension $\uparrow$ or shift isomorphism s

$$s : X \to X[1] \ \text{s.t.}\ (X[1])_d = X_{d+1} \ ,$$

induces isomorphism of graded algebras

$$s^{\otimes i} : x_1 \wedge \cdots \wedge x_i \to (-1)^{\sum_{j=1}^{i-1}(i-j)} s x_1 \vee \cdots \vee s x_i \ ,$$

and décalage isomorphism of brackets

$$l_i = (-1)^{\frac{1}{2}i(i-1)+1} s^{-1} \circ b_i \circ s^{\otimes i} \ .$$

L_∞ - coalgebra formulation

Start with graded symmetric tensor algebra

$$\mathrm{S}(X) := \bigoplus_{n=0}^{\infty} S^n X \ ,$$

and X graded vector space $X = \bigoplus_{d\in\mathbb{Z}} X_d$ over field $K =: S^0 X$.

The tensor products are graded symmetric,

$$x_1 \vee x_2 = (-1)^{|x_1||x_2|} x_2 \vee x_1, \ \ x_1, x_2 \in X \ ,$$

and the coproduct is

$$\Delta(x_1 \vee \ldots \vee x_m) = \sum_{p=0}^{m} \sum_{\sigma\in Sh(p,m-p)} \epsilon(\sigma; x)(x_{\sigma(1)} \vee \ldots \vee x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \vee \ldots \vee x_{\sigma(m)}) \ ,$$

where $\mathrm{Sh(p, m-p)} \in S_m$ denotes set of ordered permutations s.t. $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(m)$, and empty product is unit.

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Note: We include field K, ie., $\Delta(1) = 1 \otimes 1$, and $\Delta(x) = 1 \otimes x + x \otimes 1$.

L_∞ - coalgebra formulation

As a map $\Delta : \mathsf{S}(X) \rightarrow \mathsf{S}(X) \otimes \mathsf{S}(X)$ this reads:

$$\Delta \circ \mathrm{id}^{\vee m} = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p,m-p)} (\mathrm{id}^{\vee p} \otimes \mathrm{id}^{\vee (m-p)}) \circ \tau^{\sigma} \ , \ p, m \geq 0 \ ,$$

where τ^{σ} denotes action of permutations e.g. the non-identity permutation of two elements is

$$\tau^{\sigma}(x_1 \vee x_2) = (-1)^{|x_1||x_2|} x_2 \vee x_1 \ ,$$

and includes the Koszul sign.

Introduce degree 1 coderivation $D : \mathsf{S}(X) \rightarrow \mathsf{S}(X)$ of degree 1 such that:

$$\Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta \ ,$$

the co-Leibniz property is satisfied.

L_∞ - coalgebra formulation

The coderivation is

$$D = \sum_{i=0}^{\infty} b_i \ ,$$

with graded multilinear maps b_i of degree 1. The b_i act on full tensor algebra as coderivation:

$$b_i : S^j X \to S^{j-i+1} X \ ,$$
$$b_i(x_1 \vee \ldots \vee x_j) = \sum_{\sigma \in Sh(i,j-i)} \epsilon(\sigma; x) b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \ldots \vee x_{\sigma(j)}, \ j \geq i \ .$$

$$b_i(\mathrm{id}^{\vee j}) = \sum_{\sigma \in Sh(i,j-i)} (b_i \vee \mathrm{id}^{\vee (j-i)}) \circ \tau^{\sigma} \ , \ j \geq i \ ,$$

map τ^{σ} the action of transposition of elements $x_l \to \epsilon(\sigma; x) x_{\sigma(l)}$.

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Note: We include $b_0 \rightsquigarrow$ curved L_∞-algebra.

L_∞ - coalgebra formulation

Homotopy relations from $D^2 = 0$, e.g.,

$$D^2(x_1 \vee x_2) = \sum_{i=0}^{\infty} b_i \sum_{j=0}^{2} b_j(x_1 \vee x_2) =$$

$$= \sum_{i=0}^{3} b_i\big(b_0 \vee x_1 \vee x_2 + b_1(x_1) \vee x_2 + (-1)^{|x_1||x_2|} b_1(x_2) \vee x_1 + b_2(x_1, x_2)\big) = \ \ldots$$

This vanishes for

$$b_1 b_0 = 0$$
$$b_2 b_0 + b_1^2 = 0$$
$$b_3 b_0 + b_2 b_1 + b_1 b_2 = 0$$

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This vanishes for

$$\begin{aligned} b_1 b_0 &= 0 \\ b_2 b_0 + b_1^2 &= 0 \\ b_3 b_0 + b_2 b_1 + b_1 b_2 &= 0 \end{aligned}$$

Def: An L_∞-algebra is a coalgebra $(\mathsf{S}(X), \Delta)$ with coderivation $D : \mathsf{S}(X) \to \mathsf{S}(X)$ of degree 1 s.t. $\Delta \circ D = (\mathrm{id} \otimes D + D \otimes \mathrm{id}) \circ \Delta$ and $D^2 = 0$.

Example cdgla

Getzler '18

Curved dgla: Graded Lie algebra $\mathfrak{g}$, derivation d with degree 1, and curvature R of degree 2 s.t. $\forall x \in \mathfrak{g}$

$$dR = 0 \ , \quad d^2 x = [R, x] \ ,$$

the bracket satisfies graded Leibniz identity and MC element a of degree 1 is

$$R + da + \tfrac{1}{2}[a, a] = 0 \ .$$

L_∞-coalgebra: Shifted graded vector space X with maps $R \rightsquigarrow -b_0$, $d \rightsquigarrow -b_1$, the graded Lie bracket $\rightsquigarrow b_2$, satisfying the homotopy relations

$$b_1 b_0 = 0 \ , \quad b_1(b_1(x)) + b_2(b_0, x) = 0, \ x \in \mathrm{X} \ ,$$

and MC element a of degree 0 satisfies

$$b_0 + b_1(a) + \tfrac{1}{2} b_2(a, a) = 0 \ .$$

Hopf algebra

Hopf algebra is a bialgebra that admits antipode.

Start from an algebra A viewed as a vectors space over field K with multiplication $\mu : A \otimes A \rightarrow A$ and unit $\eta : K \rightarrow A$. If one can define comultiplication $\Delta : A \rightarrow A \otimes A$ and counit $\epsilon : A \rightarrow K$ such that either of two (thus both) hold

- Δ and ϵ are algebra homomorphisms
- μ and η are coalgebra homomorphisms

we have bialgebra.

If there exist antipode $S : A \rightarrow A$ such

$$\mu \circ (id \otimes S) \circ \Delta = \mu \circ (S \otimes id) \circ \Delta = \eta\epsilon$$

we have Hopf algebra.

Hopf algebra

Standard example - tensor algebra (and symmetric and exterior).

A tensor algebra $T(V)$, where V is a vector space over field K

$$\Delta(v) = v \otimes 1 + 1 \otimes v,\ \Delta(1) = 1 \otimes 1,\ v \in V$$
$$\epsilon(v) = 0,\ \epsilon(1) = 1\ ,$$
$$S(v) = -v, S(1) = 1$$

is Hopf algebra. In full algebra,

$$S(v_1 \cdot \ldots \cdot v_m) = (-1)^m v_m \cdot \ldots \cdot v_1\ ,$$
$$\Delta(v_1 \cdot \ldots \cdot v_m) = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh(p,m-p)}} \left(v_{\sigma(1)} \cdot \ldots \cdot v_{\sigma(p)}\right) \otimes \left(v_{\sigma(p+1)} \cdot \ldots \cdot v_{\sigma(m)}\right)\ ,$$

where $\mathrm{Sh(p,m-p)} \in S_m$ denotes set of ordered permutations s.t. $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(m)$, and empty product is unit.

$L_\infty \sim$ Hopf+D

cf. Schupp '93

Theorem

A graded Hopf algebra with a compatible codifferential is an L_∞-algebra. In particular, an L_∞-algebra is a bialgebra $(\mathsf{S}(X), \Delta)$ with coderivation $D : \mathsf{S}(X) \to \mathsf{S}(X)$ of degree 1 s.t. the co-Leibniz property is satisfied

$$\Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta$$

and $D^2 = 0$. It naturally inherits the structure of a Hopf algebra from graded symmetric tensor algebra, with

$$S \circ D = \widetilde{D} \circ S \ \ \& \ \ \epsilon \circ D = D \circ \epsilon \ .$$

where the codifferential $\widetilde{D}$

$$\widetilde{D} = \sum_{i=0}^{\infty} (-1)^{1-i}\, b_i \ ,$$

induces the same homotopy relations as D.

L_∞ is cocommutative and coassociative Hopf algebra $H \rightsquigarrow$ introduce non-(co)commutative deformation using Drinfel'd twist.

Using invertible twist element $\mathcal{F} =: f^k \otimes f_k \in H \otimes H$

$$(\mathcal{F} \otimes 1)(\Delta \otimes id)\mathcal{F} = (1 \otimes \mathcal{F})(id \otimes \Delta)\mathcal{F} \ ,$$
$$(\epsilon \otimes id)\mathcal{F} = 1 \otimes 1 = (id \otimes \epsilon)\mathcal{F} \ ,$$

we obtain $(H^{\mathcal{F}}, \vee, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \epsilon)$, where $H^{\mathcal{F}}$ is the same as H as vector spaces and:

$$\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \ h \in H \ ,$$

and $S^{\mathcal{F}} = S$ for Abelian twist.

Drinfel'd twist

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$\rightsquigarrow$ twisted L_∞ or $(L_\infty^{\mathcal{F}}, \vee, \Delta^{\mathcal{F}}, S, \epsilon)$

In the spirit of deformation quantisation, while twisting Hopf algebra we simultaneously twist its modules. Taking Hopf algebra L_∞ as its own module $\rightsquigarrow (L_\infty^\star, \vee_\star, \Delta_\star, S_\star, \epsilon)$:

$$x_1 \vee_\star x_2 = \bar{f}^k(x_1) \vee \bar{f}_k(x_2) \ ,$$
$$\Delta_\star(x) = x \otimes 1 + \bar{R}^k \otimes \bar{R}_k(x) \ ,$$
$$S_\star(x) = -\bar{R}^k(x)\bar{R}_k \ .$$

The $\mathcal{R}$-matrix $\mathcal{R} \in \mathsf{S}(X) \otimes \mathsf{S}(X)$ is an invertible matrix induced by the twist

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} =: R^\alpha \otimes R_\alpha \ , \mathcal{F}_{21} = f_\alpha \otimes f^\alpha \ ,$$

In the case of an Abelian twist $\mathcal{R}$ it triangular $R_\alpha \otimes R^\alpha = \bar{R}^\alpha \otimes \bar{R}_\alpha$, and $\mathcal{R} = \mathcal{F}^{-2}$.

The inverse $\mathcal{R}$-matrix controls noncommutativity of the $\vee_\star$-product and provides the representation of permutation group, e.g.,

$$\tau^\sigma_\mathcal{R}(x_1 \vee_\star x_2) = (-1)^{|x_1||x_2|}\bar{R}^\alpha(x_2) \vee_\star \bar{R}_\alpha(x_1) \ ,$$

Braided L_∞-algebra

Extend the coproduct to whole tensor algebra:

$$\Delta_\star(\mathrm{id}^{\vee_\star m}) = \sum_{\sigma\in \mathrm{Sh}(p,m-p)} (\mathrm{id}^{\vee_\star p}\otimes \mathrm{id}^{\vee_\star (m-p)})\circ \tau^\sigma_{\mathcal{R}} \ , \ p,m\geq 0 \ .$$

The compatible coderivation $D_\star = \sum_{i=0}^{\infty} b_i^\star$ is defined in terms of braided graded symmetric maps $b_i^\star$

$$b_i^\star(\mathrm{id}^{\vee_\star j}) = \sum_{\sigma\in \mathrm{Sh}(i,j-i)} (b_i^\star \vee_\star \mathrm{id}^{\vee_\star (j-i)})\circ \tau^\sigma_{\mathcal{R}} \ , \ j\geq i \ ,$$

$$b_i^\star(x_1,\dots,x_m,x_{m+1},\dots,x_i) = (-1)^{|x_m||x_{m+1}|}\, b_i^\star(x_1,\dots,\bar{R}^\alpha(x_{m+1}),\bar{R}_\alpha(x_m),\dots,x_i) \ ,$$

and the condition $D_\star^2=0$ reproduces the deformed homotopy relations.

$\rightsquigarrow$ braided L_∞-algebra obtained in Dimitrijević Ćirić et al '21.

$L^\star_\infty$ vs. $L^\mathcal{F}_\infty$

As Hopf algebras $L^\star_\infty$ and $L^\mathcal{F}_\infty$ are isomorphic Aschieri et al '05, Schenkel '12
$\exists$ map $\varphi : L^\star_\infty \to L^\mathcal{F}_\infty$ such that

$$\varphi(x_1 \vee_\star x_2) = \varphi(x_1) \vee \varphi(x_2) \ ,$$
$$\Delta_\star = (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta^\mathcal{F} \circ \varphi \ ,$$
$$S_\star = \varphi^{-1} \circ S^\mathcal{F} \circ \varphi \ .$$

$L_\infty^\star$ vs. $L_\infty^{\mathcal{F}}$

As Hopf algebras $L_\infty^\star$ and $L_\infty^{\mathcal{F}}$ are isomorphic Aschieri et al '05, Schenkel '12
$\exists$ map $\varphi : L_\infty^\star \to L_\infty^{\mathcal{F}}$ such that

$$\begin{aligned} &\varphi(x_1 \vee_\star x_2) = \varphi(x_1) \vee \varphi(x_2) \ , \\ &\Delta_\star = (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta^{\mathcal{F}} \circ \varphi \ , \\ &S_\star = \varphi^{-1} \circ S^{\mathcal{F}} \circ \varphi \ . \end{aligned}$$

On the other hand, we take $L_\infty^\star$-algebra as a module of $L_\infty^{\mathcal{F}}$ with an L_∞-action on an L_∞-algebra given by a curved L_∞-morphism Mehta, Zambon '12. Thus we obtain

$$D_\star = \varphi^{-1} D_{\mathcal{F}} \varphi \ .$$

Outlook

In the field theory context (reviewed in Hohm, Zwiebach '17, Jurčo et al. '18, '20)

- L_∞ for field theory $\rightsquigarrow$ MC equations as eoms plus compatible bilinear $\rightsquigarrow$ cyclic L_∞

Outlook

In the field theory context (reviewed in Hohm, Zwiebach '17, Jurčo et al. '18, '20)

- L_∞ for field theory $\rightsquigarrow$ MC equations as eoms plus compatible bilinear $\rightsquigarrow$ cyclic L_∞
- $Q = D^*$ $\rightsquigarrow$ BRST operator

Evaluate b_i on basis of X $\rightsquigarrow$ structure constants of L_∞-algebra:

$$b_i(\tau_{\alpha_1}, ..., \tau_{\alpha_i}) = C^{\beta}_{\alpha_1 \dots \alpha_i} \tau_\beta$$

Use to define cohomological vector Q of degree 1

$$Q = \sum_{i=0}^{\infty} \frac{1}{i!} C^{\beta}_{\alpha_1 \dots \alpha_i} z^{\alpha_1} \cdots z^{\alpha_i} \frac{\partial}{\partial z^\beta}$$

with z^{α_i} basis of $X^\star$.

In infinite dimensional case one either restricts $X^\star$ to the space spanned by z^α, or consider continuous duals in infinite-dim topological vector space. Arvanitakis et al '20

In Batalin-Vilkovisky formalism Q becomes BRST operator and z^{α_i} physical fields.

Outlook

In the field theory context (reviewed in Hohm, Zwiebach '17, Jurčo et al. '18, '20)

- L_∞ for field theory $\rightsquigarrow$ cyclic L_∞
- $Q = D^*$ becomes BRST operator
- L_∞ quasi-isomorphisms $\rightsquigarrow$ equivalent physical theories
    - If 0-bracket vanishes, 1-bracket is a differential $\rightsquigarrow$ cochain complex
    - L_∞ quasi-isomorphisms induces isomorphisms of cohomologies of respective L_∞-algebras $\rightsquigarrow$ homotopy transfer Arvanitakis et. al. '21
    - For curved L_∞? Fukaya '03, Costello '11

THANK YOU!