Spectral action for the Standard Model without fermion doubling

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The Standard Model from the spectral perspective

1. Spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$:
    - $*$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$,
    - (essentially) self-adjoint operator $\mathcal{D}$ on $\mathcal{H}$,
    - other "decorations": $\gamma, \mathcal{J}, \ldots$:
        - $\mathbb{Z}/2\mathbb{Z}$-grading γ on $\mathcal{H}$ s.th. $[\gamma, \mathcal{A}] = 0$,
        - real structure: antilinear isometry $\mathcal{J}$
    - compact resolvent, domains, relations (0th-, 1st-order), ...

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     - real structure: antilinear isometry $\mathcal{J}$
   - compact resolvent, domains, relations (0th-, 1st-order), ...
2. Spectral action principle:
   - To get an effective classical Lagrangian.
   - To compare obtained predictions with experiments.

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  What the physics tells us?
- Can the model be formulated without having too many *decorations*?
- The standard choice: almost-commutative **product** geometries

$$\big(C^\infty(M) \otimes \mathcal{A}_F, L^2(M) \otimes \mathcal{H}_F, \mathcal{D}, J, \gamma\big)$$

with $\mathcal{D} = \not{D} \otimes 1 + \gamma_5 \otimes \mathcal{D}_F$ and $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$.

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- There are too many fermions - one need to implement some projection. Is this really necessary?
- Possibility of $SU(3)$-symmetry breaking: *usual* minimal axioms do not fully eliminate unphysical models.
- Some *predictions* do not fully agree with experiments, e.g. the value of the Higgs mass.

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- Use *reverse engineering*: try to deduce the appropriate geometric formulation by looking at the **physical** Standard Model.
- Try to keep the information about the Lorentzian structure as long as possible.
- Compute the spectral action and **check** if the model *makes sense*.

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- Dirac operator for $(1,3)$-Minkowski space: $\mathcal{D} = i\gamma^\mu \partial_\mu$.

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  $\widetilde{\mathcal{D}} = \gamma^0 \mathcal{D}$ - the Krein shift of $\mathcal{D}$.
- $\widetilde{\mathcal{D}}$ - symmetric $\Leftrightarrow$ $\mathcal{D}$ - Krein-self-adjoint: $\mathcal{D} = \gamma^0 \mathcal{D} \gamma^0$
- $\mathcal{D}\gamma = -\gamma\mathcal{D}$, $\mathcal{D}\mathcal{J} = \mathcal{J}\mathcal{D}$.
- $\widetilde{\mathcal{D}}\gamma = \gamma\widetilde{\mathcal{D}}$, $\widetilde{\mathcal{D}}\mathcal{J} = -\mathcal{J}\widetilde{\mathcal{D}}$.
- ...

Finite Riemannian spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \pi_L, \pi_R)$

- $\mathcal{A}$ - finite dimensional algebra
- $\mathcal{H}$ - finite dimensional Hilbert space
- π_L - representation of $\mathcal{A}$ on $\mathcal{H}$
- π_R - representation of $\mathcal{A}^{\mathrm{op}}$ on $\mathcal{H}$
- $[\pi_L(a), \pi_R(b)] = 0$ - (0th order condition)
- $[[\mathcal{D}, \pi_L(a)], \pi_R(b)] = 0$ - (1st order condition)

Additional conditions

- spin$_c$ type geometry: $(Cl_{\mathcal{D}}(\pi_L(\mathcal{A})))' = \pi_R(\mathcal{A})$.
- Hodge condition: $(Cl_{\mathcal{D}}(\pi_L(\mathcal{A})))' = Cl_{\mathcal{D}}(\pi_R(\mathcal{A}))$,

where $CL_{\mathcal{D}}(\pi_L(\mathcal{A}))$ is the algebra generated by $\pi_L(\mathcal{A})$ and $[\mathcal{D}, \pi_L(\mathcal{A})]$.

[Dąbrowski– D'Andrea, 2016], [Dąbrowski– D'Andrea– Sitarz, 2018], [Dąbrowski– Sitarz, 2019]

Application for the Standard Model

Rephrase the universally accepted form of the **physical** Standard Model Lagrangian in the language of spectral triples being as close as possible to the Lorentzian structure.

Application for the Standard Model

- The particle content for one generation:

$$\Psi = \begin{pmatrix} \nu_R & u_R^1 & u_R^2 & u_R^3 \\ e_R & d_R^1 & d_R^2 & d_R^3 \\ \nu_L & u_L^1 & u_L^2 & u_L^3 \\ e_L & d_L^1 & d_L^2 & d_L^3 \end{pmatrix} \in M_4(H_W)$$

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- As an algebra $\mathcal{A}$ we take $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$-valued smooth functions on the spacetime, with representations:

$$\pi_L(\lambda, q, m)\Psi = \begin{pmatrix} \lambda & & \\ & \overline{\lambda} & \\ & & q \end{pmatrix} \Psi, \quad \pi_R(\lambda, q, m)\Psi = \Psi \begin{pmatrix} \overline{\lambda} & \\ & m^\dagger \end{pmatrix}.$$

Dirac operator

- At every point of the Minkowski space, linear operators on the space of particles can be encoded as a matrix from $M_4(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_4(\mathbb{C})$.
- Dirac operator: $\mathcal{D}_{SM}\Psi = \mathcal{D}\Psi + \mathcal{D}_F\Psi$, where

$$\mathcal{D} = \begin{pmatrix} & & i\widetilde{\sigma}^\mu\partial_\mu & \\ & & & i\widetilde{\sigma}^\mu\partial_\mu \\ i\sigma^\mu\partial_\mu & & & \\ & i\sigma^\mu\partial_\mu & & \end{pmatrix},$$

and $\mathcal{D}_F$ is a finite endomorphism of $M_4(H_W)$. Here $\widetilde{\sigma}^0 = \sigma^0 = 1_2$ and $\widetilde{\sigma}^j = -\sigma^j$ for $j = 1, 2, 3$.

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- As a result, $\mathcal{D}_F$ commutes with the chirality $\Gamma = \pi_L(1, -1, 1)$.
- Therefore, $\mathcal{D}_{SM} = \mathcal{D} + \mathcal{D}_F$ with $\{\mathcal{D}, \Gamma\} = 0$ and $[\mathcal{D}_F, \Gamma] = 0$.
- Krein-shifted operators behave in the opposite way.

Possible Dirac operators

Theorem

With the above assumptions,

- *Requiring* $\widetilde{\mathcal{D}_{SM}}$ *to satisfy the first order condition implies*

$$\widetilde{\mathcal{D}_F} = \begin{pmatrix} & M_l \\ M_l^\dagger & \end{pmatrix} \otimes 1_2 \otimes e_{11} + \begin{pmatrix} & M_q \\ M_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes (1_4 - e_{11}),$$

  where $M_l, M_q \in M_2(\mathbb{C})$.

- *if* M_l, M_q *are nondegenerate then* $\widetilde{\mathcal{D}_{SM}}$ *satisfies the spin*$_c$ *condition.*

The Standard Model with three generations of particles

- Hilbert space: $M_4(H_W) \otimes \mathbb{C}^3$.
- Representation enlarged diagonally.
- $M_l, M_q \in M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$:

$$M_l = \begin{pmatrix} \Upsilon_\nu & \\ & \Upsilon_e \end{pmatrix}, \qquad M_q = \begin{pmatrix} \Upsilon_u & \\ & \Upsilon_d \end{pmatrix},$$

with Υ_e, Υ_u - diagonal, $\Upsilon_\nu = U\widetilde{\Upsilon_\nu}U^\dagger$, $\Upsilon_d = V\widetilde{\Upsilon_d}V^\dagger$,

U– Pontecorvo–Maki–Nakagawa–Sakata matrix,

V– Cabibbo–Kobayashi–Maskawa matrix.

The Standard Model with three generations of particles

Theorem

The spin-c condition holds provided that for both pairs of matrices $(\Upsilon_\nu, \Upsilon_e)$ and (Υ_u, Υ_d) their eigenvalues are pairwise different.

- This is the same condition as for Hodge duality [Dąbrowski–Sitarz, 2019]

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- This is the same condition as for Hodge duality [Dąbrowski–Sitarz, 2019]
- This condition is satisfied for physical Standard Model provided that there is no massless neutrino [Dąbrowski–Sitarz, 2019]
- The model can be doubled: the resulting spectral triple satisfies the Hodge duality and is the finite part of the one studied in the almost-commutative framework.

CP violation and reality of the spectral triple

- The usual 0th order condition is not implemented by $\mathcal{J}$, but its milder version is: $\pi_R(\mathcal{A}) \subseteq \mathcal{J}\pi_L(\mathcal{A})\mathcal{J}^{-1}$.

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- The presence of the real structure implies the reality of M_l and M_q.
- One generation: fermion masses are real.
- Three generations: both the Wolfenstein parameter $\overline{\eta}$ and the CP-violating phase δ^{ν}_{CP} have to vanish.
- **CP-violation** $\Leftrightarrow$ shadow of the $\mathcal{J}$-symmetry violation in the non-doubled spectral triple.

Relation with twisted spectral triples

- $\widetilde{\mathcal{D}_{SM}}$ satisfies the order one condition,
- $\mathcal{D}_{SM}$ satisfies its twisted version:

$$[[\mathcal{D}_{SM}, \pi_L(a)]_\beta, \pi_R(b)]_\beta = 0,$$

where $[x, y]_\beta = xy - \beta y \beta^{-1} x$.

The next goal: spectral action

(1) Describe gauge transformations

(2) Find fluctuated Dirac operator

(3) Compute the spectral action

Gauge transformations

- $U_{LR} := \pi_L(U)\pi_R(U)$ for $U = (u_1, u_2, u_3) \in \mathcal{U}(\mathcal{A})$.
- They form a group $(U(1) \times SU(2) \times U(3))/(\mathbb{Z}/2\mathbb{Z})$.

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- To have $SU(3)$ rather than $U(3)$ one could impose unimodularity condition.
- The left action is already unimodular.
- The unimodularity for the right action could be be imposed either on each fundamental component or in the full representation.

Unimodularity

- In the first case: $u_1 \det u_3 = 1$ and the gauge group of the Standard Model $\left(U(1) \times SU(2) \times SU(3)\right)/(\mathbb{Z}/6\mathbb{Z})$
- In the second case: $(u_1 \det u_3)^{12} = 1$ and the group differs by a finite factor.

Fluctuated Dirac operator

- $\widetilde{\mathcal{D}_{SM}}^{\omega} = \widetilde{\mathcal{D}_{SM}} + \omega$ with

$$
\begin{aligned}
\omega = {} & A_\mu e_{11} \otimes \sigma^\mu \otimes (1_4 - e_{11}) - 2A_\mu e_{22} \otimes \sigma^\mu \otimes e_{11} \\
& - A_\mu e_{22} \otimes \sigma^\mu \otimes (1_4 - e_{11}) - A_\mu (e_{33} + e_{44}) \otimes \tilde{\sigma}^\mu \otimes e_{11} \\
& + \begin{pmatrix} 0_2 & \\ & W_\mu \end{pmatrix} \otimes \widetilde{\sigma}^\mu \otimes 1_4 + \begin{pmatrix} 1_2 & \\ & 0_2 \end{pmatrix} \otimes \sigma^\mu \otimes \begin{pmatrix} 0_1 & \\ & G_\mu \end{pmatrix} \\
& + \begin{pmatrix} 0_2 & \\ & 1_2 \end{pmatrix} \otimes \widetilde{\sigma}^\mu \otimes \begin{pmatrix} 0_1 & \\ & G_\mu \end{pmatrix} + \begin{pmatrix} & M_l \Phi \\ \Phi^\dagger M_l^\dagger & \end{pmatrix} \otimes 1_2 \otimes e_{11} \\
& + \begin{pmatrix} & M_q \Phi \\ \Phi^\dagger M_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes (1_4 - e_{11}).
\end{aligned}
$$

Physical parametrization (for one generation)

- Since $\Phi \in \mathbb{H}$ we can write $\Phi = \begin{pmatrix} \phi_1 & \phi_2 \\ -\overline{\phi_2} & \overline{\phi_1} \end{pmatrix}$.
- Define $\Phi_x := M_x(1_2 + \Phi)$, for $x = l, q$.
- The Higgs dublet $H := \begin{pmatrix} 1 + \phi_1 \\ \phi_2 \end{pmatrix}$

Simplified models

We consider two simplified versions of the full model:

- Static and spatial model,
- Wick rotated model.

Static and spatial model

We consider time-independent and spatial part of the Dirac operator.

$$\widetilde{\mathcal{D}_L} = i \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes \sigma^j \partial_j + \begin{pmatrix} & \Phi_l \\ \Phi_l^\dagger & \end{pmatrix} \otimes 1_2$$

$$+ A_j \begin{pmatrix} \sigma^3 - 1_2 & \\ & 1_2 \end{pmatrix} \otimes \sigma^j - \begin{pmatrix} 0_2 & \\ & W_j \end{pmatrix} \otimes \sigma^j .$$

$$\widetilde{\mathcal{D}_Q} = i \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes \sigma^j \partial_j \otimes 1_3 + \begin{pmatrix} & \Phi_q \\ \Phi_q^\dagger & \end{pmatrix} \otimes 1_2 \otimes 1_3$$

$$+ A_j \begin{pmatrix} \sigma^3 + \frac{1}{3} 1_2 & \\ & -\frac{1}{3} 1_2 \end{pmatrix} \otimes \sigma^j \otimes 1_3$$

$$- \begin{pmatrix} 0_2 & \\ & W_j \end{pmatrix} \otimes \sigma^j \otimes 1_3 + \begin{pmatrix} 1_2 & \\ & -1_2 \end{pmatrix} \otimes \sigma^j \otimes G_j .$$

Static and spatial model

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Static and spatial model

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- Gilkey-Seeley-DeWitt coefficients for three generations:

$$a_2 = -\frac{1}{4\pi^2} a \int d^4x \, |H|^2,$$

$$a_4 = \frac{1}{8\pi^2} \int d^4x \left[b|H|^4 + a\mathrm{Tr}|D_j H|^2 + \frac{20}{3} F^2 + 2\mathrm{Tr} W^2 + 2\mathrm{Tr} G^2 \right],$$

where

$$a = \mathrm{Tr}(\Upsilon_\nu^\dagger \Upsilon_\nu) + \mathrm{Tr}(\Upsilon_e^\dagger \Upsilon_e) + 3\mathrm{Tr}(\Upsilon_u^\dagger \Upsilon_u) + 3\mathrm{Tr}(\Upsilon_d^\dagger \Upsilon_d),$$

$$b = \mathrm{Tr}(\Upsilon_\nu^\dagger \Upsilon_\nu)^2 + \mathrm{Tr}(\Upsilon_e^\dagger \Upsilon_e)^2 + 3\mathrm{Tr}(\Upsilon_u^\dagger \Upsilon_u)^2 + 3\mathrm{Tr}(\Upsilon_d^\dagger \Upsilon_d)^2.$$

Static and spatial model

- Effective Lagrangian: $\mathcal{L} = \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{gauge}}$, where

$$\mathcal{L}_{\text{Higgs}} = \frac{bf(0)}{2\pi^2}|H|^4 - \frac{2f_2\Lambda^2 a}{\pi^2}|H|^2 + \frac{af(0)}{2\pi^2}\mathrm{Tr}|D_j H|^2,$$

$$\mathcal{L}_{\text{gauge}} = \frac{f(0)}{\pi^2}\left(\frac{10}{3}F^2 + \mathrm{Tr}\,W^2 + \mathrm{Tr}\,G^2\right).$$

- This result is consistent with taking the static part of the Lorentzian Lagrangian for the Standard Model.
- All the relations between the coefficients of the model are exactly the same as in the usual almost-commutative Euclidean formulation. Therefore, the measurable quantities will have the same values.

Wick rotated model - leptonic sector

We take the Lorentzian Dirac operator:

$$\mathcal{D}_L = i \begin{pmatrix} & 1_2 \otimes \widetilde{\sigma}^\mu \\ 1_2 \otimes \sigma^\mu & \end{pmatrix} \partial_\mu + A_\mu \begin{pmatrix} & -1_2 \otimes \widetilde{\sigma}^\mu \\ (\sigma^3 - 1_2) \otimes \sigma^\mu & \end{pmatrix}$$

$$+ \begin{pmatrix} & W_\mu \otimes \widetilde{\sigma}^\mu \\ 0_4 & \end{pmatrix} + \begin{pmatrix} \Phi_l^\dagger & \\ & \Phi_l \end{pmatrix} \otimes 1_2.$$

and Wick rotate it ($\sigma^j \to i\sigma^j$):

$$\begin{aligned}\mathcal{D}_{L,w} =& i \begin{pmatrix} & 1_2 \\ 1_2 & \end{pmatrix} \otimes 1_2 \partial_0 + i \begin{pmatrix} & -i1_2 \\ i1_2 & \end{pmatrix} \otimes \sigma^j \partial_j \\ &+ A_0 \begin{pmatrix} & -1_2 \\ (\sigma^3 - 1_2) & \end{pmatrix} \otimes 1_2 + A_j \begin{pmatrix} & i1_2 \\ i(\sigma^3 - 1_2) & \end{pmatrix} \otimes \sigma^j \\ &+ \begin{pmatrix} & W_0 \\ 0_2 & \end{pmatrix} \otimes 1_2 - \begin{pmatrix} & iW_j \\ 0_2 & \end{pmatrix} \otimes \sigma^j + \begin{pmatrix} \Phi_l^\dagger & \\ & \Phi_l \end{pmatrix} \otimes 1_2\end{aligned}$$

Wick rotated model: results

- We repeat this procedure for the quark sector.
- Next, we compute $\mathcal{D}_w^\dagger \mathcal{D}_w$ for the full Wick rotated Dirac operator.
- The Gilkey-Seeley-DeWitt coefficients:

$$a_2 = \frac{3}{4\pi^2} a \int d^4x |H|^2,$$

$$a_4 = \frac{1}{8\pi^2} \int d^4x \Big[b|H|^4 - a\mathrm{Tr}|D_\mu|^2 + \frac{20}{3} F^2 + 2\mathrm{Tr}(W^2) + 2\mathrm{Tr}(G^2)$$
$$+12\varepsilon^{jkl} F_{jk} F_{0l} - 6\varepsilon^{jkl} \mathrm{Tr}(W_{jk} W_{0l}) \Big]$$

Wick rotated model: results

- The resulting Euclidean action reads:

$$\mathcal{L}_{\text{gauge}} = \frac{f(0)}{\pi^2}\left(\frac{10}{3}F^2 + \mathrm{Tr}(W^2) + \mathrm{Tr}(G^2) + 6\varepsilon^{jkl}F_{jk}F_{0l} - 3\varepsilon^{jkl}\mathrm{Tr}(W_{jk}W_{0l})\right),$$

$$\mathcal{L}_H = \frac{bf(0)}{2\pi^2}|H|^4 + \frac{6f_2\Lambda^2}{\pi^2}a|H|^2 - \frac{af(0)}{2\pi^2}\mathrm{Tr}|D_\mu H|^2$$

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- **Spectral action** reproduces the action of the Lorentzian Standard Model with an additional electroweak "θ-term".

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- **Spectral action** reproduces the action of the Lorentzian Standard Model with an additional electroweak "θ-term".
- Potentially different coefficients do not finally affect the numerical values of the measurable parameters.

Summary

- **No fermion doubling**.
- **No** $SU(3)$ **breaking**.
- Order-one condition holds.
- Lack of real structure $\rightarrow$ **CP violation**.
- Spectral triple obeys the Morita condition of spin$_c$ geometry.
- The potentially interesting topological terms appears.

Thank you for your attention!