Notes on Spectral Geometry in NCG

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1 The Dirac operator on the sphere $\mathbb{S}^3$

We begin with a worked example of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ —an algebra $\mathcal{A}$ and an operator D act on a Hilbert space $\mathcal{H}$— where the spectrum of D can be explicitly computed. The algebra is of the form $C^\infty(M)$ where M is a compact manifold; we take $M = \mathbb{S}^3$ in this lecture.

We must determine, in turn:

- ⋆ the *Riemannian metric* g on $M = \mathbb{S}^3$, that allows us to find local orthonormal bases of vector fields $\mathfrak{X}(\mathbb{S}^3)$ and also of differential 1-forms $\mathcal{A}^1(\mathbb{S}^3)$;
- ⋆ the *spinor module* $\mathcal{S}$, together with a *Clifford action* of $\mathcal{A}^1(\mathbb{S}^3)$ on it;
- ⋆ *connections* on these $C^\infty(\mathbb{S}^3)$-modules: Levi-Civita connections ∇ on $\mathfrak{X}(\mathbb{S}^3)$ and on $\mathcal{A}^1(\mathbb{S}^3)$, and the spinor connection ∇^S on $\mathcal{S}$;
- ⋆ the *Dirac operator* $\not{D}$, acting $\mathbb{C}$-linearly on $\mathcal{S}$, and, by extension, as a self-adjoint operator on a Hilbert space completion $\mathcal{H}$ of $\mathcal{S}$;
- ⋆ a full set of *eigenvalues* and *eigenvectors* (in $\mathcal{S}$) for $\not{D}$.

The sphere $\mathbb{S}^3$ is actually the Lie group $SU(2)$, with elements

$$u = \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \in M_2(\mathbb{C}), \quad \text{with} \quad z\bar{z} + w\bar{w} = 1.$$

We also write $z = t_0 - it_3$, $w = -t_2 - it_1$, so that $t_0^2 + t_1^2 + t_2^2 + t_3^2 = 1$, so $\vec{t} \in \mathbb{S}^3 \subset \mathbb{R}^4$, and $u = t_0 \mathbf{1} - i(t_1 \sigma_1 + t_2 \sigma_2 + t_3 \sigma_3)$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{are the Pauli matrices.}$$

A *vector field* $X \in \mathfrak{X}(M)$ is just a $\mathbb{C}$-linear map $X : C^\infty(M) \to C^\infty(M)$ satisfying a *Leibniz rule*: $X(fg) = (Xf)g + f(Xg)$. These form a module over $C^\infty(M)$, usually a finitely generated projective module; but over the Lie group $\mathbb{S}^3$, all our modules will be *free* modules of finite rank, i.e., they will each have a finite basis.

1.1 A *Riemannian metric* on a manifold M is a symmetric $C^\infty(M)$-bilinear form $g\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to C^\infty(M)$. For $M=\mathbb{R}^4$, we write $\partial_j \equiv \partial/\partial t_j$ and define the *ambient metric* by $g_{\mathbb{R}^4}(\partial_k,\partial_l) := \delta_{kl}$. By restriction to $\mathbb{S}^3\subset\mathbb{R}^4$, we get a "round metric" g on $\mathbb{S}^3$. Notice that the Euler vector field $E_0 := t_0\,\partial_0 + t_1\,\partial_1 + t_2\,\partial_2 + t_3\,\partial_3$ on $\mathbb{R}^4$ is radial, i.e., normal to $\mathbb{S}^3$. We can find 3 vector fields tangent to $\mathbb{S}^3$ and orthogonal to each other by choosing:

$$\begin{aligned} E_1 &:= -t_1\,\partial_0 + t_0\,\partial_1 + t_3\,\partial_2 - t_2\,\partial_3, \\ E_2 &:= -t_2\,\partial_0 - t_3\,\partial_1 + t_0\,\partial_2 + t_1\,\partial_3, \\ E_3 &:= -t_3\,\partial_0 + t_2\,\partial_1 - t_1\,\partial_2 + t_0\,\partial_3. \end{aligned}$$

Every $C^\infty(M)$-module $\mathcal{E}$ has a *dual module* $\mathcal{E}^\sharp := \mathrm{Hom}_{C^\infty(M)}(\mathcal{E}, C^\infty(M))$. The dual module to $\mathfrak{X}(\mathbb{S}^3)$ is the module of differential 1-*forms* $\mathcal{A}^1(\mathbb{S}^3)$. In the ambient coordinates t_j, the dual basis to $\{E_1,E_2,E_3\}$ is $\{\vartheta^1,\vartheta^2,\vartheta^3\}$, given by

$$\begin{aligned} \vartheta^1 &= -t_1\,dt^0 + t_0\,dt^1 + t_3\,dt^2 - t_2\,dt^3, \\ \vartheta^2 &= -t_2\,dt^0 - t_3\,dt^1 + t_0\,dt^2 + t_1\,dt^3, \\ \vartheta^3 &= -t_3\,dt^0 + t_2\,dt^1 - t_1\,dt^2 + t_0\,dt^3. \end{aligned}$$

Using $\langle dx^i, \partial_j\rangle = \delta^i_j$, we see that $\langle \vartheta^a, E_b\rangle = \delta^a_b$.

1.2 A *connection* on a finitely generated projective [right] $C^\infty(M)$-module $\mathcal{E}$ is a map $\nabla^{\mathcal{E}}\colon \mathfrak{X}(M)\times\mathcal{E}\to\mathcal{E}$ that is $C^\infty(M)$-$\mathbb{C}$-bilinear and obeys the Leibniz rule:

$$\nabla^{\mathcal{E}}_X(sf) = (\nabla^{\mathcal{E}}_X s)f + s(Xf), \quad \text{for all} \quad s\in\mathcal{E},\ f\in C^\infty(M),\ X\in\mathfrak{X}(M).$$

The *dual connection* on $\mathcal{E}^\sharp$ is given by enforcing the Leibniz rule:

$$X(\langle\omega,s\rangle) =: \langle \nabla^{\mathcal{E}^\sharp}_X \omega, s\rangle + \langle \omega, \nabla^{\mathcal{E}}_X s\rangle.$$

We now define *Levi-Civita connections* on $\mathfrak{X}(\mathbb{S}^3)$ and dually on $\mathcal{A}^1(\mathbb{S}^3)$, both simply called ∇. For the ambient $\mathbb{R}^4$, we declare a *flat* connection on $\mathfrak{X}(\mathbb{R}^4)$ by putting $\nabla_{\partial_k}\partial_l := 0$ for all k,l. Using $\nabla_{f^i\partial_i}(h^j\,\partial_j) = f^i\nabla_{\partial_i}(h^j\,\partial_j) = f^i\partial_i(h^j)\partial_j$, we get[1]

$$\nabla_{E_a}E_b = \varepsilon^c_{ab}E_c,$$

where $\varepsilon^c_{ab} = \pm 1$ according as $(1,2,3)\mapsto(a,b,c)$ is an even or odd permutation; $\varepsilon^c_{ab} = 0$ if two indices are repeated.

[1] Repeated indices above and below have implied summation, following Einstein ["Grundlagen der allgemeinen Relativitätstheorie", *Annalen der Physik* **49** (1916), 769–822.]

Dually, since $\langle \nabla_X \alpha, Y\rangle := X(\langle \alpha, Y\rangle) - \langle \alpha, \nabla_X Y\rangle$; if $\nabla_{E_a} \vartheta^c = f^c_{ab}\vartheta^b$, then

$$f^c_{ab} = \langle \nabla_{E_a}\vartheta^c, E_b\rangle = E_a(\langle \vartheta^c, E_b\rangle) - \langle \vartheta^c, \nabla_{E_a} E_b\rangle = E_a(0 \text{ or } 1) - \varepsilon^c_{ab} = -\varepsilon^c_{ab},$$

so that $\nabla_{E_a}\vartheta^c = -\varepsilon^c_{ab}\vartheta^b$.

1.3 The (complex) Clifford algebra $\mathbb{C}\ell(M)$ is the unital algebra over $C^\infty(M)$, generated by elements $\{\, c(\alpha) : \alpha \in \mathcal{A}^1(M)\,\}$ subject to the relations

$$c(\alpha)c(\beta) + c(\beta)c(\alpha) = 2g^{-1}(\alpha,\beta)\,\mathbf{1}.$$

If $\gamma^a = c(\vartheta^a)$ for a local orthonormal basis $\{\vartheta^1,\dots,\vartheta^n\}$ of 1-forms [global for $M = \mathbb{S}^3$], we need to find a solution to the equations $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\,\delta^{ab}\,\mathbf{1}$, which involves matrices.

For $n = 2$, we can use Pauli matrices: $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^2$ which, together with $\mathbf{1}$ and $\chi = -i\gamma^1\gamma^2 = \sigma_3$, linearly generate $M_2(\mathbb{C})$.

For $n = 3$, we can use $\gamma^a = \begin{pmatrix} \sigma^a & 0 \\ 0 & -\sigma^a \end{pmatrix} \in M_4(\mathbb{C})$ for $a = 1,2,3$. These and their products generate the 8-dimensional algebra $M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. In this case, $\chi := -i\gamma^1\gamma^2\gamma^3$ is the $\mathbb{Z}/2$-grading operator $\chi = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$. To get an *irreducible* left $\mathbb{C}\ell(M)$-module, we restrict to the upper half, i.e., the $(+1)$-eigenspace of χ.

The (irreducible) *spinor module* $\mathcal{S}$, for $M = \mathbb{S}^3$, is just $C^\infty(M) \otimes \mathbb{C}^2$, a free $C^\infty(M)$-module of rank 2. The 1-form $\alpha = f_a\vartheta^a$ acts on the left by $c(\alpha) = f_a\gamma^a$.

Since $\mathcal{S}$ is now a $\mathbb{C}\ell(\mathbb{S}^3)$-$C^\infty(\mathbb{S}^3)$-bimodule, the *spin connection* ∇^S on $\mathcal{S}$ is required to satisfy *two Leibniz rules*:

$$\nabla^S_X(\psi f) = (\nabla^S_X\psi)\,f + \psi\,X(f),$$
$$\nabla^S_X(c(\alpha)\psi) = c(\nabla_X\alpha)\,\psi + c(\alpha)\,\nabla^S_X\psi,$$

for $X \in \mathfrak{X}(\mathbb{S}^3)$, $\psi \in \mathcal{S}$, $f \in C^\infty(\mathbb{S}^3)$ and $\alpha \in \mathcal{A}^1(\mathbb{S}^3)$.

Exercise 1.1. If $\psi = \binom{f_1}{f_2} =: f_1\,\psi^1 + f_2\,\psi^2 \in \mathbb{S}$ and if $\nabla^S_{E_a}\psi^c := h^c_{ab}\,\psi^b$, compute the coefficients h^c_{ab} and show that ∇^S is given by

$$\nabla^S_{E_a}\psi = E_a\psi - \frac{1}{4}\varepsilon^c_{ab}\,\sigma^b\sigma_c\,\psi$$

where $\sigma^b \equiv \sigma_b$ and E_a acts entrywise on the column vector ψ.

1.4 The *Dirac operator*, as a linear operator $\not{D}\colon \mathcal{S} \to \mathcal{S}$, is defined by

$$\not{D} = -i\,c(\vartheta^a)\,\nabla^S_{E_a} = -i\,c(dx^j)\,\nabla^S_{\partial_j}.$$

These formulas are equivalent because the transformations $\partial_j \mapsto E_a$ and $dx^j \mapsto \vartheta^a$ have inverse jacobians; so this local-looking formula is in fact global.

For $M = \mathbb{S}^3$, we compute that

$$\not{D} = -i\,\sigma^a\left(E_a - \frac{1}{4}\varepsilon^c_{ab}\,\sigma^b\sigma_c\right) = -i\,\sigma^a\,E_a - \frac{3}{2}.$$

To find eigenvectors for it, we go back to complex notation. With $z = t_0 - it_3$, $w = -t_2 - it_1$, we find:

$$\begin{aligned}
E_3 &= -iz\,\partial_z + i\bar{z}\,\partial_{\bar{z}} + iw\,\partial_w - i\bar{w}\,\partial_{\bar{w}},\\
E_+ := E_1 - i\,E_2 &= -2iw\,\partial_z + 2i\bar{z}\,\partial_{\bar{w}}\,,\\
E_- := E_1 + i\,E_2 &= \quad 2i\bar{w}\,\partial_{\bar{z}} - 2iz\,\partial_w\,,
\end{aligned}$$

and we can notice that $[E_+, E_-] = 4i\,E_3$ and $[E_3, E_\pm] = \pm 2i\,E_\pm$. We also remark that $E_3(z^k\bar{z}^l w^m\bar{w}^n) = i(l-k+n-m)\,(z^k\bar{z}^l w^m\bar{w}^n)$.

Consider the (holomorphic) polynomials, as vectors in $C^\infty(\mathbb{S}^3)$,

$$|\,j,m\rangle := z^{j+m}\,w^{j-m}$$

for $j \in \frac{1}{2}\mathbb{N} = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ and $m \in \{-j, -j+1, \dots, j-1, j\}$. It is easy to calculate that[2]

$$\begin{aligned}
E_3\,|\,j,m\rangle &= -2i\,m\,|\,j,m\rangle,\\
E_+\,|\,j,m\rangle &= -2i\,(j+m)\,|\,j,m-1\rangle,\\
E_-\,|\,j,m\rangle &= -2i\,(j-m)\,|\,j,m+1\rangle,
\end{aligned}$$

so that the E_a act (irreducibly) on $V_j := \operatorname{span}\langle|\,j,m\rangle : m = -j, \dots, j\rangle \simeq \mathbb{C}^{2j+1}$.

There is no reason to restrict to holomorphic polynomials only. If instead of $|\,j,m\rangle$ we use the (mixed) polynomials

$$|\,j,m\rangle' := z^{j_1+m_1}\,\bar{z}^{j_2-m_2}\,w^{j_1-m_1}\,\bar{w}^{j_2+m_2},$$

whenever $j_1 + j_2 = j$ and $m_1 + m_2 = m$, we get the *same* actions of the E_a, for any fixed $2j_2 \in \{0, 1, 2, \dots, 2j\}$. There *each* V_j *comes in* $(2j+1)$ *distinct copies* within $C^\infty(\mathbb{S}^3)$.

[2]The educated reader will note that we are actually using the representation theory of $SU(2)$ or, as it is more properly called, the quantum theory of angular momentum.

Now we remark that

$$D\!\!\!/ + \frac{3}{2} = -i\sigma^a E_a = -i\begin{pmatrix} E_3 & E_+ \\ E_- & -E_3 \end{pmatrix},$$

and it can be squared to a diagonal action by the following trick:[3]

$$(D\!\!\!/ + \tfrac{3}{2})^2 - 2(D\!\!\!/ + \tfrac{3}{2}) = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix},$$

where $C = -(E_3^2 - 2iE_3 + E_+E_-)$. One easily finds $C|j,m\rangle = (4j^2+4j)\,|j,m\rangle$. [$C$ stands for Casimir.]

For the eigenvalues $\lambda - \frac{3}{2}$ of $D\!\!\!/$, we must solve $\lambda^2 - 2\lambda = 4j^2 + 4j$, whose solutions are $\lambda = 2j+2$ and $\lambda = -2j$. Thus the eigenvalues of $D\!\!\!/$ are $2j + \frac{1}{2}$ [for $j > 0$ only, as it turns out] and $-2j - \frac{3}{2}$, for $j \geq 0$.

To get the *multiplicities* of these eigenvalues, we need to find the corresponding *eigenspinors*.

Exercise 1.2. Show that the full list of eigenspinors of $D\!\!\!/ + \frac{3}{2}$ in the subspace $V_j \oplus V_j$ is given by:

(a) $\psi = \begin{pmatrix} -|j,m\rangle \\ |j,m+1\rangle \end{pmatrix}$ with $\lambda = 2j+2$; for $m = -j, -j+1, \dots, j-1$;

(b) $\psi = \begin{pmatrix} (j+m+1)|j,m\rangle \\ (j-m)|j,m+1\rangle \end{pmatrix}$ with $\lambda = -2j$; for $m = -j, -j+1, \dots, j-1$;

(c) $\psi = \begin{pmatrix} |j,j\rangle \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ |j,-j\rangle \end{pmatrix}$, with $\lambda = -2j$.

If there are no more eigenvalues to be found, then —recall that each V_j is repeated $(2j+1)$ times— we get the following spectrum for $D\!\!\!/$ on $\mathbb{S}^3$:

$$\begin{aligned} &(2j+\tfrac{1}{2}), && \text{for } j > 0; && \text{with multiplicity } 2j(2j+1), \\ -&(2j+\tfrac{3}{2}), && \text{for } j \geq 0, && \text{with multiplicity } (2j+2)(2j+1). \end{aligned}$$

To repair the seeming asymmetry about 0, we relabel the first list by resetting $j \mapsto j - \frac{1}{2}$, so that now $j \geq 0$ and the positive eigenvalues are $(2j + \frac{3}{2})$. Thus we find that:

$$\operatorname{sp}(D\!\!\!/) = \{\, \pm(2j + \tfrac{3}{2}) : 2j \in \mathbb{N}, \text{ with multiplicities } (2j+2)(2j+1) \,\}.$$

[3] This trick is due to Nigel Hitchin ["Harmonic Spinors", *Adv. Math.* **14** (1974), 1–55].

To see that we have found *all* eigenspinors, we must complete $\mathcal{S} = C^\infty(\mathbb{S}^3) \oplus C^\infty(\mathbb{S}^3)$ to the Hilbert space $\mathcal{H} = L^2(\mathbb{S}^3) \oplus L^2(\mathbb{S}^3)$, with respect to the Haar measure on the compact Lie group $SU(2)$. This requires the *Peter–Weyl theorem*, which states, for $SU(2)$, that the polynomials on $\mathbb{S}^3$ form a dense subspace of $L^2(\mathbb{S}^3)$. In fact, the vectors $|j,m\rangle'$ given above form an *orthogonal basis* for $L^2(\mathbb{S}^3)$. Therefore, the spinors of the second exercise, suitably normalized, yield an *orthonormal basis for* $\mathcal{H}$, so we have found the full spectrum of $D\!\!\!/$.