Hochschild and cyclic homology

Mini-School on NCG '2010

WARSAW, IMPAN

TOMASZ MASZCZYK

Further reading:

- J.-L. Loday, "Cyclic homology"
- A. Connes, "Noncommutative Geometry"
- J. C. Varilly, H. Figueroa, J. M. Gracia-Bondia, "Elements of Noncommutative Geometry"
- J.-L. Loday, M. Wodzicki, "Cyclic homology Theory", TOK Notes 2006/07.

I. Hochschild homology

1. Categories

1.1. Pre-simplicial and simplicial categories.

Δ^{pre} - presimplicial category

Objects: $[n] := \{0 < 1 < \ldots < n\}$, $n = 0, 1, \ldots$

$$\mathrm{Mor}_{\Delta^{pre}}([m],[n]) = \{f : [m] \to [n] \mid f(0) < \ldots < f(m)\}$$

Def. (Face maps)

$$\delta_i : [n] \to [n+1], \quad \delta_i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \geq i \end{cases}$$
$i = 0, \ldots, n+1$

Proposition Δ^{pre} admits a presentation by δ_i's satisfying relations

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if} \quad i < j.$$

Example 1. $m > n \Rightarrow \mathrm{Mor}_{\Delta^{pre}}([m],[n]) = \emptyset$

2. $m \leq n \Rightarrow$ every morphism decomposes uniquely as $\delta_{i_1} \ldots \delta_{i_{n-m}}$, $i_1 \leq \ldots \leq i_{n-m}$

Δ – simplicial category

Objects: $[n] = \{0 < 1 < \dots < n\}$, $n = 0, 1, \dots$

$$\mathrm{Mor}_\Delta([m],[n]) = \{f : [m] \to [n] \mid f(0) \leq \dots \leq f(m)\}$$

Face maps as for Δ^{pre}

Def. (Degeneracy maps)

$$\sigma_j : [n] \to [n-1], \quad \sigma_j(x) = \begin{cases} x & \text{if } x \leq j \\ x-1 & \text{if } x > j \end{cases}$$

$j = 0, \dots, n-1$

Proposition Δ admits a presentation by δ_i's and σ_j's satisfying relations

$$\delta_j \delta_i = \delta_i \delta_{j-1} \qquad \text{if } i < j$$

$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \qquad \text{if } i \leq j$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j \\ \mathrm{id}_{[n]} & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & \text{if } i > j+1. \end{cases}$$

Exercise 1. Every $f \in \mathrm{Mor}_\Delta([m],[n])$ decomposes uniquely as $\delta_{i_1} \dots \delta_{i_{n-m+s}} \sigma_{j_1} \dots \sigma_{j_s}$

1.2. (Pre) simplicial modules

Def. (Pre) simplicial object in $\mathcal{C}$ is a functor $((\Delta^{pre})^{op} \longrightarrow \mathcal{C})$
$\Delta^{op} \longrightarrow \mathcal{C}$.

Fix k a commutative ring ($\mathbb{Z}$, any field etc)

(Pre) simplicial object in k-Mod = (Pre) simplicial module.

$$\delta_i \rightsquigarrow \partial_i$$
$$\sigma_j \rightsquigarrow s_j$$

and opposite order of composition.

Example A a k-algebra: $A \otimes_k A \xrightarrow{\mu} A$ in k-Mod
$k \xrightarrow{\eta} A$

$$\begin{tikzcd} A \otimes_k A \otimes_k A \arrow[r, "\mu \otimes A"] \arrow[d, "A \otimes \mu"'] & A \otimes_k A \arrow[d, "\mu"] \\ A \otimes_k A \arrow[r, "\mu"'] & A \end{tikzcd}$$ associativity

$$\begin{tikzcd} A = k \otimes_k A \arrow[r, "\eta \otimes A"] \arrow[dr, equal] & A \otimes_k A \arrow[d] & A \otimes_k k = A \arrow[l, "A \otimes \eta"'] \arrow[dl, equal] \\ & A & \end{tikzcd}$$ unitality

M an A-bimodule: $A \otimes_k M \xrightarrow{\lambda} M$ in k-Mod
$M \otimes_k A \xrightarrow{\rho} M$

$$\begin{tikzcd}
A \otimes_k A \otimes_k M \arrow[r, "A \otimes \lambda"] \arrow[d, "\mu \otimes M"'] & A \otimes_k M \arrow[d, "\lambda"] \\
A \otimes_k M \arrow[r, "\lambda"'] & M
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes_k M \otimes_k A \arrow[r, "A \otimes \rho"] \arrow[d, "\lambda \otimes A"'] & A \otimes_k M \arrow[d, "\lambda"] \\
M \otimes_k A \arrow[r, "\rho"'] & M
\end{tikzcd}$$

$$\begin{tikzcd}
M \otimes_k A \otimes_k A \arrow[r, "M \otimes \mu"] \arrow[d, "\rho \otimes A"'] & M \otimes_k A \arrow[d, "\rho"] \\
M \otimes_k A \arrow[r, "\rho"'] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
M = k \otimes_k M \arrow[r, "\eta \otimes M"] \arrow[dr, equal] & A \otimes_k M \arrow[d, "\lambda"] & M \otimes_k A \arrow[d, "\rho"'] & M \otimes_k k = M \arrow[l, "M \otimes \eta"'] \arrow[dl, equal] \\
& M & M &
\end{tikzcd}$$

$$C_n(A,M) := M \otimes_k A^{\otimes_k n} = M \otimes A^{\otimes n} \qquad (\otimes_k = \otimes)$$

$$\partial_0(m, a_1, \dots, a_n) = (m a_1, a_2, \dots, a_n)$$

$$\partial_i(m, a_1, \dots, a_n) = (m, a_1, \dots, a_i a_{i+1}, \dots, a_n) \qquad i = 1, \dots, n-1$$

$$\partial_n(m, a_1, \dots, a_n) = (a_n m, a_1, \dots, a_{n-1})$$

$$s_j(m, a_1, \dots, a_n) = (m, a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n) \quad j = 0, \dots, n$$

__Exercise__. Check that it is a simplicial module.

__Definition__ Let $(\{C_n\}, \partial_i, s_j)$ $((C_n), \partial_i)$ be a (pre) simplicial module. Boundary map

$$\partial : C_n \longrightarrow C_{n-1}$$

$$\partial := \sum_{i=0}^{n} (-1)^i \partial_i$$

__Exercise.__ $\partial^2 = 0$.

Name "boundary" comes from Algebraic Topology

For a simplicial module as above special notation $\partial = b$.

<u>Corollary</u> $((C_n), \partial)$ for a simplicial module is a complex.

It leads to homology:

$$H_n(C_\bullet) = \frac{\ker(\partial : C_n \to C_{n-1})}{\mathrm{im}(\partial : C_{n+1} \to C_n)}.$$

<u>Definition</u> $H_n(C_\bullet(A,M), b) = H_n(A,M)$

n-th <u>Hochschild homology of A with coefficients in M</u>.

$$HH_n(A) := H_n(A,A).$$

<u>Properties</u> 1) <u>Base ring plays a role</u>:

$k = \mathbb{C} \Rightarrow HH_1(\mathbb{C}) = 0$

$k = \mathbb{Q} \Rightarrow HH_1(\mathbb{C}) \neq 0.$

2. <u>Functoriality</u>:

$B \to A \rightsquigarrow H_n(B,M) \to H_n(A,M)$

· M - A-bimodule

$M \to M' \rightsquigarrow H_n(A,M) \to H_n(A,M')$

A - bimodules

3. <u>HH_n respects products</u>

$$HH_n(A\times B) \cong HH_n(A)\times HH_n(B)$$

(special case of the Künneth formula for Tor's)

4. <u>$H_n(A,M)$ is a $Z(A)$-module</u>.

<u>Low dimensional homology:</u>

$$H_0(A,M) = M/\operatorname{im} b = M/(am-ma) = M/[A,M]$$

In particular: $HH_0(A) = A/[A,A]$

<u>Exercise</u>. $HH_0(k) = k$

$HH_n(k) = 0 \quad , \quad n>0.$

<u>Interesting property of $A/[A,A]$</u>

$$Tr : M_r(A) \longrightarrow A/[A,A] \quad (a_{ij}) \longmapsto \sum_i a_{ii} + [A,A]$$

$\ker Tr = [M_r(A), M_r(A)]$ (use elementary matrices)

$$\Rightarrow \quad M_r(A)/[M_r(A),M_r(A)] \xrightarrow[Tr]{\cong} A/[A,A].$$

(This is a special case of Morita invariance.)

<u>Proposition</u> If A is commutative, M - A-module (symm. bimodule) $a \cdot m = am$, $m \cdot a = am$, $a, a' \in A$, $c \in k$

$$H_1(A, M) = M \otimes_A \Omega^1_{A/k}$$

where $\Omega^1_{A/k} = A \cdot (da) \big/ \big(d(aa') - a\,da' - a'\,da,\ d(c)\big)$

A-module of universal k-linear derivations of A with values in A-modules

$$\begin{tikzcd} A \arrow[r, "\forall \delta"] \arrow[d, "d"', "k\text{-linear}"] & M \\ \Omega^1_{A/k} \arrow[ur, dashed, "\exists!\ A\text{-linear}"'] & \end{tikzcd}$$

<u>Proof</u> $(M = A)$

$$C_1(A, A) = A \otimes A \xrightarrow{b} C_0(A, A) = A$$

$$a_0 \otimes a_1 \longmapsto a_0 a_1 - a_1 a_0 = 0$$

$$\Rightarrow \quad HH_1(A) = A \otimes A \big/ (a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1)$$

$$\downarrow\uparrow \qquad \begin{matrix} a_0 \otimes a_1 \\ \updownarrow \\ a_0\, da_1 \end{matrix}$$

$$\Omega^1_{A/k}$$

<u>Exercise</u> They are well defined, and inverse one to each other.

<u>Exercise</u> Prove it for arbitrary A-module M.

More advanced understanding of $H_n(A,M)$ needs homological algebra

$\boxed{\text{It is a derived functor of } M \rightsquigarrow M/[A,M]}$

<u>Def.</u> (Bar complex)

$$B_n(A) = A \otimes A^{\otimes n} \otimes A = A^{\otimes n+2}$$

A-bimodule

$$a(a_0 \otimes \dots \otimes a_{n+1}) = aa_0 \otimes \dots \otimes a_{n+1}$$

$$(a_0 \otimes \dots \otimes a_{n+1})a = a_0 \otimes \dots \otimes a_{n+1}a$$

$$b' := \sum_{i=0}^{n-1} (-1)^i \partial_i \quad , \quad b'b' = 0$$

<u>Proposition</u> The map of complexes

$$\begin{tikzcd}
B_0(A) = A \otimes A \arrow[r, "\mu"] & A \\
B_1(A) \arrow[u, "b'"] \arrow[r] & 0 \arrow[u] \\
B_2(A) \arrow[u, "b'"] \arrow[r] & 0 \arrow[u] \\
\vdots \arrow[u] & \arrow[u]
\end{tikzcd}$$

is a chain homotopy equivalence. (Resolution)

<u>Proof</u>. Homotopy $s: \underset{B_{n-1}(A)}{A^{\otimes n+1}} \to \underset{B_n(A)}{A^{\otimes n+2}}$

$$s(a_0 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes \ldots \otimes a_n .$$

then $\quad \partial_i s = s \partial_{i-1} \quad , \quad i \geq 1$

$$\partial_0 s = id .$$

$\Rightarrow \quad b' s + s b' = id \quad , \quad \mu s = id$

$$\ldots \xrightarrow{b'} B_1(A) \xrightarrow{b'} B_0(A) \xrightarrow{\mu} A \to 0 \to \ldots$$

contractible to 0. $\square$

Deriving a functor F means applying it to an F-acyclic resolution, i.e. a resolution on whose homogenous pieces the functor F behaves as exact.

If A is k-flat (i.e. $A \otimes_k -$ is exact) then $B_n(A)$ is $A \otimes_k A^{op} =: A^e$ - flat.

$$C_n(A,M) = M \otimes_k A^{\otimes n} \simeq M \otimes_{A^e} (A \otimes A^{\otimes n} \otimes A) =$$

$$= M \otimes_{A^e} B_n(A)$$

$\Rightarrow H_n(A,M)$ is a derived functor of

$$M \rightsquigarrow M \otimes_{A^e} A = M/[A,M] \quad \square$$

Kähler differentials

A a commutative k-algebra

$$\Omega^\bullet_{A/k} = \frac{(a, da)}{(d(aa') - a\,da' - a'\,da,\ d(x))} \quad \text{- } k\text{-algebra}$$

→ make it a skew-commutative dg k-algebra

Universal skew-commutative differential k-algebra

$\forall$ k - - - - algebra maps on A

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\text{universal algebra map}"'] & \Omega^\bullet \\ \Omega^\bullet_{A/k} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$\exists!$ dg algebra map of skew-commutative dg algebras

1) elements: $a_0\,da_1 \dots da_n$ span a k-module $\Omega^n_{A/k}$.

2) $d(a_0\,da_1 \dots da_n) = da_0 \dots da_n$ well defined.

Def. A is smooth over k if $\Omega^1_{A/k}$ is a direct summand in a free A-module of finite rank and A is finitely generated as a k-algebra and flat as a k-module.

Motivation $A = \mathbb{C}[X]$ - polynomial functions on a complex alg. variety $X \subset \mathbb{C}^n$ $\Rightarrow$ A-smooth $/\mathbb{C}$ $\Leftrightarrow$ X-manifold.

$$H^n(\Omega^\bullet_{A/k}, d) =: H^n_{DR}(A).$$

De Rham cohomology

Motivation: $A = \mathbb{C}[X]$ - polynomial functions on a complex algebraic manifold $X \subset \mathbb{C}^n$ $\Rightarrow$ $H^n_{DR}(A) = H^n(X, \mathbb{C})$.

Hochschild – Kostant – Rosenberg Theorem
for Hochschild homology commutative

If A is smooth commutative over $k \supset \mathbb{Q}$ and M is an A-module, then

$$H_n(A,M) \rightleftarrows M \otimes_A \Omega^n_{A/k} .$$

$$m \otimes a_1 \otimes \ldots \otimes a_n \longmapsto m \otimes da_1 \ldots da_n$$

$$\frac{1}{n!} \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \cdot m \otimes a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)} \longleftarrow m \otimes da_1 \ldots da_n$$

well defined

and inverse one to each other.

Relative Hochschild homology $HH_n(A,I)$

Def. $I \triangleleft A$

$$HH_n(A,I) := H_n(\ker(C_\bullet(A,A) \to C_\bullet(A/I, A/I)))$$

Proposition Long exact sequence

$$\cdots \to HH_n(A,I) \to HH_n(A) \to HH_n(A/I) \to HH_{n-1}(A,I) \to \cdots$$

Morita invariance

$$M_r(M) := M_r(k) \otimes M$$

$$M_\infty(M) := \varinjlim_r M_r(M)$$

$$Tr : M_r(M) \otimes M_r(A)^{\otimes n} \longrightarrow M \otimes A^{\otimes n}$$

$$(m_{ij}) \otimes (a_{1,ij}) \otimes \ldots \otimes (a_{n,ij}) \mapsto$$

$$\sum_{1 \leq i_0, \ldots, i_n \leq r} m_{i_0 i_1} \otimes a_{1, i_1 i_2} \otimes \ldots \otimes a_{n-1, i_{n-1} i_n} \otimes a_{n, i_n i_0}$$

Dennis trace map

<u>Proposition</u> The Dennis trace map gives a morphism of complexes

$$C_n(M_r(A), M_r(M)) \to C_n(A, M)$$

<u>Proof</u>. $Tr \circ \partial_i = \partial_i \circ Tr$ $\square$

<u>Def^n</u> $i : M \hookrightarrow M_r(M)$

$$m \longmapsto \begin{pmatrix} m & & 0 \\ & 0 & \\ 0 & & \ddots \; 0 \end{pmatrix}.$$

<u>Theorem</u> $Tr \circ i = id$ on $C_n(A, M)$

$i \circ Tr \sim id$ on $C_n(M_r(A), M_r(M))$

(chain homotopy equivalence)

<u>Proof</u>. (Loday)

<u>Corollary</u> Tr and i define isomorphisms inverse to each other

$$Tr_n : H_n(M_r(A), M_r(M)) \to H_n(A, M)$$

$$i_n : H_n(A, M) \longrightarrow H_n(M_r(A), M_r(M)).$$

Morita equivalence

A, B k-algebras

P is an (B,A)-bimodule if

$$B\otimes P \xrightarrow{\lambda} P, \quad P\otimes A \xrightarrow{\rho} P$$

& commutative diagrams as above for A-bimodules.

<u>Def.</u> $A\sim B$ (Morita equivalent) if there exist P a (B,A)-bimodule and Q an (A,B)-bimodule s.t.

$$P\otimes_A Q \cong B \quad \text{as } B\text{-bimodules}$$

$$Q\otimes_B P \cong A \quad \text{as } A\text{-bimodules}$$

<u>Example</u> $A = M_r(B) \sim B$

$$P = B^r \quad (\text{rows } (b_1,\dots,b_r))$$

$$Q = B^r \quad \left(\text{columns } \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}\right)$$

<u>Exercise</u> show that $P\otimes_{M_r(B)} Q \cong B$

$$Q\otimes_B P \cong M_r(B).$$

<u>Theorem</u> $A \sim B$ via P, Q as above

N B-bimodule

$M := Q \otimes_B N \otimes_B P$ A-bimodule

$$\Rightarrow \quad H_n(A, M) \cong H_n(B, N).$$

<u>Proof</u> (Loday)

<u>More-advanced understanding of Morita equivalence needs categories and functors</u>

$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "P \otimes_A -"] & B\text{-Mod} \arrow[l, bend left, "Q \otimes_B -"] \end{tikzcd}$$

A pair (P, Q) defines equivalences inverse to each other.

<u>Theorem</u> (Morita) Every equivalence between A-Mod and B-Mod comes from some pair (P, Q).

Cyclic homology

Cyclic category Λ

Objects : $[n] = \{ 0 < 1 < 2 < \cdots < n < 0 \}$ cyclic order

$$\mathrm{Mor}_\Lambda([m],[n]) = \{ f : [m] \to [n] \mid f(0) \leq f(1) \leq f(2) \leq \cdots \leq f(m) \leq f(0) \}$$

Face and degeneracy maps as for Δ

Def. (cyclic maps)

$$\tau_n : [n] \longrightarrow [n] \qquad \tau_n = \begin{cases} x-1 & x \geq 1 \\ n & x = 0 \end{cases}$$

(counter clock wise rotation by one)

Proposition Λ admits a presentation by δ_i's, σ_j's, τ_n's satisfying relations as for Δ and $\tau_n^{n+1} = \mathrm{id}_{[n]}$,

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \qquad 1 \leq i \leq n$$
$$\tau_n \delta_0 = \delta_n,$$
$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \qquad 1 \leq i \leq n$$
$$\tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 .$$

Def. A <u>cyclic module</u> is a functor $\Lambda^{op} \longrightarrow k\text{-Mod}$.

$$\delta_i \longmapsto \partial_i$$
$$\sigma_j \longmapsto s_j \qquad \text{and opposite order of composition}$$
$$\tau_n \longmapsto t_n$$

Def 1 Let $((C_n), \partial_i, s_j, t_n)$ be a cyclic module. The <u>associated cyclic bicomplex</u> $(CC_\bullet, b, b', 1-t, N)$ is

$$\begin{tikzcd}
{} \arrow[d, "b"'] & {} \arrow[d, "-b'"] & {} \arrow[d, "b"] & {} \arrow[d, "-b'"] & \\
C_2 \arrow[d, "b"'] & C_2 \arrow[l, "1-t"] \arrow[d, "-b'"] & C_2 \arrow[l, "N"] \arrow[d, "b"] & C_2 \arrow[l, "N"] \arrow[d, "-b'"] & \arrow[l, "N"] \\
C_1 \arrow[d, "b"'] & C_1 \arrow[l, "1-t"] \arrow[d, "-b'"] & C_1 \arrow[l, "N"] \arrow[d, "b"] & C_0 \arrow[l, "1-t"] \arrow[d, "-b'"] & \arrow[l, "N"] \\
C_0 & C_0 \arrow[l, "1-t"] & C_0 \arrow[l, "N"] & C_0 \arrow[l, "1-t"] & \arrow[l, "N"]
\end{tikzcd}$$

$$b = \sum_{i=0}^{n} (-1)^i \partial_i, \quad b' = \sum_{i=0}^{n-1} (-1)^i \partial_i, \quad N = \sum_{i=0}^{n} t_n^i$$

$t = t_n$.

Def. 2 (Connes) The complex $(C^\lambda_\bullet, b)$

$$C^\lambda_n = C_n/(1-t_n)$$

$$\cdots \xrightarrow{b} C^\lambda_1 \xrightarrow{b} C^\lambda_0 \to 0 \to 0 \to \cdots$$

Def. 3 Mixed complex BC

$$\begin{tikzcd}
\arrow[d, "b"] & \arrow[d, "b"] & \arrow[d, "b"] \\
C_2 \arrow[d, "b"] & C_1 \arrow[l, "B"'] \arrow[d, "b"] & C_0 \arrow[l, "B"'] \\
C_1 \arrow[d, "b"] & C_0 \arrow[l, "B"'] & \\
C_0 & &
\end{tikzcd}$$

$$B := (1-t)sN \quad , \quad s := (-1)^n t_n s_n : C_n \to C_{n+1}$$

theorem $k \supset \mathbb{Q} \Rightarrow$

$$H_n(\mathrm{Tot}(CC_{\bullet\bullet})) \cong H_n(C^\lambda_\bullet) \cong H_n(\mathrm{Tot}(BC_{\bullet\bullet}))$$

Common name: $HC_n(C_\bullet)$
(homologie cyclique)

Proof (Loday)

<u>Example</u>. A a k-algebra

$$[n] : \left(n \to 0 \to 1 \to 2 \to \cdots \to n \right) \rightsquigarrow \left(a_n \otimes a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes \right)$$

∂_i's, s_j's as for $C_\bullet(A,A)$

$$t_n \left(a_n \otimes a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes \right) = (-1)^n \left(a_{n-1} \otimes a_n \otimes a_0 \otimes a_1 \otimes \cdots \otimes \right)$$

clock wise rotation

(remember: cyclic module is a contravariant functor)

<u>Exercise</u> Show that $(C_\cdot(A,A), \partial_i, s_j, t_n)$ is a cyclic module.

$$HC_n(C_\cdot(A,A)) =: HC_n(A)$$

<u>cyclic homology</u> of a k-algebra A.

<u>Theorem</u> (Connes) There is a long exact sequence (Connes' Periodicity)

$$\cdots \to HC_{n-1}(A) \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} \cdots$$

where S is induced by

$$C_n(A,A) \longrightarrow C_{n-2}(A,A), \quad n > 1$$

$$x \longmapsto \frac{-1}{n(n-1)} \sum_{0 \le i,j \le n} (-1)^{i+j} \partial_i \partial_j (x)$$

(Periodicity map)

I is induced by the projection

$$C_n(A,A) \longrightarrow C_n(A,A)/(1-t_n),$$

$B = (1-t)sN$ can be computed as

$$C_n(A,A) \longrightarrow C_{n+1}(A,A)$$

$$B(a_0,\dots,a_n) = \sum_{i=0}^{n-1} \Big((-1)^{ni} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1}) - (-1)^{n(i-1)} (a_{i-1}, 1, a_i, \dots, a_n, a_0, \dots, a_{i-2}) \Big)$$

Example $HC_n(k) = \begin{cases} k & n \text{ even} \\ 0 & n \text{ odd}. \end{cases}$

Example $HC_0(A) \cong HH_0(A) \cong A/[A,A]$

Proposition A - commutative $\Rightarrow$ $HC_1(A) \cong \Omega^1_{A/k}/dA$.

Kind of Hochschild-Kostant-Rosenberg Theorem for cyclic homology:

A smooth over k

$$HC_{2n}(A) \cong \Omega^{2n}_{A/k}/d\Omega^{2n-1}_{A/k} \oplus H^{2n-2}_{DR}(A) \oplus \dots \oplus H^0_{DR}(A)$$

$$HC_{2n+1}(A) \cong \Omega^{2n+1}_{A/k}/d\Omega^{2n}_{A/k} \oplus H^{2n-1}_{DR}(A) \oplus \dots \oplus H^1_{DR}(A).$$