Quantized Minimal Surfaces

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Introduction

In this talk I will present a number of different approaches to solving discrete versions of minimal surface equations.

There are several motivations for doing this. First, there are physical motivations since such equations appear in matrix models, like in the IKKT model and Membrane theory. Secondly, it is of interest in mathematics to understand in what sense there is a nice theory of noncommutative minimal surfaces in analogy with the classical situation.

(Joint work with J. Choe, J. Hoppe, G. Huisken, M. Kontsevich)

Outline

1. Poisson algebraic formulation of Kähler geometry, Laplace operators and the relation to double commutator equations.
2. Discrete Minimal Surface Algebras - studying solutions of
$$\sum_{j=1}^{m}[[X^i, X^j], X^j] = \mu_i X^i.$$
3. Noncommutative Minimal Surfaces from the Weyl algebra:
$$\big[[X^i, U], U\big] + \big[[X^i, V], V\big] = 0$$
where $[U, V] = i\hbar\mathbb{1}$.
4. A noncommutative catenoid: $\sum_{j=1}^{3}\big[[X^i, X^j], X^j\big] = 0$

Note that these type of equations have been algebraically studied under the name (inhomogeneous) Yang-Mills algebras.

[Connes, Dubois-Violette, *Lett. Math. Phys 61 (2002)*], [Berger, Dubois-Violette, *Lett. Math. Phys 76 (2006)*]
[Herscovich, Solotar, *Ann. Math. 173 (2011)*]

Poisson algebraic formulation of Kähler geometry

In order to motivate the different approaches that we've taken, let me quickly review how one may formulate Kähler geometry in terms of the Poisson algebra generated by (isometric) embedding coordinates into an ambient space. (For an arbitrary *n*-dimensional Riemannian manifold, one uses multilinear *n*-brackets instead.)

[A., Hoppe, Huisken, *J. Diff Geo. 91 (2012)*], [A., Huisken, *Lett. Math. Phys 104 (2014)*]

On a Kähler manifold, the fact that the Poisson/symplectic structure is compatible with the metric has the following particular consequence (in local coordinates)

$$\gamma^2 g^{ab} = \theta^{ap}\theta^{bq} g_{pq} \tag{1}$$

where θ^{ab} is the Poisson bivector and $\gamma = 1$. The introduction of γ seems superfluous at this point, but this freedom turns out to be useful. If the Poisson structure is compatible with the metric, in the sense of (1), there exists a Poisson algebraic formulation of the Riemannian geometry.

Embedded surfaces

For simplicity (and as our examples will be of this form), let us consider the case of a surface Σ embedded in $\mathbb{R}^m$ (via the embedding coordinates $x^1, \ldots, x^m$), with a metric induced from the Euclidean metric. For an arbitrary density ρ,

$$\{f, h\} = \frac{1}{\rho}\epsilon^{ab}(\partial_a f)(\partial_b h)$$

defines a Poisson structure on Σ. The "natural" (Kähler-)choice corresponds to $\rho = \sqrt{g}$. However, setting $\gamma = \sqrt{g}/\rho$ one finds that

$$\gamma^2 g^{ab} = \theta^{ap}\theta^{bq}g_{pq}$$

(since the right-hand-side is simply the cofactor expansion of the inverse of the matrix g_{ab}).

The Laplace operator

The Laplace-Beltrami operator on Σ:

$$\Delta(f) = \frac{1}{\sqrt{g}}\partial_a\Big(\sqrt{g}g^{ab}\partial_b f\Big).$$

can be written as

$$\Delta(f) = \gamma^{-1}\sum_{i=1}^{m}\{\gamma^{-1}\{f,x^i\},x^i\}$$
$$\Delta(f) = \gamma^{-1}\{\gamma^{-1}\{f,u^a\}g_{ab},u^b\}.$$

where $\{x^i(u^1,u^2)\}_{i=1}^m$ are the embedding coordinates of Σ, and $u^1 = u, u^2 = v$ is a set of local coordinates on Σ.

Note that such reformulations have also been considered elsewehere; e.g. in [Blaschke, Steinacker, *Class. Quant. Grav. (2010)*]

The Laplace operator

For $\gamma = 1$ (i.e. $\rho = \sqrt{g}$) the first forumla becomes

$$\Delta(f) = \sum_{i=1}^{m} \{\{f, x^i\}, x^i\}.$$

For a conformal metric $g_{ab} = E\delta_{ab}$ and $\rho = 1$ (giving $\{u, v\} = 1$ and $\gamma = \sqrt{g} = E$) the second formula becomes

$$\Delta(f) = \frac{1}{E}\Big[\{\{f, u^1\}, u^1\} + \{\{f, u^2\}, u^2\}\Big].$$

Some remarks

As claimed, one can express all objects of Riemannian geometry in a similar way; for instance, the Gaussian curvature of a surface embedded in $\mathbb{R}^m$ can be computed as

$$\sum_{j,k,l=1}^{m}\left(\frac{1}{2}\{\{x^j,x^k\},x^k\}\{\{x^j,x^l\},x^l\}-\frac{1}{4}\{\{x^j,x^k\},x^l\}\{\{x^j,x^k\},x^l\}\right).$$

Morevover (however, unrelated to this talk), it is natural to turn the question around and ask: Can one do Riemannian geometry in a Poisson algebra without any reference to an underlying manifold? It turns out that one may find simple conditions for a Poisson algebra to allow for such a formulation.

[A., Al-Shujary, *J. Geom. Phys. 136 (2018)*]

Minimal surfaces in $\mathbb{R}^m$

With the help of the reformulations of the Laplace operator, one can formulate the equations for minimal surfaces in the following way.

A surface embedded in $\mathbb{R}^m$, via the embedding functions x^i, is minimal if (assuming $\rho = \sqrt{g}$)

$$\Delta(x^i) = \sum_{j=1}^{m} \{\{x^i, x^j\}, x^j\} = 0$$

or (assuming $\rho = 1$)

$$\Delta(x^i) = \{\{x^i, u\}, u\} + \{\{x^i, v\}, v\} = 0$$

when the metric is conformal; i.e. $\vec{x}'_u \cdot \vec{x}'_v = 0$ and $\vec{x}'_u \cdot \vec{x}'_u = \vec{x}'_v \cdot \vec{x}'_v$.

Minimal surfaces in S^d

We have also been interested in noncommutative analogues of minimal surfaces in S^{d-1}, which can be found by considering embeddings in $\mathbb{R}^d$ such that $|\vec{x}|^2 = 1$ and

$$\Delta(x^i) = \sum_{j=1}^{d} \{\{x^i, x^j\}, x^j\} = -2x^i$$

Noncommutative analogues of these equations also appear in Membrane theory, when constructing solutions to the equations of motion. The above equations are already very rich for $d = 4$ as there exist surfaces of arbitrary genus in S^3. [Lawson, *Ann. Math. 92 (1970)*]

We will in the following consider noncommutative/discrete versions starting from the correspondence $\{\cdot,\cdot\} \leftrightarrow [\cdot,\cdot]/(i\hbar)$.

Discrete Minimal Surface Algebras

[A., Hoppe, *SIGMA 6 (2010)*]

In this paper, we study properties of the equation

$$\Delta_{\mathcal{X}}(x_i) = \sum_{j=1}^{m} \left[[x_i, x_j], x_j\right] = \mu_i x_j. \tag{2}$$

where $\mathcal{X} = \{x_1, x_2, \ldots, x_m\}$. The above equation makes sense in a Lie algebra, as well as a (noncommutative) associate algebra. However, I will in this talk not focus so much on algebraic properties, but rather on solutions.

The set $\{\mu_1, \ldots, \mu_m\}$ will be called the *spectrum*.

Clifford algebra solutions

Let $\mathrm{Cl}_{p,q}$ be a Clifford algebra generated by $e_1, e_2, \ldots, e_{p+q}$ with

$$\begin{aligned} e_i^2 &= 1 \qquad \text{for } i = 1, \ldots, p \\ e_i^2 &= -1 \qquad \text{for } i = p+1, \ldots, p+q \\ e_i e_j &= -e_j e_i \qquad \text{when } i \neq j. \end{aligned}$$

It is then easy to check that

$$\sum_{j=1}^{p+q} \big[[e_i, e_j], e_j\big] = \begin{cases} 4(p-q-1)e_i & \text{if } i \in \{1, \ldots, p\} \\ 4(p-q+1)e_i & \text{if } i \in \{p+1, \ldots, p+q\}. \end{cases}$$

Lie algebra solutions in $\mathfrak{sl}_n$

It is easy to see that if $\{x_1, \ldots, x_d\}$ is an orthonormal basis (with respect to the Killing form) of a semi-simple Lie algebra, then $\{x_i\}_{i=1}^d$ is a solution to the equations.

Let $\alpha_1, \ldots, \alpha_{n-1}$ denote the simple roots of $\mathfrak{sl}_n$ and for every positive root α, we choose elements $e_\alpha, e_{-\alpha}, h_\alpha$ such that

$$[h, e_\alpha] = \alpha(h)e_\alpha$$
$$[e_\alpha, e_{-\alpha}] = h_\alpha,$$

and h_α is the element of the Cartan subalgebra $\mathfrak{h}$ such that $\alpha(h) = K(h_\alpha, h)$ for all $h \in \mathfrak{h}$. Moreover, let $l^2 = \alpha(h_\alpha)$ denote the length of a root.

For every positive root α in $\mathfrak{sl}_n$, we set

$$e_\alpha^+ = ic\big(e_\alpha + e_{-\alpha}\big) \qquad \text{and} \qquad e_\alpha^- = c\big(e_\alpha - e_{-\alpha}\big),$$

for arbitrary $c \in \mathbb{R}$.

Lie algebra solutions in $\mathfrak{sl}_n$

Then the following holds

1. $\big[[e^+_\alpha, e^+_\beta], e^+_\beta\big] = \big[[e^+_\alpha, e^-_\beta], e^-_\beta\big] = -\frac{1}{2}c^2 l^2 e^+_\alpha$
(when $\alpha \pm \beta$ is a root)
2. $\big[[e^-_\alpha, e^+_\beta], e^+_\beta\big] = \big[[e^-_\alpha, e^-_\beta], e^-_\beta\big] = -\frac{1}{2}c^2 l^2 e^-_\alpha$
(when $\alpha \pm \beta$ is a root)
3. $\big[[e^\pm_\alpha, e^\mp_\alpha], e^\mp_\alpha\big] = -2c^2 l^2 e^\pm_\alpha$
4. $\big[[e^\pm_\alpha, h_\beta], h_\beta\big] = (\alpha, \beta)^2 e^\pm_\alpha$
5. $\big[[h_\alpha, e^\pm_\beta], e^\pm_\beta\big] = \mp 2c^2(\alpha, \beta) h_\beta$

Let $\mathcal{X} = \{e^\pm_{\beta_1}, \ldots, e^\pm_{\beta_d}\}$ for any positive roots β_i. In this case, $\big[[x_i, x_j], x_j\big]$ is proportional to x_i for all $x_i, x_j \in \mathcal{X}$.

Let $\mathcal{X} = \{h_{\beta_1}, \ldots, h_{\beta_k}, e^+_{\gamma_1}, e^-_{\gamma_1}, \ldots, e^+_{\gamma_l}, e^-_{\gamma_l}\}$. Now, $\big[[h_{\beta_i}, e^+_{\gamma_j}], e^+_{\gamma_j}\big]$ might not be proportional to h_{β_i}. However, since both $e^+_{\gamma_j}, e^-_{\gamma_j} \in \mathcal{X}$ this term will cancel against $\big[[h_{\beta_i}, e^-_{\gamma_j}], e^-_{\gamma_j}\big]$. Thus, $\Delta_{\mathcal{X}}(h_{\beta_i}) = 0$ for $i = 1, \ldots, k$.

Noncommutative minimal surfaces in the Weyl algebra

[A., Choe, Hoppe, *Lett. Math. Phys. 106 (2016)*]

Let us return to the equations defining a minimal surface in $\mathbb{R}^m$. We think of a parametrized minimal surface given by $\vec{x}(u,v) : U \to \mathbb{R}^3$ such that

$$\Delta(x^i) = \{\{x^i, u\}, u\} + \{\{x^i, v\}, v\} = 0$$

when the metric is conformal; i.e. $\vec{x}'_u \cdot \vec{x}'_v = 0$ and $\vec{x}'_u \cdot \vec{x}'_u = \vec{x}'_v \cdot \vec{x}'_v$. (Recall that the above formula is valid for $\{u, v\} = 1$.)

What happens, if one naively translates these conditions to noncommutative algebras? Can one find noncommutative minimal surfaces this way? There are many explicitly known minimal surfaces in $\mathbb{R}^3$.

Of course, we do not solve the (perhaps) more relevant physical equations, but one may gain insight on what to expect (and the problem turns out to be interesting in itself).

Since $\{u, v\} = 1$, it is natural to start with a noncommutative algebra generated by U and V, satisfying $[U, V] = i\hbar\mathbb{1}$. That is, the Weyl algebra, which we denote by $\mathcal{A}_\hbar$ and (for technical reasons) its fraction field $\mathcal{F}_\hbar$. Let us start from the following extremely naive definition.

Definition

An element $X = (X^1, \ldots, X^n) \in \mathfrak{F}_\hbar^n$ is called a *noncommutative minimal surface* if $(X^i)^* = X^i$ and

$$\Delta(X^i) = \big[[X^i, U], U\big] + \big[[X^i, V], V\big] = 0 \qquad \text{for } i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} (\partial_u X^i)(\partial_u X^i) = \sum_{i=1}^{n} (\partial_v X^i)(\partial_v X^i)$$

$$\sum_{i=1}^{n} \Big[(\partial_u X^i)(\partial_v X^i) + (\partial_v X^i)(\partial_u X^i)\Big] = 0$$

$\partial_u(a) = [a, V]/(i\hbar)$ and $\partial_v(a) = -[a, U]/(i\hbar)$

Noncommutative Weierstrass representation

Theorem

Let $X \in \mathfrak{F}_\hbar^3$ be a minimal with $\partial(X^1 - iX^2) \neq 0$. Then there exist r-holomorphic elements $f, g \in \mathfrak{F}_\hbar$ (i.e. $\bar{\partial} f = \bar{\partial} g = 0$) together with $x^i \in \mathbb{R}$ (for $i = 1, 2, 3$), such that

$$
\begin{aligned}
X^1 &= x^1 \mathbb{1} + \operatorname{Re} \int \frac{1}{2} f(\mathbb{1} - g^2) d\Lambda \\
X^2 &= x^2 \mathbb{1} + \operatorname{Re} \int \frac{i}{2} f(\mathbb{1} + g^2) d\Lambda \\
X^3 &= x^3 \mathbb{1} + \operatorname{Re} \int f g d\Lambda.
\end{aligned}
\tag{3}
$$

Conversely, for any r-holomorphic f and g such that $f(1 - g^2)$, $f(1 + g^2)$ and fg are integrable, (3) *defines a minimal surface.*

Note: Integration is just the "anti-derivative" in Λ.

Another classical representation formula

Proposition

Let $F \in \mathfrak{F}_\hbar$ *be r-holomorphic and assume that*

$$\Phi^1 = \big(1 - \Lambda^2\big)F, \quad \Phi^2 = i\big(1 + \Lambda^2\big)F, \quad \Phi^3 = 2\Lambda F$$

are integrable. Then $(X^1, X^2, X^3) \in \mathfrak{F}_\hbar^3$*, defined by*

$$X^i = x^i \mathbb{1} + \mathrm{Re} \int \Phi^i d\Lambda,$$

is a minimal surface for arbitrary $x^1, x^2, x^3 \in \mathbb{R}$*.*

Thus, given any polynomial $F(\Lambda)$ the above result constructs a minimal surface.

Algebraic minimal surfaces

The previous result gives a class of algebraic minimal surfaces

$$
\begin{aligned}
X^1 &= \mathrm{Re}\left[(n-1)\Big(n\Lambda^{n-2} - (n-2)\Lambda^n\Big)\right]\\
X^2 &= \mathrm{Re}\left[i(n-1)\Big(n\Lambda^{n-2} + (n-2)\Lambda^n\Big)\right]\\
X^3 &= \mathrm{Re}\left[2n(n-2)\Lambda^{n-1}\right].
\end{aligned}
$$

and the case $n = 3$ corresponds to the Enneper surface:

$$
\begin{aligned}
X^1 &= U + UV^2 - \frac{1}{3}U^3 - i\hbar V\\
X^2 &= -V - U^2V + \frac{1}{3}V^3 + i\hbar U\\
X^3 &= U^2 - V^2.
\end{aligned}
$$

A change of coordinates?

[A., Hoppe, Kontsevich, arXiv:1903.10792]

Recall that for $\{f, h\} = \frac{1}{\sqrt{g}}\epsilon^{ab}(\partial_a f)(\partial_b)$:

$$\sum_{j=1}^{m}\{\{x^i, x^j\}, x^j\} = 0 \tag{4}$$

and for $\{f, h\} = \epsilon^{ab}(\partial_a f)(\partial_b)$:

$$\{\{x^i, u\}, u\} + \{\{x^i, v\}, v\} = 0 \tag{5}$$

(in conformal parametrization) are the equations for the embedding coordinates of a minimal surface in $\mathbb{R}^m$.

For the corresponding noncommutative equations, we have an infinite number of explicit examples for the latter, but far less for the former. Is there a way to obtain solutions to (4) from solutions to (5)? How does one do it in ordinary geometry?

A change of coordinates?

Assume that $\vec{x}(u, v)$ is a parametric minimal surface and assume that $\tilde{u}(u, v)$ and $\tilde{v}(u, v)$ are such that

$$\left|\frac{\partial(\tilde{u}, \tilde{v})}{\partial(u, v)}\right| = \sqrt{g}.$$

If $\{\tilde{u}, \tilde{v}\} = 1$ then

$$\sum_{j=1}^{m} \{\{x^i, x^j\}, x^j\} = 0.$$

Can we make use of this in the noncommuative setting? Let us consider the case of the catenoid.

The catenoid

Parametrizing the catenoid in $\mathbb{R}^3$ as

$$\vec{x}(u,v) = \big(\cosh(v)\cos(u), \cosh(v)\sin(u), v\big),$$

implying $w := x^1 + ix^2 = \cosh(v)e^{iu}$ and reparametrizing as

$$\tilde{u} = u, \quad \tilde{v}(v) = \frac{1}{2}v + \frac{1}{4}\sinh(2v),$$

with

$$\left|\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\right| = \cosh^2(v) = \sqrt{g},$$

gives

$$\begin{aligned} w &= \cosh\big(v(\tilde{v})\big)e^{i\tilde{u}} \\ z &= v(\tilde{v}). \end{aligned}$$

In analogy with $\tilde{U}(f)(\varphi) = \varphi f(\varphi)$ and $\tilde{V}(f)(\varphi) = -i\hbar\frac{\partial}{\partial\varphi}f(\varphi)$, giving $e^{i\tilde{U}}(f)(\varphi) = e^{i\varphi}f(\varphi)$, $[\tilde{U}, \tilde{V}] = i\hbar\mathbb{1}$ and

$$
\begin{aligned}
e^{i\tilde{U}}e^{-in\varphi} &= e^{-i(n-1)\varphi}\\
\tilde{V}e^{-in\varphi} &= -\hbar n\, e^{-in\varphi},
\end{aligned}
$$

one makes the following Ansatz

$$
W\big|n\rangle = w_n\big|n-1\rangle \qquad Z\big|n\rangle = z_n\big|n\rangle
$$

for the operators corresponding to the functions

$$
w = \cosh\big(v(\tilde{v})\big)e^{i\tilde{u}} \qquad z = v(\tilde{v}).
$$

In terms of $W = X^1 + iX^2$ and $Z = X^3$, the equations

$$\sum_{j=1}^{3} \left[[X^i, X^j], X^j\right] = 0 \qquad (i = 1, 2, 3)$$

are equivalent to

$$\frac{1}{2}\left[[W, W^\dagger], W\right] + \left[[W, Z], Z\right] = \frac{1}{2}\left[[Z, W], W^\dagger\right] + \frac{1}{2}\left[[Z, W^\dagger], W\right] = 0.$$

In terms of our Ansatz, these equations are equivalent to

$$r_n - \tfrac{1}{2} r_{n+1} - \tfrac{1}{2} r_{n-1} + (z_n - z_{n-1})^2 = 0$$
$$r_n(z_n - z_{n-1}) = r_{n+1}(z_{n+1} - z_n)$$

for $r_n = |w_n|^2$ and $n \in \mathbb{Z}$. One immediately notes that for every solution to the above recursion relations, $c := r_n(z_n - z_{n-1})$ is constant.

Since $r_n = |w_n|^2$ we are interested in positive solutions to the recursion relations. These can be constructed as follows.

For $c \neq 0$, $r_0 > 0$ and $r_0 \leq r_1 \leq r_0 + 2c^2/r_0^2$ set

$$r_n = 2r_{n-1} - r_{n-2} + \frac{2c^2}{r_{n-1}^2} \qquad \text{for } n \geq 2 \tag{6}$$

$$r_n = 2r_{n+1} - r_{n+2} + \frac{2c^2}{r_{n+1}^2} \qquad \text{for } n \leq -1 \tag{7}$$

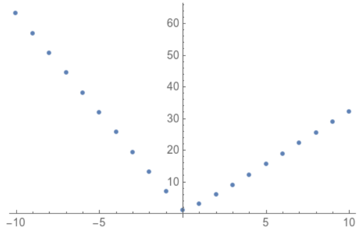
$$z_n = z_{n-1} + \frac{c}{r_n} \qquad \text{for } n \geq 1 \tag{8}$$

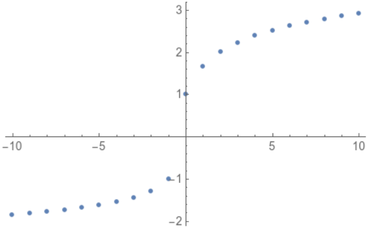
$$z_n = z_{n+1} - \frac{c}{r_{n+1}} \qquad \text{for } n \leq -1. \tag{9}$$

It is easy to see that with the initial conditions given as above, $r_n > 0$. Hence, one defines the operators

$$W|n\rangle = \sqrt{r_n}|n\rangle \qquad Z|n\rangle = z_n|n\rangle$$

Example of r_n (top) and z_n (bottom):





Summary

- I've tried to give an overview of the different approaches to quantum/noncommutative minimal surfaces we've taken.
- The algebras we construct and the equations we try to solve are motivated both from physics and mathematics.
- In particular, I've presented two different ways (in terms of different Poisson structures) to obtain noncommutative equations for minimal surfaces.
- One may find solutions to these equations, and in the Weyl algebra one obtains infinitely many explicit noncommutative minimal surfaces.
- At the end, an idea to connect the two approaches was presented, with the hope of being able to construct solutions to the equations which are more relevant in physics.

Thank you for your attention!