LECTURE I: QUANTUM SU(2)

- $q \in [-1,1] \setminus \{0\}$

- FOR H – HILBERT SPACE AND $\dot{\alpha}, \dot{\gamma} \in B(H)$ WE SAY THAT $(\dot{\alpha}, \dot{\gamma})$ SATISFY $S_qU(2)$ RELATIONS IF

$$\dot{\alpha}\dot{\gamma} = q\dot{\gamma}\dot{\alpha}$$
$$\dot{\gamma}^*\dot{\gamma} = \dot{\gamma}\dot{\gamma}^* \qquad (\dot{\alpha}\dot{\gamma} = q\dot{\gamma}\dot{\alpha})$$
$$\dot{\alpha}^*\dot{\alpha} + \dot{\gamma}^*\dot{\gamma} = \mathbb{1}$$
$$\dot{\alpha}\dot{\alpha}^* + q^2\dot{\gamma}^*\dot{\gamma} = \mathbb{1}$$

- $\mathcal{F}$:= FREE $*$-ALGEBRA GENERATED BY SYMBOLS $\alpha, \gamma, \underset{\text{UNIT}}{\mathbb{1}}$

- FOR $a \in \mathcal{F}$ WE LET

$$\|a\| := \sup \|\pi(a)\|$$

OVER ALL $*$-HOMOMORPHISMS $\pi : \mathcal{F} \to B(H)$ (H – HILBERT SPACE) SUCH THAT $(\pi(\alpha), \pi(\gamma))$ SATISFY $S_qU(2)$ RELATIONS

FACT: $\|\cdot\|$ IS A C^*-SEMINORM ON $\mathcal{F}$

EXERCISE: $(\dot{\alpha}, \dot{\gamma})$ SATISFY $S_qU(2)$ RELATIONS IFF

$$\begin{pmatrix} \dot{\alpha} & -q\dot{\gamma}^* \\ \dot{\gamma} & \dot{\alpha}^* \end{pmatrix} \in M_2 \otimes B(H) \cong B(H \oplus H)$$

IS UNITARY.

EXERCISE: FOR ANY $a \in \mathcal{F}$ $\|a\|$ IS FINITE.

- $A :=$ COMPLETION OF $\mathcal{F}/\{a \mid \|a\|=0\}$

- FOR ANY PAIR $(\dot{\alpha},\dot{\gamma})$ OF OPERATORS ON A HILBERT SPACE H SUCH THAT $(\dot{\alpha},\dot{\gamma})$ SATISFY $S_qU(2)$ RELATIONS THERE IS A UNIQUE $\pi \in \mathrm{Rep}(A,H)$ SUCH THAT
$$\pi(\alpha)=\dot{\alpha}, \quad \pi(\gamma)=\dot{\gamma}.$$

- EXAMPLE OF A REPRESENTATION OF A:
  - H - HILBERT SPACE WITH ONB $(e_{n,k})_{\substack{n\in\mathbb{Z}_+\\ k\in\mathbb{Z}}}$
  - $$\pi(\alpha)\, e_{n,k} = \sqrt{1-q^{2n}}\, e_{n-1,k}$$
    $$\pi(\gamma)\, e_{n,k} = q^n\, e_{n,k+1}$$

THIS REPRESENTATION IS FAITHFUL, SO WE MAY REGARD A AS EMBEDDED INTO $B(H)$.

- MATRICES

$$\begin{pmatrix} \alpha\otimes 1 & -q\gamma^*\otimes 1 \\ \gamma\otimes 1 & \alpha^*\otimes 1 \end{pmatrix} \quad \text{AND} \quad \begin{pmatrix} 1\otimes\alpha & -q\, 1\otimes\gamma^* \\ 1\otimes\gamma & 1\otimes\alpha^* \end{pmatrix}$$

ARE UNITARY, SO THEIR PRODUCT

$$\begin{pmatrix} \alpha\otimes\alpha - q\gamma^*\otimes\gamma & -q\gamma^*\otimes\alpha^* - q\,\alpha\otimes\gamma^* \\ \gamma\otimes\alpha + \alpha^*\otimes\gamma & \alpha^*\otimes\alpha^* - q\,\gamma\otimes\gamma^* \end{pmatrix}$$

IS UNITARY.

THEREFORE $\exists!$ UNITAL $*$- HOMOMORPHISM

$$\Delta : A \longrightarrow A\otimes A$$

SUCH THAT

$$\Delta(\alpha) = \alpha\otimes\alpha - q\gamma^*\otimes\gamma$$

$$\Delta(\gamma) = \gamma\otimes\alpha + \alpha^*\otimes\gamma$$

- NOTICE THAT FOR

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2(A) \cong M_2\otimes A$$

WE HAVE

$$(id_{M_2}\otimes\Delta)\, u = u_1 u_2 ,$$

WHERE

$$u_1 = \begin{pmatrix} \alpha\otimes 1 & -q\gamma^*\otimes 1 \\ \gamma\otimes 1 & \alpha^*\otimes 1 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 1\otimes\alpha & -q\, 1\otimes\gamma^* \\ 1\otimes\gamma & 1\otimes\alpha^* \end{pmatrix}.$$

<u>EXERCISE:</u> $(id_A\otimes\Delta)\circ\Delta = (\Delta\otimes id_A)\circ\Delta$

- ASSUME WE HAVE $v\in M_n\otimes A$ S.T. $v^*v = vv^* = 1$ AND

$$(id_{M_n}\otimes\Delta)v = v_1 v_2 ,$$

WHERE

$$v_1 = \begin{pmatrix} v_{11}\otimes 1 & \cdots & v_{1n}\otimes 1 \\ \vdots & & \vdots \\ v_{n1}\otimes 1 & \cdots & v_{nn}\otimes 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1\otimes v_{11} & \cdots & 1\otimes v_{1n} \\ \vdots & & \vdots \\ 1\otimes v_{n1} & \cdots & 1\otimes v_{nn} \end{pmatrix}$$

- WE HAVE $\left((\mathrm{id}\otimes\Delta)v\right)v_2^{*} = v_1$ I.E.

$$\begin{pmatrix} \Delta(v_{11}) & \cdots & \Delta(v_{1n}) \\ \vdots & & \vdots \\ \Delta(v_{n1}) & \cdots & \Delta(v_{nn}) \end{pmatrix} \begin{pmatrix} 1\!\!1\otimes v_{11}^{*} & \cdots & 1\!\!1\otimes v_{n1}^{*} \\ \vdots & & \vdots \\ 1\!\!1\otimes v_{1n}^{*} & \cdots & 1\!\!1\otimes v_{nn}^{*} \end{pmatrix} = \begin{pmatrix} v_{11}\otimes 1\!\!1 & \cdots & v_{1n}\otimes 1\!\!1 \\ \vdots & & \vdots \\ v_{n1}\otimes 1\!\!1 & \cdots & v_{nn}\otimes 1\!\!1 \end{pmatrix}$$

OR

$$v_{kl}\otimes 1\!\!1 = \sum_{p=1}^{n} \Delta(v_{kp})(1\!\!1\otimes v_{lp}^{*})$$

QUESTION: ARE THERE MANY SUCH MATRICES v ?

- TAKE $\left.\begin{array}{l} v\in M_n\otimes A \\ w\in M_m\otimes A \end{array}\right\}$ UNITARY

WITH

$$(\mathrm{id}_{M_n}\otimes\Delta)v = v_1 v_2, \quad (\mathrm{id}_{M_m}\otimes\Delta)w = w_1 w_2.$$

$$v = \sum_k m_k\otimes a_k, \quad v_1 = \sum_k m_k\otimes a_k\otimes 1\!\!1, \quad v_2 = \sum_k m_k\otimes 1\!\!1\otimes a_k$$

$$w = \sum_l n_l\otimes b_l, \quad w_1 = \sum_l n_l\otimes b_l\otimes 1\!\!1, \quad w_2 = \sum_l n_l\otimes 1\!\!1\otimes b_l$$

DEFINE

$$v \ominus w = \sum_{k,l} m_k\otimes n_l\otimes a_k b_l \in M_n\otimes M_m\otimes A \cong M_{n\cdot m}\otimes A$$

NOTE THAT $v\ominus w$ IS UNITARY BECAUSE

$$v\ominus w = \underbrace{\left(\sum_k m_k\otimes 1\!\!1\otimes a_k\right)\left(\sum_l 1\!\!1\otimes n_l\otimes b_l\right)}_{\text{UNITARY}}$$

EXERCISE: PROVE THAT

$$(\mathrm{id}_{M_{n\cdot m}}\otimes\Delta)(v\ominus w) = (v\ominus w)_1\,(v\ominus w)_2.$$

- TAKE
$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in M_2 \otimes A$$
AND PUT
$$u \text{\textcircled{\tiny T}} u = u^{\text{\textcircled{\tiny T}} 2} = \begin{pmatrix} \alpha\alpha & -q\alpha\gamma^* & -q\gamma^*\alpha & q^2\gamma^*\gamma^* \\ \alpha\gamma & \alpha\alpha^* & -q\gamma^*\gamma & -q\gamma^*\alpha^* \\ \gamma\alpha & -q\gamma\gamma^* & \alpha^*\alpha & -q\alpha^*\gamma^* \\ \gamma\gamma & \gamma\alpha^* & \alpha^*\gamma & \alpha^*\alpha^* \end{pmatrix} \in M_4 \otimes A$$

$$u^{\text{\textcircled{\tiny T}} 3} = u \text{\textcircled{\tiny T}} u \text{\textcircled{\tiny T}} u = \begin{pmatrix} \alpha\, u^{\text{\textcircled{\tiny T}} 2} & -q\gamma^* u^{\text{\textcircled{\tiny T}} 2} \\ \gamma\, u^{\text{\textcircled{\tiny T}} 2} & \alpha^* u^{\text{\textcircled{\tiny T}} 2} \end{pmatrix} \in M_8 \otimes A$$

$$\vdots$$

ANY MONOMIAL IN $\alpha, \gamma, \alpha^*, \gamma^*$ IS A MATRIX ELEMENT OF SOME $u^{\text{\textcircled{\tiny T}} n}$.

<u>COROLLARY:</u> $\operatorname{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}$ IS DENSE IN $A \otimes A$.

<u>PROOF:</u> RECALL THAT IF $v \in M_n \otimes A$ IS UNITARY AND
$$(id_{M_n} \otimes \Delta) v = v_1 v_2$$
THEN $v_{kl} \otimes 1\!\!1 \in \operatorname{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}$,

SO $\forall\, c \in A$
$$v_{kl} \otimes c \in \operatorname{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}.$$

v_{kl} CAN BE ANY MONOMIAL IN THE GENERATORS OF A, SO FOR ANY x IN THE (DENSE) $*$-ALGEBRA GENERATED BY α, γ AND ANY $c \in A$
$$x \otimes c \in \operatorname{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\}. \qquad \blacksquare$$

<u>FACT:</u> $\operatorname{span}\{(a \otimes 1\!\!1)\Delta(b) \mid a, b \in A\}$ IS DENSE IN A.

(SIMILAR PROOF)

THEOREM (S.L. WORONOWICZ, A. VAN DAELE):

LET A BE A UNITAL C^*-ALGEBRA, $\Delta : A \to A \otimes A$ A UNITAL $*$-HOMOMORPHISM SUCH THAT

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta .$$

ASSUME THAT

$$\mathrm{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\} \text{ AND } \mathrm{span}\{(a \otimes 1\!\!1)\Delta(b) \mid a, b \in A\}$$

ARE DENSE IN $A \otimes A$.

THEN THERE EXISTS A UNIQUE STATE h ON A SUCH THAT

$$\left.\begin{aligned} (id \otimes h)\,\Delta(a) &= h(a) 1\!\!1 \\ (h \otimes id)\,\Delta(a) &= h(a) 1\!\!1 \end{aligned}\right. \qquad \forall a \in A$$

- IN OUR CASE WE HAVE

$$h(a) = (1-q^2) \sum_{n=0}^{\infty} q^{2n} \left(e_{n,0} \mid \pi(a)\, e_{n,0} \right)$$

$$\begin{aligned} \pi(\alpha)\, e_{n,k} &= \sqrt{1-q^{2n}}\, e_{n-1,k} \\ \pi(\gamma)\, e_{n,k} &= q^n\, e_{n,k+1} \end{aligned} \qquad (n \in \mathbb{Z}_+,\ k \in \mathbb{Z})$$

EXERCISE: DEFINE

$$\alpha_k = \begin{cases} \alpha^k & k \geq 0 \\ (\alpha^*)^{-k} & k < 0 \end{cases}$$

(a) $\mathrm{span}\{\alpha_k \gamma^n (\gamma^*)^m \mid k \in \mathbb{Z},\ m, n \in \mathbb{Z}_+\}$ IS DENSE IN A

(b) $h(\alpha_k \gamma^n (\gamma^*)^m) = \delta_{k,0}\, \delta_{m,n} \dfrac{1-q^2}{1-q^{2n+2}}$.

- THERE EXISTS A UNIQUE CHARACTER $e: A \to \mathbb{C}$ SUCH THAT $(id \otimes e) \circ \Delta = id = (e \otimes id) \circ \Delta$
- LET $\mathcal{A} = \text{span}\{\alpha_k \gamma^n (\gamma^*)^m \mid k \in \mathbb{Z},\ m, n \in \mathbb{Z}_+\} \subset A$. THEN $\mathcal{A}$ IS A DENSE UNITAL $*$-SUBALGEBRA OF A ($\mathcal{A}$ IS THE $*$-ALGEBRA GENERATED BY α AND γ) THERE EXISTS A UNIQUE LINEAR ANTI-MULTIPLICATIVE
$$\kappa : \mathcal{A} \to \mathcal{A}$$
SUCH THAT
$$\kappa(\alpha) = \alpha^*, \qquad \kappa(\alpha^*) = \alpha,$$
$$\kappa(\gamma) = -q\gamma, \qquad \kappa(\gamma^*) = -q^{-1}\gamma^*.$$
WE HAVE
$$(id_{M_2} \otimes \kappa)\, u = u^* \quad (= u^{-1}).$$
- FOR $n \in \frac{1}{2}\mathbb{N}$ LET $T_n = \{-n, -n+1, \dots, n\}$. FOR $k \in T_n$ DEFINE $x_k = \alpha^{n+k}(\gamma^*)^{n-k}$. IT CAN BE SHOWN THAT
$$\Delta(x_k) = \sum_{i \in T_n} x_i \otimes w_{i,k}$$
FOR UNIQUE $w_{i,k} \in \mathcal{A}$ $(i, k \in T_n)$. WE HAVE THE $(2n+1) \times (2n+1)$ MATRIX
$$w = \begin{pmatrix} w_{-n,-n} & \cdots & w_{-n,n} \\ \vdots & & \vdots \\ w_{n,-n} & \cdots & w_{n,n} \end{pmatrix}$$
AND
$$(id_{M_{2n+1}} \otimes \Delta)\, w = w_1 w_2 .$$

LECTURE II: COMPACT QUANTUM GROUPS AND REPRESENTATION THEORY

DEFINITION: A COMPACT QUANTUM GROUP IS

A PAIR (A, Δ) SUCH THAT

A IS A UNITAL C^*-ALGEBRA

$\Delta : A \to A \otimes A$ IS A UNITAL $*$-HOMOMORPHISM

AND
- $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$
- $\left.\begin{array}{l} \operatorname{span}\{\Delta(a)(1\!\!1 \otimes b) \mid a, b \in A\} \\ \operatorname{span}\{(a \otimes 1\!\!1)\Delta(b) \mid a, b \in A\} \end{array}\right\}$ ARE DENSE IN $A \otimes A$.

EXAMPLES: (1) $S_qU(2)$

(2) G — COMPACT GROUP

$A := C(G)$

$\Delta : C(G) \to C(G) \otimes C(G) \cong C(G \times G)$

$f \in C(G)$, $\Delta(f) \in C(G \times G)$

$$(\Delta(f))(x, y) = f(xy)$$

EXERCISE: CHECK THAT

$\left.\begin{array}{l} \operatorname{span}\{\Delta(f)(1\!\!1 \otimes g) \mid f, g \in C(G)\} \\ \operatorname{span}\{(f \otimes 1\!\!1)\Delta(g) \mid f, g \in C(G)\} \end{array}\right\}$ ARE DENSE IN $C(G \times G)$

(3) Γ — DISCRETE GROUP, $A := C^*(\Gamma)$

$\Gamma \subset \mathbb{C}\Gamma \subset A$, $\exists!\ \Delta : A \to A \otimes A$

$$\Delta(\gamma) = \gamma \otimes \gamma \quad (\gamma \in \Gamma).$$

FACT: LET (A, Δ) BE A COMPACT QUANTUM GROUP WITH A - COMMUTATIVE. THEN THERE IS A COMPACT SPACE G AND A CONTINUOUS $\mu: G \times G \longrightarrow G$ SUCH THAT

$$A \cong C(G)$$

$$(\Delta(f))(x,y) = f(\mu(x,y))$$

AND (EXERCISE)

- $\mu(x, \mu(y,z)) = \mu(\mu(x,y), z) \quad \forall\, x,y,z \in G$
- FOR $x,y \in G$

$$[\exists z \;\; \mu(x,z) = \mu(y,z)] \Rightarrow [x = y],$$

$$[\exists z \;\; \mu(z,x) = \mu(z,y)] \Rightarrow [x = y].$$

EXERCISE: A COMPACT TOPOLOGICAL SEMIGROUP WITH CANCELLATION PROPERTIES IS A TOPOLOGICAL GROUP.

THEOREM (S.L. WORONOWICZ, A. VAN DAELE):
LET (A, Δ) BE A COMPACT QUANTUM GROUP. THEN THERE EXISTS A UNIQUE STATE h ON A SUCH THAT

$$(\mathrm{id} \otimes h)\Delta(a) = h(a)\mathbb{1} = (h \otimes \mathrm{id})\Delta(a) \qquad \forall\, a \in A.$$

DEFINITION: THIS STATE h IS THE HAAR MEASURE OF (A, Δ).

- IF $A = C(G)$ THEN $h(f) = \int_G f(x)\,dx$ (HAAR MEASURE)

- IF $A = C^*(\Gamma)$ THEN $h(a) = (\delta_e \mid \lambda(a)\,\delta_e)$, WHERE $\lambda: C^*(\Gamma) \longrightarrow B(\ell^2(\Gamma))$ IS THE REGULAR REPRESENTATION.

<u>DEFINITION:</u> LET (A, Δ) BE A COMPACT QUANTUM GROUP. A FINITE-DIMENSIONAL REPRESENTATION OF (A, Δ) IS AN INVERTIBLE MATRIX $v \in M_n \otimes A$ SUCH THAT $(id_{M_n} \otimes \Delta) v = v_1 v_2$

<u>EXERCISE:</u> $v \in M_n \otimes A$, $v = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{nn} \end{pmatrix}$ IS A REPRESENTATION OF (A, Δ) IFF

- v IS INVERTIBLE

AND

- $\Delta(v_{kl}) = \sum_{p=1}^{n} v_{kp} \otimes v_{pl}$, $\quad k, l = 1, \dots, n$.

- $M_n \otimes A$ IS A $*$-ALGEBRA (C^*-ALGEBRA)
- WE SAY THAT A REPRESENTATION $v \in M_n \otimes A$ IS UNITARY IF $v^* v = v v^* = 1\!\!1$ $(= 1\!\!1_{M_n} \otimes 1\!\!1_A)$.

FACT: IF $A = C(G)$ THEN $v \in M_n \otimes A \cong C(G, M_n)$ IS A REPRESENTATION OF (A, Δ) IFF

$$v \in C(G, M_n)$$

HAS RANGE IN THE SET OF INVERTIBLE MATRICES AND $v(x) v(y) = v(xy)$ FOR ALL $x, y \in G$.

FACT: IF $A = C^*(\Gamma)$ AND $v \in M_n \otimes A$ IS A REPRESENTATION OF (A, Δ) THEN THERE IS AN INVERTIBLE MATRIX $C \in M_n(\mathbb{C})$ AND $\gamma_1, \dots, \gamma_n \in \Gamma$ SUCH THAT

$$v = C \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{pmatrix} C^{-1}$$

<u>THEOREM:</u> LET (A,Δ) BE A COMPACT QUANTUM GROUP AND LET $v \in M_n \otimes A$ BE A REPRESENTATION OF (A,Δ). THEN THERE EXISTS AN INVERTIBLE MATRIX $T \in M_n(\mathbb{C})$ SUCH THAT

$$(T \otimes 1\!\!1)\, v\, (T^{-1} \otimes 1\!\!1)$$

IS A UNITARY REPRESENTATION OF (A,Δ).

<u>PROOF:</u> $v \in M_n \otimes A$ IS INVERTIBLE, SO v^*v IS POSITIVE AND INVERTIBLE. THEREFORE $\mathrm{Sp}\, v^*v$ IS SEPARATED FROM 0. THUS $v^*v \geq \delta 1\!\!1_{M_n \otimes A}$ FOR SOME $\delta > 0$.
LET $Q = (\mathrm{id} \otimes h)(v^*v) \in M_n(\mathbb{C})$. h IS POSITIVE, SO

$$Q \geq \delta (\mathrm{id} \otimes h) 1\!\!1 = \delta 1\!\!1_{M_n}.$$

IT FOLLOWS THAT Q IS INVERTIBLE. LET $T = Q^{1/2}$.
PUT $w = (T \otimes 1\!\!1)\, v\, (T^{-1} \otimes 1\!\!1)$.
LET US CHECK THAT w IS UNITARY:

$$w^*w = (T^{-1} \otimes 1\!\!1)\, v^* (Q \otimes 1\!\!1)\, v\, (T^{-1} \otimes 1\!\!1)$$

NOW

$$\begin{aligned}
Q \otimes 1\!\!1 &= \left[(\mathrm{id} \otimes h)(v^*v)\right] \otimes 1\!\!1 = (\mathrm{id} \otimes h \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)(v^*v) \\
&= (\mathrm{id} \otimes h \otimes \mathrm{id})\left((v_1 v_2)^* v_1 v_2\right) \\
&= (\mathrm{id} \otimes h \otimes \mathrm{id})(v_2^* v_1^* v_1 v_2) \\
&= v^* \left[(\mathrm{id} \otimes h \otimes h)(v_1^* v_1)\right] v \\
&= v^* \left[\left((\mathrm{id} \otimes h)(v^*v)\right) \otimes 1\!\!1\right] v = v^*(Q \otimes 1\!\!1)\, v
\end{aligned}$$

SO THAT

$$w^*w = (T^{-1} \otimes 1\!\!1)(Q \otimes 1\!\!1)(T^{-1} \otimes 1\!\!1) = 1\!\!1$$

SIMILARLY $\quad ww^* = (T \otimes 1\!\!1) v (Q^{-1} \otimes 1\!\!1) v^* (T \otimes 1\!\!1)$.

WE KNOW ALREADY THAT $\quad Q \otimes 1\!\!1 = v^* (Q \otimes 1\!\!1) v$, SO

$$(v^*)^{-1} (Q \otimes 1\!\!1) v^{-1} = Q \otimes 1\!\!1.$$

TAKING INVERSES GIVES

$$v (Q^{-1} \otimes 1\!\!1) v^* = Q^{-1} \otimes 1\!\!1$$

AND SO

$$w w^* = (T \otimes 1\!\!1)(Q^{-1} \otimes 1\!\!1)(T \otimes 1\!\!1) = 1\!\!1.$$

THE LAST THING TO CHECK IS

$$\begin{aligned}
(id \otimes \Delta) w &= (id \otimes \Delta)\left((T \otimes 1\!\!1)\, v\, (T^{-1} \otimes 1\!\!1)\right) \\
&= (T \otimes 1\!\!1 \otimes 1\!\!1)\, v_1 v_2\, (T^{-1} \otimes 1\!\!1 \otimes 1\!\!1) \\
&= (T \otimes 1\!\!1 \otimes 1\!\!1)\, v_1\, (T^{-1} \otimes 1\!\!1 \otimes 1\!\!1)\, (T \otimes 1\!\!1 \otimes 1\!\!1)\, v_2\, (T^{-1} \otimes 1\!\!1 \otimes 1\!\!1) \\
&= \left[(T \otimes 1\!\!1) v (T^{-1} \otimes 1\!\!1)\right]_1 \left[(T \otimes 1\!\!1) v (T^{-1} \otimes 1\!\!1)\right]_2 = w_1 w_2 .
\end{aligned}$$

$\blacksquare$

- LET $v \in M_n \otimes A$ AND $w \in M_m \otimes A$ BE REPRESENTATIONS OF (A, Δ). THE <u>DIRECT SUM</u> OF v AND w IS
$$v \oplus w = \begin{pmatrix} v & 0 \\ 0 & w \end{pmatrix} \in M_{n+m} \otimes A$$

$v \oplus w$ IS A REPRESENTATION OF (A, Δ) AND IF v AND w ARE UNITARY THEN SO IS $v \oplus w$.
LET $P = \begin{pmatrix} \mathbb{1}_n & 0 \\ 0 & 0 \end{pmatrix}$. THEN P IS A PROJECTION AND
$$(P \otimes \mathbb{1})(v \oplus w) = (v \oplus w)(P \otimes \mathbb{1}).$$

- IF $u \in M_k \otimes A$ IS A UNITARY REPRESENTATION OF (A, Δ) AND $P \in M_k(\mathbb{C})$ IS A PROJECTION SUCH THAT
$$(P \otimes \mathbb{1}) u = u (P \otimes \mathbb{1})$$
THEN u IS EQUIVALENT TO A DIRECT SUM $u \sim v \oplus w$ OF TWO UNITARY REPRESENTATIONS

> <u>EQUIVALENCE OF REPRESENTATIONS:</u>
> $$\left(u \sim w \right) \iff \left(\begin{array}{l} \exists \text{ INVERTIBLE } T \\ \text{SUCH THAT } (T \otimes \mathbb{1}) u = w (T \otimes \mathbb{1}) \end{array} \right)$$

<u>DEFINITION:</u> A REPRESENTATION $u \in M_n \otimes A$ OF (A, Δ) IS <u>IRREDUCIBLE</u> IF

$$\left(\begin{array}{c} P \in \mathrm{Proj}(M_n(\mathbb{C})) \\ (P \otimes \mathbb{1}) u = u (P \otimes \mathbb{1}) \end{array} \right) \implies \left(P = 0 \ \vee \ P = \mathbb{1} \right).$$

THEOREM (S.L. WORONOWICZ):

ANY REPRESENTATION OF A COMPACT QUANTUM GROUP IS EQUIVALENT TO A DIRECT SUM OF IRREDUCIBLE REPRESENTATIONS.

REMARK: THE ABOVE THEOREM IS TRUE ALSO FOR INFINITE-DIMENSIONAL REPRESENTATIONS (UNITARY, STRONGLY CONTINUOUS) OF COMPACT QUANTUM GROUPS:

- (A, Δ) – COMPACT QUANTUM GROUP
- H – HILBERT SPACE
- $U \in M(\mathcal{K}(H) \otimes A)$ UNITARY

$$(\mathrm{id} \otimes \Delta) U = U_1 U_2$$

$$U_1 = U \otimes \mathbb{1} \in M(\mathcal{K}(H) \otimes A \otimes A)$$

$$U_2 = \Phi(U), \quad \Phi \in \mathrm{Mor}(\mathcal{K}(H) \otimes A, \mathcal{K}(H) \otimes A \otimes A),$$

$$\Phi(m \otimes a) = m \otimes \mathbb{1} \otimes a.$$

FACT: ANY IRREDUCIBLE REPRESENTATION OF A COMPACT QUANTUM GROUP IS FINITE-DIMENSIONAL.

THEOREM (S.L. WORONOWICZ):

LET (A, Δ) BE A COMPACT QUANTUM GROUP AND LET $\mathcal{A}$ BE THE SPAN OF MATRIX ELEMENTS OF ALL IRREDUCIBLE REPRESENTATIONS OF (A, Δ). THEN $\mathcal{A}$ IS A DENSE UNITAL $*$-SUBALGEBRA OF A. MOREOVER $\Delta(\mathcal{A}) \subset \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$ AND $(\mathcal{A}, \Delta|_{\mathcal{A}})$ IS A HOPF $*$-ALGEBRA.

- THE PROOF OF THE LAST THEOREM IS BASED ON THE NOTION OF THE REGULAR REPRESENTATION.

THIS CONSTRUCTION LIES OUTSIDE THE SCOPE OF THESE NOTES. LET US ONLY MENTION ITS SIMPLIFIED VERSION

ASSUME THAT THE HAAR MEASURE h OF (A, Δ) IS FAITHFUL. THEN A EMBEDDS AS A DENSE SUBSET OF THE GNS HILBERT SPACE H FOR h. ALSO $A \subset B(H)$. ONE CAN PROVE THAT THE MAPPING

$$A \otimes_{alg} A \ni a \otimes b \longmapsto \Delta(a)(\mathbb{1} \otimes b) \in A \otimes A$$

EXTENDS TO A UNITARY $U \in B(H \otimes H)$. MOREOVER $U \in M(\mathcal{K}(H) \otimes A)$ AND

$$(\mathrm{id} \otimes \Delta) U = U_1 U_2 .$$

REPRESENTATIONS OF $S_qU(2)$:

- $s \in \frac{1}{2}\mathbb{N}$, $T_s := \{-s, -s+1, \ldots, s\}$
- $w^s := \left(w^s_{a,b}\right)_{a,b \in T_s}$ DEFINED BY

$$\Delta\left(\alpha^{s+k}(\gamma^*)^{s-k}\right) = \sum_{i \in T_s} \alpha^{s+i}(\gamma^*)^{s-i} \otimes w^s_{i,k}$$

- $w^0 := \mathbb{1}$
- $\{w^s\}_{s \in \frac{1}{2}\mathbb{Z}_+}$ IS A COMPLETE LIST OF IRREDUCIBLE REPRESENTATIONS OF $S_qU(2)$ UP TO EQUIVALENCE.

LECTURE III: ACTIONS OF COMPACT QUANTUM GROUPS

DEFINITION: (A,Δ) – COMPACT QUANTUM GROUP
B – UNITAL C^*-ALGEBRA

AN ACTION OF (A,Δ) ON B IS A UNITAL $*$-HOMOMORPHISM $\alpha : B \longrightarrow B \otimes A$ SUCH THAT

- $(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$,
- $\operatorname{span}\{\alpha(b)(1\!\!1 \otimes a) \mid b \in B,\ a \in A\}$ IS DENSE IN $B \otimes A$.

CLASSICAL CASE: AN ACTION OF A COMPACT GROUP G ON A COMPACT SPACE X IS A CONTINUOUS MAP $X \times G \longrightarrow X$ (WRITTEN $(x,g) \longmapsto xg$) SUCH THAT

(i) $(x g_1) g_2 = x(g_1 g_2) \quad \forall x \in X,\ g_1, g_2 \in G$

(ii) $xe = x \quad \forall x \in X$

APPLICATION OF THE FUNCTOR $C(\cdot)$ YIELDS

$$\alpha : C(X) \longrightarrow C(X \times G) \cong C(X) \otimes C(G)$$

EXERCISE: (i) IS EQUIVALENT TO

$$(\alpha \otimes \mathrm{id}) \circ \alpha = (\mathrm{id} \otimes \Delta) \circ \alpha$$

WHERE $\Delta : C(G) \longrightarrow C(G) \otimes C(G)$ IS THE STANDARD COMULTIPLICATION ON $C(G)$.

PROPOSITION: B – UNITAL C^*-ALGEBRA

G – COMPACT GROUP

$\alpha : B \longrightarrow B \otimes C(G)$ UNITAL $*$-HOMOMORPHISM

$(\alpha \otimes id) \circ \alpha = (id \otimes \Delta) \circ \alpha$

THEN THE FOLLOWING ARE EQUIVALENT:

(1) THERE IS AN ACTION OF G ON B
(I.E. A HOMOMORPHISM $\tilde{\alpha} : G \longrightarrow \mathrm{Aut}(B)$ SUCH THAT $\forall\, b \in B$ THE MAP $G \ni t \mapsto \tilde{\alpha}_t(b) \in B$ IS CONTINUOUS)
SUCH THAT $(\alpha(b))(t) = \tilde{\alpha}_t(b)$ $\forall\, b \in B,\ t \in G$

(2) $\mathrm{span}\{\alpha(b)(\mathbb{1} \otimes f) \mid b \in B,\ f \in C(G)\}$ IS DENSE IN $B \otimes C(G)$.

PROOF: $(1) \Rightarrow (2)$

WE IDENTIFY $B \otimes C(G)$ WITH $C(G, B)$. THE MAP

$$\Phi_0 : B \otimes_{alg} C(G) \ni b \otimes f \longmapsto \alpha(b)(\mathbb{1} \otimes f) \in B \otimes C(G)$$

EXTENDS TO A UNITAL $*$-HOMOMORPHISM

$$\Phi : B \otimes C(G) \longrightarrow B \otimes C(G).$$

WE HAVE FOR $F \in B \otimes C(G) = C(G, B)$

$$(\Phi(F))(t) = \tilde{\alpha}_t(F(t)) \qquad (t \in G).$$

THUS FOR ANY $F \in C(G, B)$ WE HAVE $F = \Phi(\tilde{F})$, WHERE $\tilde{F}(t) = \tilde{\alpha}_{t^{-1}}(F(t))$. THIS MEAN THAT Φ IS ONTO. IT IS ALSO CLEARLY INJECTIVE, SO $\Phi \in \mathrm{Aut}(C(G) \otimes B)$

IT FOLLOWS THAT

$$\mathrm{span}\{\alpha(b)(\mathbb{1} \otimes f) \mid b \in B,\ f \in C(G)\}$$

IS DENSE IN $B \otimes C(G)$ AS THE IMAGE OF $B \otimes_{alg} C(G)$ UNDER AN AUTOMORPHISM.

$(2) \Rightarrow (1)$ DEFINE $\tilde{\alpha}_t = (id \otimes \delta_t) \circ \alpha : B \to B$

EVALUATION FUNCTIONAL $C(G) \ni f \mapsto f(t) \in \mathbb{C}$

THEN $\tilde{\alpha}_t \circ \tilde{\alpha}_s = \tilde{\alpha}_{ts}$ $\forall\, t,s \in G$. THIS IS AN ACTION OF G IFF THE IDEMPOTENT ENDOMORPHISM $\tilde{\alpha}_e : B \to B$ IS THE IDENTITY.

ASSUME THAT $\mathrm{span}\{\alpha(b)(1 \otimes f) \mid b \in B,\ f \in C(G)\}$ IS DENSE IN $B \otimes C(G)$. THEN FOR ANY $b \in B$ THE TENSOR $b \otimes 1$ CAN BE APPROXIMATED BY ELEMENTS FROM THIS SET, I.E.

$$b \otimes 1 = \lim_{n \to \infty} \sum_{i=1}^{N_n} \alpha(b_i^n)(1 \otimes f_i^n).$$

APPLYING $id \otimes \delta_e$ TO BOTH SIDES GIVES

$$b = \lim_{n \to \infty} \sum_{i=1}^{N_n} \tilde{\alpha}_e(b_i^n)\, f_i^n(e)$$

$$= \lim_{n \to \infty} \tilde{\alpha}_e \left(\sum_{i=1}^{N_n} f_i^n(e)\, b_i^n \right),$$

SO $\mathrm{Ran}\, \tilde{\alpha}_e = B$ AND $\tilde{\alpha}_e = id$. ■

EXAMPLES OF ACTIONS:

<u>(0)</u> ACTION OF A COMPACT GROUP ON A UNITAL C^*-ALGEBRA (PARTICULAR CASE: ACTION ON A COMPACT SPACE)

(1) FIX $n \in \mathbb{N}$. $X := \{1, \dots, n\}$, $G := S_n$ PERMUTATION GROUP. (4)

$X \times G \longrightarrow X$ — (ALMOST) STANDARD ACTION.

THIS GIVES RISE TO AN ACTION OF $(A = C(G), \Delta)$ ON $B = C(X) \cong \mathbb{C}^n$.

LET'S LOOK AT $G = S_n$ AS THE GROUP OF PERMUTATION MATRICES

$$\begin{pmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$(a_{ij})^2 = a_{ij} \quad (= \overline{a_{ij}}) \qquad \forall\, i,j$$

$$\sum_{j=1}^{n} a_{ij} = 1 \qquad \forall\, i$$

$$\sum_{i=1}^{n} a_{ij} = 1 \qquad \forall\, j$$

THUS

$$C(G) = C^*\left(a_{ij} \;\middle|\; a_{ij} = (a_{ij})^2 = \overline{a_{ij}},\ \sum_{i=1}^{n} a_{ij} = 1\!\!1,\ \sum_{j=1}^{n} a_{ij} = 1\!\!1,\ [a_{ij}, a_{kl}] = 0\right)$$

SIMILARLY

$$C(X) = C^*\left(e_i \;\middle|\; i = 1, \dots, n,\ e_i = e_i^* = e_i^2\ \forall i,\ \sum_{j=1}^{n} e_j = 1\!\!1\right)$$

THE ACTION IS DESCRIBED BY

$$\alpha_0 : C(X) \longrightarrow C(X) \otimes C(G)$$

$$\alpha_0(e_j) = \sum_{i=1}^{n} e_i \otimes a_{ij} \qquad (j = 1, \dots, n).$$

LET US CONSIDER A DIFFERENT C^*-ALGEBRA

$$A := C^*\left(a_{ij} \mid ij=1,\dots,n,\ a_{ij}=a_{ij}^2=a_{ij}^*,\ \sum_{i=1}^{n} a_{ij}=1,\ \sum_{j=1}^{n} a_{ij}=1\right)$$

WE HAVE A UNITAL $*$-HOMOMORPHISM

$$\alpha : B = C(X) \ni e_j \longmapsto \sum_{i=1}^{n} e_i \otimes a_{ij} \in B \otimes A.$$

THEOREM (S. WANG):

(1) THERE IS A UNIQUE $\Delta : A \to A \otimes A$ MAKING (A, Δ) A COMPACT QUANTUM GROUP SUCH THAT

$$\alpha : B \to B \otimes A$$

IS AN ACTION OF (A, Δ) ON B.

(2) IF (C, Δ_C) IS A COMPACT QUANTUM GROUP AND $\beta : B \to B \otimes C$ IS AN ACTION OF (C, Δ_C) ON B THEN THERE IS A UNIQUE $\Phi : B \to C$ SUCH THAT

$$\beta = (id \otimes \Phi) \circ \alpha.$$

MOREOVER $(\Phi \otimes \Phi) \circ \Delta = \Delta_C \circ \Phi$.

REMARK: POINT (2) ABOVE CHARACTERIZES THE COMPACT QUANTUM GROUP (A, Δ) UNIQUELY.

THEOREM (S. WANG): FOR $n = 1, 2, 3$ $A \cong C(S_n)$. FOR $n \geq 4$ A IS NON-COMMUTATIVE AND $\dim A = \infty$.

THE COMPACT QUANTUM GROUPS (A, Δ) DISCUSSED ABOVE ARE CALLED THE QUANTUM PERMUTATION GROUPS.

NOTE: LET $\mu : B \to \mathbb{C}$ BE THE FUNCTIONAL ("MEASURE") WHICH MAPS EACH e_i TO $\frac{1}{n}$. THEN

$$(\mu \otimes id)\,\alpha(b) = \mu(b)\,\mathbb{1} \qquad \forall\, b \in B$$

(EXERCISE: CHECK THIS).

THIS PROPERTY IS CALLED INVARIANCE OF μ WITH RESPECT TO α (μ IS PRESERVED BY α).

(2) LET US CONSIDER $B = M_2(\mathbb{C})$ AND FOR $q \in]0,1]$ LET ω_q BE THE FUNCTIONAL

$$B \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{a + q^2 d}{1 + q^2} \in \mathbb{C}$$

(THIS IS THE POWERS STATE ON M_2).

PROPOSITION: $B := M_2$, $(A, \Delta) := S_qU(2)$. THERE IS A UNIQUE MAP $\Psi_q : B \to B \otimes A$ SUCH THAT

$$\Psi_q\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} -q\alpha\gamma & \alpha^2 \\ -q\gamma^2 & q^{-1}\alpha\gamma \end{pmatrix} \in M_2(A) = B \otimes A.$$

MOREOVER Ψ_q IS AN ACTION OF $S_qU(2)$ ON B PRESERVING THE POWERS STATE.

A FEW MORE DETAILS:

- THE 3-DIMENSIONAL IRREDUCIBLE REPRESENTATION OF $S_qU(2)$

$$w' = \begin{pmatrix} 1 - (1+q^2)\gamma^*\gamma & \sqrt{1+q^2}\,\gamma\alpha & -\sqrt{1+q^2}\,\gamma^*\alpha^* \\ -\sqrt{1+q^2}\,\alpha\gamma^* & \alpha^2 & q^2\gamma^{*2} \\ \sqrt{1+q^2}\,\alpha^*\gamma & \gamma^2 & \alpha^{*2} \end{pmatrix}$$

- LET C BE THE C^*-SUBALGEBRA OF A GENERATED BY MATRIX ELEMENTS OF w'. THEN $\Delta(C) \subset C \otimes C$ AND WITH $\Delta_C = \Delta|_C$ THE PAIR (C, Δ_C) IS A COMPACT QUANTUM GROUP.
THIS IS THE <u>QUANTUM SO(3) GROUP</u> $(S_qO(3))$.

- THE ACTION Ψ_q OF $S_qU(2)$ ON $B = M_2$ IS REALLY AN ACTION OF $S_qO(3)$, I.E. $\Psi_q(B) \subset B \otimes C$.

<u>THEOREM:</u> LET (D, Δ_D) BE A COMPACT QUANTUM GROUP AND LET $\varphi: M_2 \to M_2 \otimes D$ BE AN ACTION OF (D, Δ_D) ON M_2 PRESERVING THE POWERS STATE ω_q.
THEN THERE EXISTS A UNIQUE UNITAL $*$-HOMOMORPHISM $\Phi: C \to D$ SUCH THAT

$$\varphi = (\mathrm{id} \otimes \Phi) \circ \Psi_q.$$

<u>FACT:</u> LET (D, Δ_D) BE A COMPACT QUANTUM GROUP ACTING ON A FINITE-DIMENSIONAL C^*-ALGEBRA N. THEN THERE EXISTS A FAITHFUL STATE ON N INVARIANT FOR THE ACTION OF (D, Δ_D).

THEOREM: LET (D, Δ_D) BE A COMPACT QUANTUM GROUP ACTING ON M_2:

$$\varphi : M_2 \longrightarrow M_2 \otimes D$$

THEN THERE EXIST

- $q \in \left]0,1\right]$
- UNITARY $u \in M_2$
- $\Phi : C \longrightarrow D$, WHERE $(C, \Delta_C) = S_qO(3)$

SUCH THAT

$$\varphi(m) = (id \otimes \Phi)\Big((u \otimes 1)\, \Psi_q(u^* m u)\,(u^* \otimes 1)\Big) \qquad \forall\, m \in M_2 .$$