Horizon Europe Weeks meeting on

Noncommutative Geometry and Physics: Quantum Spacetimes

29.11-1.12 2021, Wieliczka (Kraków)

The Levi-Civita connection in noncommutative Riemannian geometry: braided symmetric algebras

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Plan: present a canonical construction of Noncommutative Riemannian Geometry, including existence and uniqueness of the Levi-Civita connection, on a wide class of noncommutative algebras (e.g. algebras of coordinate functions on noncommutative manifolds). *[P.A. e-Print: 2006.02761]*

Data:

- (H, R) triangular Hopf algebra, or quantun group
  (a subalgebra of the UEA of infinitesimal quantum diffeomorphisms on the NC manifold).

  We use the notation $R^{-1} = \bar{R}^{\alpha} \otimes \bar{R}_{\alpha} \in H \otimes H$

  Triangularity: $R^{-1} = R_{21}$.

- A an H-module algebra

  A carries a representation of (is symmetric under) the quantum group H.

- The product in A is braided commutative:

$$ab = (\bar{R}^{\alpha} \triangleright b)(\bar{R}_{\alpha} \triangleright b) \ .$$

Easiest examples of NC spacetimes are

$$(I) \qquad [x^i, x^j] = i\theta^{ij} \qquad \textit{canonical}$$

$$(II) \qquad [x^i, x^j] = if^{ij}{}_k x^k \qquad \textit{Lie algebra}$$

$$(III) \qquad x^i x^j - q x^j x^i = 0 \qquad \textit{quantum plane}$$

(I) and (III) have a triangular Hopf algebra symmetry (with $H = \mathcal{O}(T^d)$). Some examples from (II), e.g. $x^0x^i - x^ix^0 = \frac{i}{\kappa}x^i$, $x^ix^j - x^jx^i = 0$, that is κ-Minkowski spacetime, have a triangular Hopf algebra symmetry.

Further examples:

- All NC algebras arising as Drinfeld twist (2-cocycle) deformations of commutative algebras are of this kind: e.g. NC-torus; Connes-Landi spheres, Connes–Dubois-Violette NC manifolds....

- Any cotriangular Hopf algebra, for example Sweedler Hopf algebra H_4.

The present work genralizes previous studies.

For the Moyal-Weyl NC plane, $\mathbb{R}^n_\theta$, with relations $[x^\mu, x^\nu] = i\theta^{\mu\nu}$ the partial derivatives ∂_μ are derivations of the algebra. Easy differential calculus. Levi-Civita connection, Ricci tensor and scalar curvature were studied in [Wess et al. 2005].

The partial derivatives ∂_μ are derivations also of the NC torus algebra, hence the same expression for the Levi-Civita connection applies also to T^n_θ, cf. [Rosenberg 2013].

Further cases have been considered in [Arnlind et al. 2017], here too relying on the existence of a "big enough" set of derivations. See also [Bhowmick, Goswami, Landi 2020] and [P.A. Castellani 2010].

In the present study there is no assumption on the existence of derivations of the algebra, and no use of special coordinates. Indeed we use a global, coordinate independent, approach.

We retrive the results in [Wess et al. 2005] by considering coordinates x^μ and partial derivatives ∂_μ.

We complement the results in [Wess et al. 2006] where we used an arbitrary twist but we did not have an explicit formula for the Levi-Civita connection.

We consider a categorical approach that clarifies the requirements for the construction of NC Riemmanian geometry.

Triangular Hopf algebras and representations

$(H,\mathcal{R})$ a triangular Hopf algebra over a field $\Bbbk$ ($\mathbb{C}$ or $\mathbb{C}[[\hbar]]$ power series in $\hbar$)

${}^H\mathscr{M}$ category of left H-modules, objects in ${}^H\mathscr{M}$ are H-modules V.

H is a bialgebra $\Rightarrow$ $V\otimes W$ is still an H-module (${}^H\mathscr{M}$ is a monoidal category).

(H,R) triangular $\Rightarrow$ $V\otimes W$ isomorphic to $W\otimes V$:

$$\tau_{V,W} : V\otimes W \longrightarrow W\otimes V\ , \qquad v\otimes w \longmapsto \left(\bar{R}^\alpha \triangleright w\right)\otimes\left(\bar{R}_\alpha \triangleright v\right) \tag{1}$$

where $\mathcal{R}^{-1} = \bar{R}^\alpha\otimes\bar{R}_\alpha$. The category ${}^H\mathscr{M}$ is braided symmetric $\tau^2 = id$.

H is a Hopf algebra $\Rightarrow$ $\mathrm{hom}_\Bbbk(V,W)$ is in ${}^H\mathscr{M}$; for all $L : V\to W$ and $h\in H$,

$$(h\triangleright L)(v) = h_{(1)}\triangleright(L(S(h_{(2)})\triangleright v))\,, \tag{2}$$

${}^H\mathscr{M}$ is a braided closed monoidal category

Another H-action on linear maps $L : V\to W$

$$(h\triangleright^{cop}\tilde{L})(v) := h_{(2)}\triangleright(\tilde{L}(S^{-1}(h_{(1)})\triangleright v))$$

We have the H-module ${}_\Bbbk\mathrm{hom}(V,\cdot)$.

Summary:
$L \in \mathrm{hom}_{\Bbbk}(V,W)$, $\tilde{L} \in \mathrm{hom}_{\Bbbk}(V,W)$.

$({}^{H}\mathscr{M}, \otimes, \mathrm{hom}_{\Bbbk})$ and $({}^{H}\mathscr{M}, \otimes, {}_{\Bbbk}\mathrm{hom})$ related via the braiding.

Tensor products [Majid 1994]

Given linear maps $L \in \mathrm{hom}_{\Bbbk}(V,W)$, $L' \in \mathrm{hom}_{\Bbbk}(V',W')$

$$L \otimes_{\mathcal{R}} L' := (L \circ \bar{R}^{\alpha} \triangleright) \otimes (\bar{R}_{\alpha} \triangleright L') \in \mathrm{hom}_{\Bbbk}(V \otimes W, \tilde{V} \otimes \widetilde{W}) \ ,$$

In particular $L \otimes_{\mathcal{R}} \mathrm{id} = L \otimes \mathrm{id}$, $\mathrm{id} \otimes_{\mathcal{R}} L' = \bar{R}^{\alpha} \triangleright \ \otimes \bar{R}_{\alpha} \triangleright L'$

Given linear maps $\tilde{L} \in {}_{\Bbbk}\mathrm{hom}(V,W)$, $\tilde{L}' \in {}_{\Bbbk}\mathrm{hom}(V',W')$

$$\tilde{L} \,\tilde{\otimes}_{\mathcal{R}} \tilde{L}' := (\bar{R}^{\alpha} \triangleright^{cop} \tilde{L}) \otimes (\tilde{L}' \circ \bar{R}_{\alpha} \triangleright) \in {}_{\Bbbk}\mathrm{hom}(V \otimes W, \tilde{V} \otimes \widetilde{W}) \ ,$$

* * *

Consider now an H-module algebra A that is braided commutative:

$$ab = (\bar{R}^\alpha \triangleright b)(\bar{R}_\alpha \triangleright a) \ .$$

Category ${}^H_A\mathscr{M}^{\mathsf{sym}}_A$ of H-modules A-bimodules that are braided symmetric:

for all $v \in V$, $\quad a\,v = (\bar{R}^\alpha \triangleright v)(\bar{R}_\alpha \triangleright a)$

$({}^H_A\mathscr{M}^{\mathsf{sym}}_A, \otimes_A)$ is a braided monoidal category.

Moreover, as for ${}^H\mathscr{M}$, we have two closed monoidal structures

$$({}^H_A\mathscr{M}_A{}^{\mathsf{sym}}, \otimes_A, \mathsf{hom}_A) \ , \ \ ({}^H_A\mathscr{M}_A{}^{\mathsf{sym}}, \otimes_A, {}_A\mathsf{hom}) \ .$$

$\mathsf{hom}_A(V,W)$ module of right A-linear maps $L(va) = L(v)a$.

${}_A\mathsf{hom}(V,W)$ module of left A-linear maps: $\tilde{L}(av) = \tilde{a}\tilde{L}(v)$.

Classical example: A-bimodule of 1-forms or of vector fields on a manifold.

- We further consider only **finitely generated and projective modules**, hence every A-bimodule V has a dual A-bimodule ${}^*V = {}_A\mathsf{hom}(V,A)$.

We thus work in a rigid braided symmetric monoidal category, i.e. compact closed category.

Differential and Cartan Calculus

[T. Weber 2019]
(twist deformation case in
[P.A, Dimitrievich, Meyer, Wess '06])

The module $\mathsf{Der}(A)$ in ${}^H_A\mathscr{M}^{\mathsf{sym}}_A$ of *braided* derivations.

$u \in \mathsf{Der}(A) \subset \mathsf{hom}_{\mathbb{k}}(A, A)$ with braided Leibniz rule

$$u(ab) = u(a)b + (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright u)(b) \ .$$

$h \triangleright u$ and $au(b) := a\,u(b)$ are again braided derivations.

The braided commutator

$$[\ ,\ ] : \mathsf{Der}(A) \otimes \mathsf{Der}(A) \to \mathsf{Der}(A) \ , \quad u \otimes v \mapsto uv - (\bar{R}^\alpha \triangleright v)(\bar{R}_\alpha \triangleright u)$$

structures $\mathsf{Der}(A)$ as a braided Lie algebra,

$$[u, v] = -[\bar{R}^\alpha \triangleright v, \bar{R}_\alpha \triangleright u]$$

$$[u, [v, z]] = [[u, v], z] + [\bar{R}^\alpha \triangleright v, [\bar{R}_\alpha \triangleright u, z]] \ .$$

<u>*Braided derivations*</u> *based differential calculus*

Dual module of 1-forms

$$\Omega(A) := {}_A\mathsf{hom}(\mathsf{Der}(A), A)$$

(left A-linear maps). Pairing:

$$u \otimes \omega \mapsto \langle u, \omega \rangle$$

Exterior derivative

$$\langle u, \mathsf{d}a \rangle = u(a) \ ,$$

Contraction operator

$$\mathsf{i}_u(\omega) = \langle u, \omega \rangle \ . \tag{3}$$

Generalize the pairing to the tensor algebra

$$\langle \nu \otimes_A u, \omega_1 \otimes_A \omega_2 \ldots \omega_p \otimes_A v_1 \otimes_A \ldots v_q \rangle = \langle \nu \, , \langle u_1, \omega_1 \rangle \, \omega_2 \ldots \otimes_A v_1 \otimes_A v_q \rangle \ .$$

Exterior product

$$\omega \wedge \omega' := \omega \otimes_A \omega' - \bar{R}^\alpha \triangleright \omega' \otimes_A \bar{R}_\alpha \triangleright \omega \ , \tag{4}$$

is braided antisymmetric.

Lie derivative

$$\mathscr{L}_u(a) := u(a)\ ,\ \ \mathscr{L}_u(v) := [u,v]\ .$$

$\mathscr{L}$ is H-equivariant. Extended to the tensor algebra via:

$$\mathscr{L}_u(\nu \otimes_A \nu') = \mathscr{L}_u(\nu) \otimes_A \nu' + \bar{R}^\alpha \triangleright \nu \otimes_A \mathscr{L}_{\bar{R}_\alpha \triangleright u}(\nu')$$

and on contravariant tensor fields is canonically defined by duality,

$$\mathscr{L}_u\langle \nu, \theta\rangle = \langle \mathscr{L}_u \nu, \theta\rangle + \langle \bar{R}^\alpha \triangleright \nu, \mathscr{L}_{\bar{R}_\alpha \triangleright u}\theta\rangle \tag{5}$$

Theorem (Braided Cartan calculus) [T. Weber]

$$\begin{aligned} [\mathscr{L}_u, \mathscr{L}_v] &= \mathscr{L}_{[u,v]_\mathcal{R}}, & [\mathsf{i}_u, \mathsf{i}_v] &= 0\,, \\ [\mathscr{L}_u, \mathsf{i}_v] &= \mathsf{i}_{[u,v]_\mathcal{R}}, & [\mathsf{i}_u, \mathsf{d}] &= \mathscr{L}_u, \\ [\mathscr{L}_u, \mathsf{d}] &= 0\,, & [\mathsf{d}, \mathsf{d}] &= 0\,, \end{aligned}$$

where $[L, L'] = L \circ L' - (-1)^{|L||L'|}\bar{R}^\alpha(L') \circ \bar{R}_\alpha(L)$ is the graded braided commutator of $\mathbb{k}$-linear maps L, L' on $\Omega^\bullet(A)$ of degree $|L|$ and $|L'|$.

Connections and Cartan equation

Def. A *right* connection on a module Γ in ${}^H_A\mathscr{M}_A{}^{\mathsf{sym}}$ is a $\mathbb{k}$-linear map

$$\nabla : \Gamma \to \Gamma \otimes_A \Omega \tag{6}$$

which satisfies the Leibniz rule, for all $s \in \Gamma$, $a \in A$,

$$\nabla(sa) = \nabla(s)a + s \otimes_A \mathrm{d}a \ . \tag{7}$$

A *left* connection on Γ is a $\mathbb{k}$-linear map

$$\nabla : \Gamma \to \Omega \otimes_A \Gamma \tag{8}$$

which satisfies the Leibniz rule,

$$\nabla(as) = \mathrm{d}a \otimes_A s + a\nabla(s) \ . \tag{9}$$

$$\mathsf{Con}_A(\Gamma) = \{ \text{ right connections } \} \ , \quad {}_A\mathsf{Con}_A(\Gamma) = \{ \text{ left connections } \}$$

Since Γ is a braided commutative A-bimodule a right connection ∇ on Γ is also a braided left connection [P.A, Schenkel '14] (similarly a left connection ∇ on Γ is also a braided right connection):

$$\nabla(as) = (\bar{R}^\alpha \triangleright a)(\bar{R}_\alpha \triangleright \nabla)(s) + \bar{R}^\alpha \triangleright s \otimes_A \bar{R}_\alpha \triangleright \mathrm{d}a$$

If ∇ is H-equivariant we recover the notion of connection on central A-bimodules studied in [Dubois-Violette, Michor].

Given connections $\nabla \in \mathsf{Con}_A(\Gamma)$, $\widehat{\nabla} \in \mathsf{Con}_A(\widehat{\Gamma})$ on the modules Γ and $\widehat{\Gamma}$ in ${}^H_A\mathscr{M}_A{}^{\mathsf{sym}}$ we consider the connection $\nabla \oplus \widehat{\nabla} \in {}_A\mathsf{Con}(\Gamma \otimes_A \widehat{\Gamma})$ **well defined** by

$$\begin{aligned} \nabla \oplus \widehat{\nabla} : \Gamma \otimes_A \widehat{\Gamma} &\longrightarrow \Gamma \otimes_A \widehat{\Gamma} \otimes_A \Omega(A) \\ s \otimes_A \widehat{s} &\longmapsto \tau_{23} \circ (\nabla(s) \otimes_A \widehat{s}) + (\bar{R}^\alpha \triangleright s) \otimes_A (\bar{R}_\beta \triangleright \widehat{\nabla})(\widehat{s}) \end{aligned}$$

where we have used the braiding isomorphism $\tau_{23} : \Gamma \otimes_A \Omega(A) \otimes_A \widehat{\Gamma} \to \Gamma \otimes_A \widehat{\Gamma} \otimes \Omega(A)$.
(a similar formula holds for left connections).

Extend ∇ to

$$\mathsf{d}_\nabla : \Gamma \otimes_A \Omega^\bullet(A) \longrightarrow \Gamma \otimes_A \Omega^{\bullet+1}(A) \ ,$$

by

$$\mathsf{d}_\nabla(s \otimes_A \theta) = \nabla(s) \otimes_A \theta + s \otimes_A \mathsf{d}\theta \ ,$$

d_∇ satisfies the *unbraided* Leibniz rule,

$$\mathsf{d}_\nabla(\varsigma \wedge \vartheta) = \mathsf{d}_\nabla\varsigma \wedge \vartheta + (-1)^k \varsigma \wedge \mathsf{d}\vartheta$$

Curvature

The curvature of $\nabla \in {}_A\mathsf{Con}(\Gamma)$ is

$$\mathsf{d}_\nabla^2 = \mathsf{d}_\nabla \circ \mathsf{d}_\nabla \ .$$

It is a left $\Omega^\bullet(A)$-linear map,

$$\mathsf{d}_\nabla^2 \in {}_{\Omega^\bullet(A)}\mathsf{hom}(\Omega^\bullet(A) \otimes_A \Gamma, \Omega^{\bullet+2} \otimes_A \Gamma) \ .$$

Torsion

For $\Gamma = \Omega(A)$,

$$\theta \mapsto (\mathrm{d} + \wedge \circ \nabla)\theta \ .$$

All other expression of curvature and torsion are equivalent due to Cartan structure equations.

Dual connections & Cartan structure equation for curvature and torsion

Let ∇ now denote the connection dual to ∇, i.e.

$$\mathrm{d}\langle u, \theta\rangle = \langle \mathrm{d}_{\nabla} u, \theta\rangle + \langle u, \mathrm{d}_{\nabla}\theta\rangle \ .$$

Then

$$\langle R_{\nabla}(u, v, z), \theta\rangle = \langle u \otimes_A v \otimes_A z, \mathrm{d}_{\nabla}^2 \theta\rangle$$

$$\langle T_{\nabla}(u, v), \theta\rangle = -\langle u \otimes_A v, (\mathrm{d} + \wedge \circ \nabla)\, \theta\rangle$$

Braided Riemaniann geometry

Braided-symmetric elements. Let $\mathrm{g} \in \Omega(A) \otimes_A \Omega(A)$.
Def. g is braided symmetric if $\tau(\mathrm{g}) = \mathrm{g}$

Example: $\omega \otimes_\star \omega' + (\bar{R}^\alpha \triangleright \omega') \otimes_\star (\bar{R}_\alpha \triangleright \omega)$ is braided symmetric.

Given $\mathrm{g} \in \Omega(A) \otimes_A \Omega(A)$ we write $\mathrm{g} = \mathrm{g}^a \otimes_A \mathrm{g}_a$ and define

$$\mathrm{g}^\flat : \mathfrak{X}(A) \to \Omega(A) \ , \quad v \mapsto \mathrm{g}^\flat(v) = \langle v, \mathrm{g}^a \rangle \, \mathrm{g}_a$$

Def. A pseudo-Riemannian metric on $\mathfrak{X}A$ in ${}^H_A\mathscr{M}_A^{\mathrm{sym}}$ is a braided symmetric element $\mathrm{g} \in \Omega(A) \otimes_A \Omega(A)$, with $\mathrm{g}^\flat$ that is an isomorphism.

Let $\mathrm{g} \in \Omega(A) \otimes_A \Omega(A)$ be a pseudo-Riemannian metric. A connection $\nabla \in \mathrm{Con}_A(\Omega(A))$ is metric compatible if it satisfies $\nabla(\mathrm{g}) = 0$. A connection $\nabla \in {}_A\mathrm{Con}(\mathfrak{X}(A))$ is metric compatible if its dual $\nabla \in \mathrm{Con}_A(\Omega(A))$ is metric compatible. It follows

$$\mathrm{d}\langle v \otimes_A z, \mathrm{g} \rangle = \langle \nabla(v \otimes_A z), \mathrm{g} \rangle$$

A Levi-Civita connection is a metric compatible and torsion free connection.

Existence and uniqueness of Levi-Civita connection is proven, similarly to the classical case, via a **braided Koszul formula**.

For all $u, v, z \in \mathrm{Der}(A)$, (braiding omitted)

$$\mathscr{L}_u\langle v \otimes_A z, \mathrm{g}\rangle = \langle \nabla_u(v \otimes_A z), \mathrm{g}\rangle$$
$$= \langle z \otimes_A \nabla_v u \,, \mathrm{g}\rangle + \langle [u, v] \otimes_A z \,, \mathrm{g}\rangle + \langle v \otimes_A \nabla_u z \,, \mathrm{g}\rangle$$

Summing $\mathscr{L}_u\langle v \otimes_A z, \mathrm{g}\rangle - \mathscr{L}_z\langle u \otimes_A v, \mathrm{g}\rangle + \mathscr{L}_v\langle z \otimes_A u, \mathrm{g}\rangle$ (braiding omitted) we obtain

$$2\langle {}^{\alpha}v \otimes_A \nabla_{{}_{\alpha}u}\, z \,, \mathrm{g}\rangle = \mathscr{L}_u\langle v \otimes_A z, \mathrm{g}\rangle - \mathscr{L}_{{}^{\alpha}v}\langle {}_{\alpha}u \otimes_A z, \mathrm{g}\rangle + \mathscr{L}_{{}^{\alpha\beta}z}\langle {}_{\alpha}u \otimes_A {}_{\beta}v, \mathrm{g}\rangle$$
$$- \langle [u, v] \otimes_A z \,, \mathrm{g}\rangle + \langle u \otimes_A [v, z] \,, \mathrm{g}\rangle + \langle [u, {}^{\beta}z] \otimes_A {}_{\beta}v \,, \mathrm{g}\rangle \,.$$

were ${}^{\alpha}v := \bar{R}^{\alpha} \triangleright v$ and ${}_{\alpha}u := \bar{R}_{\alpha} \triangleright u$. Now, since u, v, z are arbitrary, the pairing is nondegenerate and the metric is also nondegenerate, knowledge of the l.h.s. uniquely defines the Levi-Civita connection.

Conclusions

Existence and uniqueness of the Levi-Civita connection ∇ *for arbitrary metric* g *(braided symmetric nondegenerate covariant 2-tensor* g*).*

We have formulated in vacuo Einstein equations and defined Enistein NC manifolds.

Gauss–Bonnet?

Possible to relax the triangular Hopf algebra condition? See talk of Thomas.