K-THEORY

WOJCIECH SZYMAŃSKI

1. Lecture 1

We begin by recalling some general facts and constructions regarding C^*-algebras. Let A be an arbitrary C^*-algebra. One can check that $\widetilde{A} := A \oplus \mathbb{C}$ is a unital C^*-algebra with multiplication $(a, \lambda)(b, \mu) = (\mu a + \lambda b + ab, \lambda\mu)$, adjoint given by $(a, \lambda)^* = (a^*, \overline{\lambda})$ and unit $1_{\widetilde{A}} = (0, 1)$. We call $\widetilde{A}$ the *minimal unitization of* A.

For any $n \in \mathbb{N}$, we write $M_n(A)$ for the C^* of $n \times n$ square matrices with entries in A. We have the canonical inclusion $M_n(A) \subset M_{n+1}(A)$ via the map

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

When A is unital, we say that an element $u \in A$ is a *unitary* if $uu^* = u^*u = 1_A$, and denote the set of unitary elements of A by $\mathcal{U}(A)$.

An element p of an arbitrary C^*-algebra A is called a *projection* if it satisfies $p^2 = p^* = p$. Equivalently, p is a projection if it is self-adjoint and the spectrum $\mathrm{Sp}(p) \subseteq \{0, 1\}$. We denote the set of all projections in A by $\mathcal{P}(A)$ and write $\mathcal{P}_n(A)$ for $\mathcal{P}(M_n(A))$.

Definition 1.1. Let A be a C^*-algebra, $p, q \in \mathcal{P}(A)$. Then we have the following equivalence relations on $\mathcal{P}(A)$.

- Murray-von Neumann equivalence: $p \sim q \iff$ there exists some $w \in A$ such that $p = ww^*$ and $q = w^*w$.
- Unitary equivalence: $p \sim_u q \iff$ there exists a unitary $u \in \mathcal{U}(\widetilde{A})$ such that $p = uqu^*$
- Homotopy equivalence: $p \sim_h q \iff$ there exists a norm-continuous map $h : [0, 1] \to \mathcal{P}(A)$ such that

$$h(0) = p \text{ and } h(1) = q.$$

Proposition 1.2. *We have the following implications. For any $p, q \in \mathcal{P}(A)$,*

$$(p \sim_h q) \Longrightarrow (p \sim_u q) \Longrightarrow (p \sim q).$$

Proof. Fix $p, q \in \mathcal{P}(A)$ and suppose $p \sim_h q$. Then we can find projections h_i for $0 \le i \le k$ such that $h_0 = p, h_k = q$ and $\|h_i - h_{i+1}\| < \frac{1}{2}$ for $0 \le i \le k-1$. Hence it suffices to show that

$$\|p - q\| < \frac{1}{2} \Longrightarrow p \sim_u q.$$

To this end, fix $z = pq + (1-p)(1-q)$. One can check that $pz = zq$ and $\|z-1\| < 1$, so z is invertible. So z admits a polar decomposition $z = u|z|$ where $|z| = (z^*z)^{1/2}$. Then $p = uqu^*$ and we have $p \sim_u q$.

Now suppose $p \sim_u q$, with $p = uqu^*$. Then $w = pu$ satisfies $ww^* = p$ and $w^*w = q$ so $p \sim q$ as required.

□

Proposition 1.3. *Fix $p, q \in \mathcal{P}(A)$. Then*

- $p \sim q$ *in* $A \Longrightarrow p \sim_u q$ *in* $M_2(A)$.
- $p \sim_u q$ *in* $A \Longrightarrow p \sim_h q$ *in* $M_2(A)$.

Proof. Suppose $p \sim q$ and let $p = vv^*$ and $q = v^*v$. Then one can check that

$$w = \begin{pmatrix} v^* & 1-p \\ 1-q & v \end{pmatrix} \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$$

is a unitary satisfying

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} = w \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} w^*$$

so we have $p \sim_u q$ in $M_2(A)$.

Now suppose $p \sim_u q$ with $q = upu^*$. We can find a norm-continuous path of unitaries $\{w_t\}_{t\in[0,1]}$ in $M_2(A)$ such that

$$w_0 = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \text{ and } w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\left\{ w_t \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} w_t^* \right\}_{t\in[0,1]}$ is a norm-continuous path of projections from $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$, so $p \sim_h q$ in $M_2(A)$. □

In light of the previous two propositions, we see that the three equivalence relations $\sim, \sim_u$ and $\sim_h$ are equivalent on the the set

$$\mathcal{P}_\infty(A) := \bigcup_{n=1}^{\infty} \mathcal{P}_n(A).$$

We set $\mathcal{D}(A) := \mathcal{P}_\infty(A)/\sim$. Then with addition defined as

$$[p] \oplus [q] := \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \right]$$

the set $\mathcal{D}(A)$ becomes an abelian semigroup.

To define K_0 we must first recall the Grothendieck group associated to an abelian semigroup. Let $(S, +)$ be an abelian semigroup. We define an equivalence relation $\approx$ on the set $S \times S$ by

$$(x_1, y_1) \approx (x_2, y_2) \iff \text{there exists } z \in S \text{ such that } x_1 + y_2 + z = x_2 + y_1 + z.$$

We write $[x, y]$ for the equivalence class of (x, y).

Then $G(S) := S \times S/ \approx$ is an abelian group with addition

$$[x_1, y_1] + [x_2, y_2] = [x_1 + x_2, y_1 + y_2]$$

identity element $0 = [x, x]$ and inverse given by

$$-[x, y] = [y, x].$$

There is an additive map $\delta : S \to G(S)$ satisfying $\delta(x) = [x, x + y]$, which is independent of the choice of $y \in S$. Notice that this map is not necessarily injective.

We are now ready to define the K_0 group of A. We begin with the case where A is unital.

Definition 1.4. Let A be a unital C^*-algebra. Define $K_0(A)$ to be the Grothendieck group $G(\mathcal{D}(A))$.

Remark 1.5. In replacing $\mathcal{D}(A)$ with $K_0(A)$ we lose some information, but $K_0(A)$ has better functorial properties.

Now suppose A is non-unital, and let $\widetilde{A}$ be the minimal unitization of A. The we have a split exact sequence

$$0 \longrightarrow A \longrightarrow \widetilde{A} \overset{\pi}{\longleftrightarrow} \mathbb{C} \longrightarrow 0$$

We will see shortly that K_0 is a covariant functor, so the map $\pi : \widetilde{A} \to \mathbb{C}$ yields a group homomorphism

$$K_0(\pi) : K_0(\widetilde{A}) \to K_0(\mathbb{C}) \cong \mathbb{Z}.$$

So we may make the following definition.

Definition 1.6. Let A be a non-unital C^*-algebra. Define

$$K_0(A) := \ker(K_0(\pi))$$

considered as a subgroup of $K_0(\widetilde{A})$.

1.1. Properties of K_0.

1.1.1. *Functoriality.* Let A, B be C^*-algebras and $\alpha : A \to B$ a C^*-homomorphism. The there exists a group homomorphism

$$K_0(\alpha) : K_0(A) \to K_0(B).$$

The K_0 becomes a covariant functor from the category of C^*-algebras to the category of abelian groups.

1.1.2. *Half-exactness.* If we have a short-exact sequence (*)

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

then we get an exact sequence (**)

$$K_0(J) \longrightarrow K_0(A) \longrightarrow K_0(B)$$

If (*) is split short exact, then (**) is split exact.

1.1.3. *Homotopy invariance.* If $\alpha, \beta : A \to B$ are homotopic C^*-homomorphisms, then $K_0(\alpha) = K_0(\beta)$.

1.1.4. *Stability.* Fix $n \in \mathbb{N}$ and suppose $j : A \to M_n(A)$ is the canonical inclusion. Then $K_0(j) : K_0(A) \to K_0(M_n(A))$ is an isomorphism.

Furthermore, suppose $\mathcal{K} = \mathcal{K}(\mathcal{H})$ is the C^*-algebra of compact operators on some Hilbert space $\mathcal{H}$, and let $p \in \mathcal{K}$ be a minimal projection. Then there is an injection $j : A \to A \otimes \mathcal{K}$ satisfying

$$j(a) = a \otimes p$$

and $K_0(j) : K_0(A) \to K_0(A \otimes \mathcal{K})$ is an isomorphism.

1.1.5. *Continuity.* Suppose

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \ldots$$

is an increasing limit of C^*-algebras, and define

$$A = \overline{\bigcup_{n=1}^{\infty} A_n}.$$

Then we have a directed system

$$K_0(A_1) \to K_0(A_2) \to \cdots \to K_0(A_n) \to \ldots$$

and

$$\lim_{n\to\infty} K_0(A_n) \cong K_0(A).$$

This can be generalised to inductive limits of C^*-algebras.

1.2. **AF-algebras.** Recall that a C^*-algebra A is finite-dimensional if it is of the form

$$A = M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C}).$$

Suppose we have an increasing sequence of finite dimensional C^*-algebras

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n \subseteq \ldots$$

Then

$$A := \overline{\bigcup_{n=0}^{\infty} A_n}$$

is called an *AF-algebra*.

Example 1.7. Let A be the C^*-algebra with Bratelli diagram

$$\begin{array}{ccccccccc} \bullet & \xrightarrow{\;1\;} & \bullet & \xrightarrow{\;1\;} & \bullet & \xrightarrow{\;1\;} & \bullet & \xrightarrow{\;1\;} & \cdots \\ (1) & & (2) & & (3) & & (4) & & \end{array}$$

Then $A \cong \mathcal{K}$, the compact operators.

Example 1.8. Let A be the C^*-algebra with Bratelli diagram

$$\begin{array}{ccccccccc} \bullet & \xrightarrow{\;n\;} & \bullet & \xrightarrow{\;n\;} & \bullet & \xrightarrow{\;n\;} & \bullet & \xrightarrow{\;n\;} & \cdots \\ (1) & & (n) & & (n^2) & & (n^3) & & \end{array}$$

Then $A \cong$UHF(n, ∞).

By the continuity of K_0, we have

$$K_0(A) = \lim_{k\to\infty} K_0(A_k).$$

We also have a partial order $\leq$ on $K_0(A)$ induced by the partial order on A.

Define the *scale*

$$\mathcal{S} := \{[p] : p \in A\}$$

i.e the set of classes of projections in A(as opposed to $M_n(A)$). Then we have the following theorem.

Theorem 1.9 (G. A. Elliott, 1976)**.** *The triple*

$$(K_0, \leq, \mathcal{S})$$

is a complete isomorphism invariant for an AF C^*-algebra.

2. Lecture 2

Let A be a unital C^*-algebra. Recall that an element $u \in A$ is unitary if and only if $u^*u = uu^* = 1_A$. We say that two unitaries $u, w \in \mathcal{U}(A)$ are *homotopic*, and write $u \sim_h w$, if there exists a continuous map

$$v : [0, 1] \to \mathcal{U}(A)$$

such that $v(0) = u$ and $v(1) = w$. Then define

$$\mathcal{U}_0(A) := \{u \in A : u \sim_h 1_A\}.$$

Proposition 2.1. *We have the following properties of $\mathcal{U}_0(A)$.*

- *$\mathcal{U}_0(A)$ is a normal subgroup of $\mathcal{U}(A)$,*
- *$\mathcal{U}_0(A)$ is open and closed in $\mathcal{U}(A)$,*
- *$u \in \mathcal{U}_0(A)$ if and only if there exists self-adjoint elements $h_1, \dots, h_k$ such that*

$$u = e^{ih_1} \dots e^{ih_k}.$$

Lemma 2.2. *Let A be a unital C^*-algebra.*

- *If $u \in \mathcal{U}(A)$ satisfies $\mathrm{Sp}(u) \neq \mathbb{T}$, then $u \in \mathcal{U}_0(A)$,*
- *If $u, w \in \mathcal{U}(A)$ satisfy $\|u - w\| < 2$, then $u \sim_h w$,*
- *For $u, w \in \mathcal{U}(A)$, we have*

$$\begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix} \sim_h \begin{pmatrix} uw & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} wu & 0 \\ 0 & 1 \end{pmatrix} \sim_h \begin{pmatrix} w & 0 \\ 0 & u \end{pmatrix}.$$

In particular,

$$\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \sim_h \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proposition 2.3. *Let A, B be unital C^*-algebras and $\phi : A \to B$ be a surjective C^*-homomorphism. Then*

$$\phi(\mathcal{U}_0(A)) = U_0(B).$$

Thus, every unitary homotopic to 1_B has a unitary inverse image in A which is also homotopic to 1.

Now, fix an arbitrary C^*-algebra A and let $\widetilde{A}$ be its minimal unitization. We define an equivalence relation $\sim$ on $\mathcal{U}(\bigcup_{n=1}^{\infty} M_n(\widetilde{A}))$ as follows. For $u \in \mathcal{U}(M_n(\widetilde{A}))$ and $w \in \mathcal{U}(M_m(\widetilde{A}))$ we say $u \sim w$ if and only if there exists $k \geq \max\{m, n\}$ such that

$$u \oplus 1_{k-n} \sim_h w \oplus 1_{k-m} \text{ in } \mathcal{U}(M_k(\widetilde{A})).$$

Definition 2.4. We define the K_1 group as

$$K_1(A) := \mathcal{U}\left(\bigcup_{n=1}^{\infty} M_n(\widetilde{A})\right) / \sim .$$

This is again an abelian group with

$$u \oplus w = \begin{pmatrix} u & 0 \\ 0 & w \end{pmatrix}, \quad -u = u^*.$$

We also have the same functorial properties of K_1; i.e. K_1 satisfies functoriality, half-exactness, homotopy invariance, stability and continuity.

2.1. **6-term exact sequence of K-theory.** Suppose we have a short exact sequence of C^*-algebras

$$0 \longrightarrow J \xrightarrow{\ j\ } A \xrightarrow{\ \pi\ } B \longrightarrow 0$$

Then the following sequence of abelian groups is exact

$$
\begin{array}{ccccc}
K_0(J) & \xrightarrow{K_0(j)} & K_0(A) & \xrightarrow{K_0(\pi)} & K_0(B) \\
\partial_{\text{ind}} \Big\uparrow & & & & \Big\downarrow \partial_{\text{exp}} \\
K_1(B) & \xleftarrow{K_1(\pi)} & K_1(A) & \xleftarrow{K_1(j)} & K_1(J)
\end{array}
$$

The two difficult maps are

- ∂_{ind} - the index map, and
- ∂_{exp} - the exponential map

2.2. **The index map.** Fix $u \in \mathcal{U}(M_n(\widetilde{B}))$ and let $m \geq n$ be such that there exists some partial isometry $w \in M_m(\widetilde{A})$ with

$$\pi(w) = \begin{pmatrix} u & 0 \\ 0 & 0_{m-n} \end{pmatrix}.$$

Then there exist projection $p, q \in \mathcal{P}_m(\widetilde{J})$ such that

$$1 - w^*w = j(p), \quad 1 - ww^* = j(q).$$

Then

$$\partial_{\text{ind}}([u]) = [p] - [q] \in K_0(J).$$

Remark 2.5. If $m = 2n$, then such a partial isometry will exist. First take $a \in M_n(\widetilde{A})$ with $\|a\| = 1$ and $\pi(a) = u$. Then

$$w = \begin{pmatrix} a & 0 \\ \sqrt{1 - a^*a} & 0 \end{pmatrix}$$

will do.

3. Lecture 3

3.1. **The Toeplitz algebra.** Let $S \in \mathcal{B}(\ell^2(\mathbb{N}))$ be the unilateral shift and let $A = C^*(S)$ be the C^*-algebra generated by S. We have the well-known short exact sequence

We know that

$$K_0(\mathcal{K}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$$
$$K_1(\mathcal{K}) \cong K_1(\mathbb{C}) \cong 0.$$

The associated 6-term exact sequence is

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & K_0(A) & \longrightarrow & \mathbb{Z} \\ \partial_{\text{ind}} \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & K_1(A) & \longleftarrow & 0 \end{array}$$

To calculate ∂_{ind}, first note that the isometry S is a lift of the unitary generator z of $C(S^1)$. Then

$$\begin{aligned} \partial_{\text{ind}}([z]) &= [1 - S^*S] - [1 - SS^*] \\ &= -[p] \end{aligned}$$

where $p \in \mathcal{K}$ is some minimal projection. So we have $\partial_{\text{ind}} : \mathbb{Z} \to \mathbb{Z}$ satisfies $\partial_{\text{ind}}(k) = -k$. It follows that

$$\begin{aligned} &K_0(A) = \mathbb{Z}, \text{ generated by}[1_A] \\ &K_1(A) = 0. \end{aligned}$$

3.2. **The universal group algebra of $\mathbb{F}_2$ (J. Cuntz, 1982).** Fix $n = 2$. This is only for simplicity. Then

$$\mathbb{F}_2 = \langle a, b \rangle$$

is the free group on 2 generators a and b. We define $C^*(\mathbb{F}_2)$ to be the C^*-algebra universal for unitary representations of $\mathbb{F}_2$.

Theorem 3.1. *We have*

$$\begin{aligned} K_0(C^*(\mathbb{F}_2)) &\cong \mathbb{Z} \\ K_1(C^*(\mathbb{F}_2)) &\cong \mathbb{Z}^2. \end{aligned}$$

Proof. Define

$$w_t := \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}.$$

Then for each $t \in [0, 1]$ w_t is a unitary, and

$$w_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For each t, let $\gamma_t : C^*(\mathbb{F}_2) \to M_2(C^*(\mathbb{F}_2))$ be the C^*-homomorphism satisfying

$$\begin{aligned} \gamma_t(a) &= \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \\ \gamma_t(b) &= w_t \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} w_t^*. \end{aligned}$$

Also, let $\psi : C^*(\mathbb{F}_2) \to C^*(\mathbb{F}_2)$ be the C^*-homomorphism satisfying $\psi(a) = 1 = \psi(b)$.

On the level of K-theory,

$$K_*(\gamma_0) = \text{id} + K_*(\psi)$$

and since γ_1 is homotopic to γ_0,

$$K_*(\gamma_1) = \mathrm{id} + K_*(\psi).$$

Let

$$D := \gamma_1(C^*(\mathbb{F}_2)) \cong \{(x,y) \in C^*(a) \oplus C^*(b) : \psi(x) = \psi(y)\}.$$

We have that $D \cong C(X)$ is abelian, where X is a figure-eight. One can check that

$$\begin{aligned} K_0(D)) &\cong \mathbb{Z} \\ K_1(D) &\cong \mathbb{Z}^2. \end{aligned}$$

so it suffices to show that $K_*(C^*(\mathbb{F}_2)) \cong K_*(D)$.

To this end, consider the embedding $j : D \to C^*(\mathbb{F}_2)$, the map $k : C^*(\mathbb{F}_2) \to D$ given by γ_1 and $\widetilde{\psi} = \psi|_D$. Then we claim that the maps

$$\begin{aligned} K_*(k) &: K_*(C^*(\mathbb{F}_2)) \to K_*(D) \\ K_*(j) - K_*(\widetilde{\psi}) &: K_*(D) \to K_*(C^*(\mathbb{F}_2)) \end{aligned}$$

are mutually inverse. We have

$$\begin{aligned} (K_*(j) - K_*(\widetilde{\psi})) \circ K_*(k) &= K_*(j \circ k) - K_*(\widetilde{\psi} \circ k) \\ &= K_*(\gamma_1) - K_*(\psi) \\ &= \mathrm{id} + K_*(\psi) - K_*(\psi) \\ &= \mathrm{id}. \end{aligned}$$

For the other direction, we have

$$(k \circ j)(x,y) = \begin{pmatrix} (x, \psi(y)) & (0,0) \\ (0,0) & (\psi(x), y) \end{pmatrix} \in M_2(D).$$

By conjugating with the unitaries $(1, w_t)$ for $t \in [0,1]$, we get a homotopy from $(k \circ j)$ to ϕ, where

$$\phi(x,y) = \begin{pmatrix} (x,y) & (0,0) \\ (0,0) & (\psi(x), \psi(y)) \end{pmatrix}.$$

Thus,

$$K_*(k \circ j) = K_*(\phi) = \mathrm{id} + K_*(k \circ \widetilde{\psi})$$

so that

$$K_*(k) \circ (K_*(j) - K_*(\widetilde{\psi})) = \mathrm{id}$$

as required. □

3.3. **Cuntz-Krieger algebras.** Fix n and let A be an $n \times n$ square matrix with entries in $\{0, 1\}$, such that no row or column consists of all zeroes.

Definition 3.2. Define $\mathcal{O}_A$ to be the universal C^*-algebra generated by partial isometries $S_1, \ldots, S_n$ such that

(1) $S_i^* S_j = 0$ if $i \neq j$; i.e. the S_i's have mutually orthogonal range projections,
(2) $S_i^* S_i = \sum_{j=1}^n A(i,j) S_j S_j^*$ for $1 \leq i \leq n$.

Theorem 3.3 (J. Cuntz)**.** *We have*

$$K_0(\mathcal{O}_A) \cong \mathrm{coker}(1 - A)$$
$$K_1(\mathcal{O}_A) \cong \ker(1 - A).$$