

# Hochschild and cyclic homology (0)

Mini-school on NCG '2010

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## Further reading:

- J.-L. Loday, "Cyclic homology"
- A. Connes, "Noncommutative Geometry"
- J. C. Varilly, H. Figueroa, J. M. Gracia-Bondia, "Elements of Noncommutative Geometry"
- J.-L. Loday, M. Wodzicki, "Cyclic homology Theory", TOK Notes 2006/07.

# I. Hochschild homology

## 1. Categories

### 1.1. Pre-simplicial and simplicial categories.

#### $\Delta^{pre}$ - pre-simplicial category

Objects:  $[n] := \{0 < 1 < \dots < n\}$ ,  $n = 0, 1, \dots$

$\text{Mor}_{\Delta^{pre}}([m], [n]) = \{f : [m] \rightarrow [n] \mid f(0) < \dots < f(m)\}$

Def. (Face maps)

$$\delta_i : [n] \rightarrow [n+1], \delta_i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \geq i \end{cases}$$

$i = 0, \dots, n+1$

Proposition  $\Delta^{pre}$  admits a presentation by  $\delta_i$ 's satisfying relations

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if } i < j.$$

Exercise 1.  $m > n \Rightarrow \text{Mor}_{\Delta^{pre}}([m], [n]) = \emptyset$

2.  $m \leq n \Rightarrow$  every morphism decomposes

uniquely as  $\delta_{i_1} \dots \delta_{i_{n-m}}, i_1 \leq \dots \leq i_{n-m}$

$\Delta$ -simplicial category

Objects:  $[n] = \{0 < 1 < \dots < n\}$ ,  $n = 0, 1, \dots$

$\text{Mor}_{\Delta}([m], [n]) = \{f: [m] \rightarrow [n] \mid f(0) \leq \dots \leq f(m)\}$

Face maps as for  $\Delta^{\text{pre}}$

Def. (Degenometry maps)

$$\sigma_j: [n] \rightarrow [n-1], \sigma_j(x) = \begin{cases} x & \text{if } x \leq j \\ x-1 & \text{if } x > j \end{cases}$$

$j = 0, \dots, n-1$

Proposition  $\Delta$  admits a presentation by  $\delta_i$ 's and  $\sigma_j$ 's satisfying relations

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{if } i < j$$

$$\sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad \text{if } i \leq j$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j \\ \text{id}_{[m]} & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & \text{if } i > j+1. \end{cases}$$

Exercise 1. Every  $f \in \text{Mor}_{\Delta}([m], [n])$  decomposes uniquely as  $\delta_{i_1} \dots \delta_{i_{n-m+s}} \sigma_{j_1} \dots \sigma_{j_s}$

(Pre)

1.2. Prebialgebras

Def. (Pre)bialgebra object in  $\mathcal{C}$   
 is a functor  $((\Delta^{pre})^{op} \rightarrow \mathcal{C})$   
 $\Delta^{op} \rightarrow \mathcal{C}$ .

for  $k$  a commutative ring ( $\mathbb{Z}$ , any field etc)

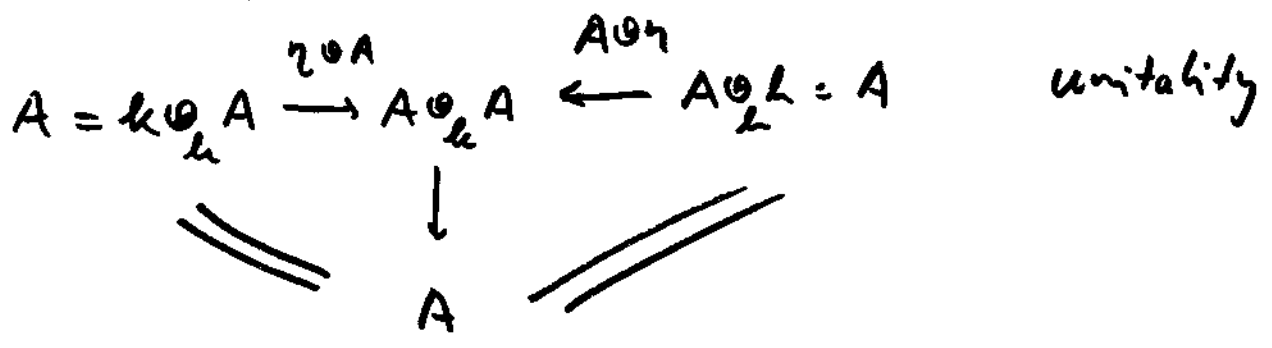
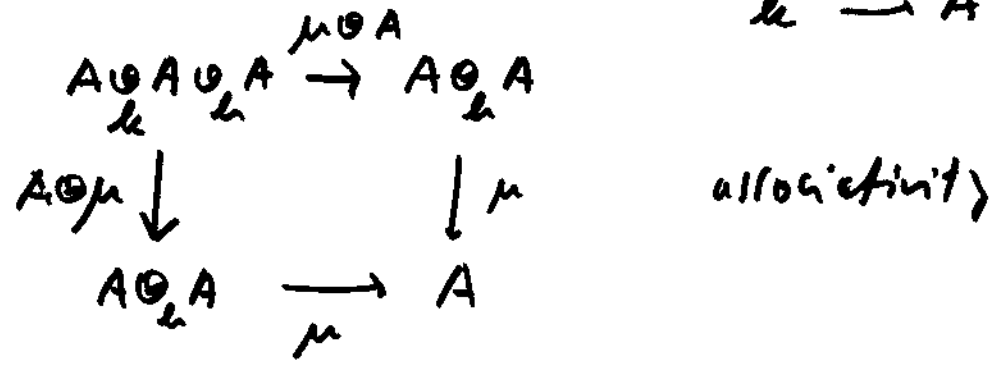
(Pre)bialgebra object in  $k\text{-Mod} =$   
 $=$  (Pre)bialgebra module.

$$\delta_i \rightsquigarrow d_i$$

$$\sigma_j \rightsquigarrow s_j$$

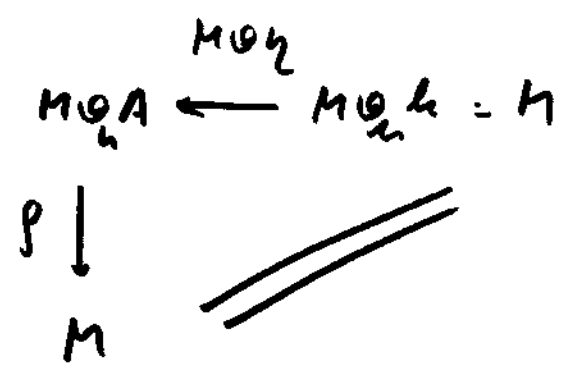
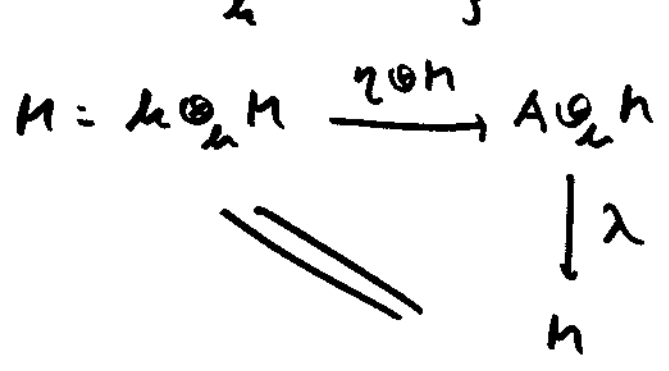
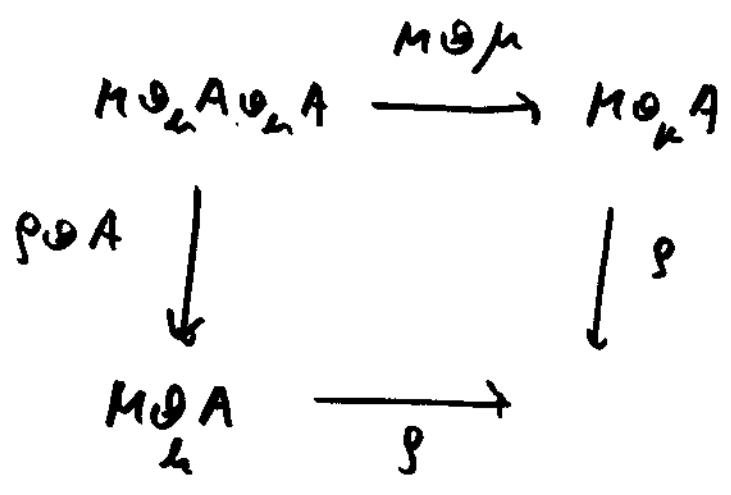
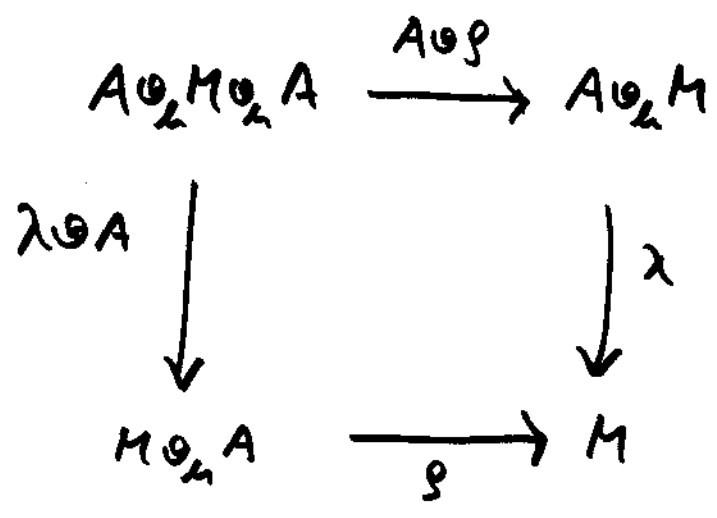
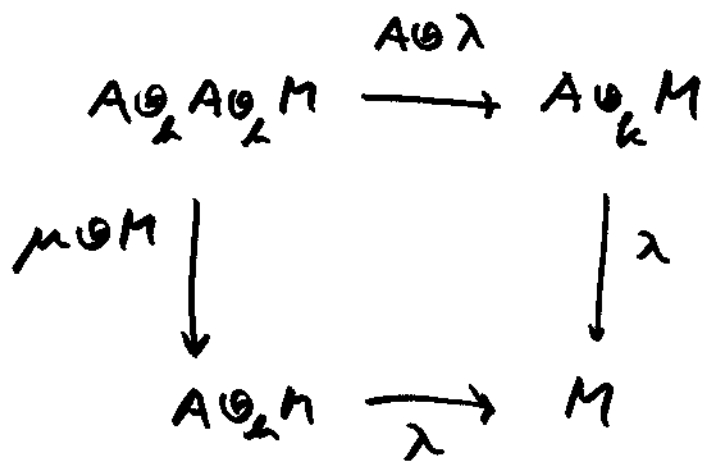
and opposite order of composition.

Example  $A$  a  $k$ -algebra:  $A \otimes_k A \xrightarrow{\mu} A$  in  $k\text{-Mod}$   
 $k \xrightarrow{\eta} A$



④

M an A-bimodule:  $A \otimes_k M \xrightarrow{\lambda} M$  in  $k\text{-Mod}$   
 $M \otimes_k A \xrightarrow{\rho} M$



$$C_n(A, M) := M \otimes_k A^{\otimes_k n} = M \otimes A^{\otimes n} \quad (\otimes_k = \otimes) \quad (5)$$

$$\partial_0(m, a_1, \dots, a_n) = (ma_1, a_2, \dots, a_n)$$

$$\partial_i(m, a_1, \dots, a_n) = (m, a_1, \dots, a_i a_{i+1}, \dots, a_n) \quad i=1, \dots, n-1$$

$$\partial_n(m, a_1, \dots, a_n) = (a_n m, a_1, \dots, a_{n-1})$$

$$s_j(m, a_1, \dots, a_n) = (m, a_1, \dots, a_j, 1, a_{j+1}, \dots, a_n) \quad j=0, \dots, n$$

Exercise. Check that it is a simplicial module.

$(\{C_n\}, \partial_i)$

Definition Let  $(\{C_n\}, \partial_i, s_j)$  be a

(pre) simplicial module. Boundary map

$$\partial : C_n \rightarrow C_{n-1}$$

$$\partial := \sum_{i=0}^n (-1)^i \partial_i$$

Exercise.  $\partial^2 = 0$ .

Name "boundary" comes from Algebraic Topology

For a simplicial module as above

special notation  $\partial = \partial$ .

Corollary  $((C_n), \partial)$  for a simplicial module is a complex.

It leads to homology:

$$H_n(C_\bullet) = \frac{\ker(\partial: C_n \rightarrow C_{n-1})}{\text{im}(\partial: C_{n+1} \rightarrow C_n)}$$

Definition  $H_n(C_\bullet(A, M), b) = H_n(A, M)$

$n$ -th Hochschild homology of  $A$  with coefficients in  $M$ .

$$HH_n(A) := H_n(A, A)$$

Properties 1) Base ring plays a role:

$$k = \mathbb{C} \Rightarrow HH_n(\mathbb{C}) = 0$$

$$k = \mathbb{Q} \Rightarrow HH_n(\mathbb{C}) \neq 0$$

2. Functoriality:

$$B \rightarrow A \rightsquigarrow H_n(B, M) \rightarrow H_n(A, M)$$

$M$ - $A$ -bimodule

$$M \rightarrow M' \rightsquigarrow H_n(A, M) \rightarrow H_n(A, M')$$

$A$ -bimodules

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3. HH<sub>n</sub> respects products

$$HH_n(A \times B) \cong HH_n(A) \times HH_n(B)$$

(special case of the Künneth formula for Tor's)

4. H<sub>n</sub>(A, M) is a Z(A)-module

Low dimensional homology:

$$H_0(A, M) = M / \text{im } b = M / (am - ma) = M / [A, M]$$

In particular:  $HH_0(A) = A / [A, A]$

Exercise,  $HH_0(k) = k$

$$HH_n(k) = 0, \quad n > 0.$$

Interesting property of  $A / [A, A]$

$$\text{Tr} : M_r(A) \longrightarrow A / [A, A] \quad (a_{ij}) \longmapsto \sum_i a_{ii} + [A, A]$$

$$\ker \text{Tr} = [M_r(A), M_r(A)] \quad (\text{use elementary matrices})$$

$$\Rightarrow M_r(A) / [M_r(A), M_r(A)] \xrightarrow[\text{Tr}]{\cong} A / [A, A].$$

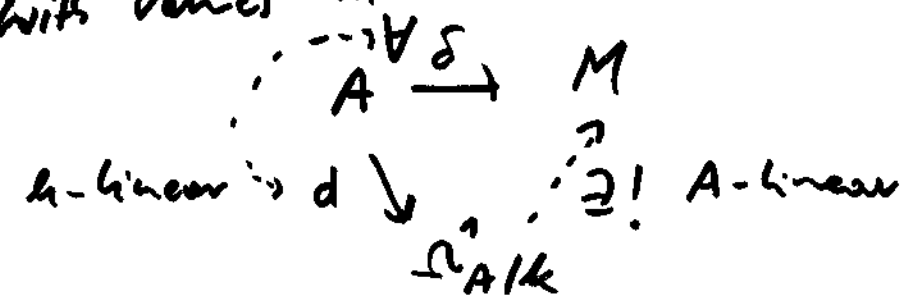
(This is a special case of)   
 Morita invariance.

Proposition If  $A$  is commutative,  $M$  -  $A$ -module   
 (symm. bimodule)   
 $a \cdot m = am$    
 $m \cdot a = am$    
 $a, a' \in A, c \in k$

$$H_1(A, M) = M \otimes_A \Omega_{A/k}^1$$

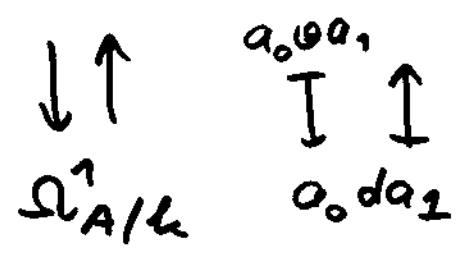
where  $\Omega_{A/k}^1 = A \cdot (da) / (d(aa') - ada' - a'da), d(c)$

$A$ -module of universal  $k$ -linear derivations of  $A$    
 with values in  $A$ -modules



Proof.  $(M=A)$   $C_1(A, A) = A \otimes A \xrightarrow{b} K_0(A, A) = A$    
 $a_0 \otimes a_1 \mapsto a_0 a_1 - a_1 a_0 = 0$

$$\Rightarrow HH_1(A) = A \otimes A / (a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1)$$



Exercise They are well defined, and inverse one to each other.

Exercise Prove it for arbitrary  $A$ -module  $M$ .

More advanced understanding of  $H_n(A, M)$  needs homological algebra

It is a derived functor of  $M \mapsto M/[A, M]$

Def. (Bar complex)

$$B_n(A) = A \otimes A^{\otimes n} \otimes A = A^{\otimes n+2}$$

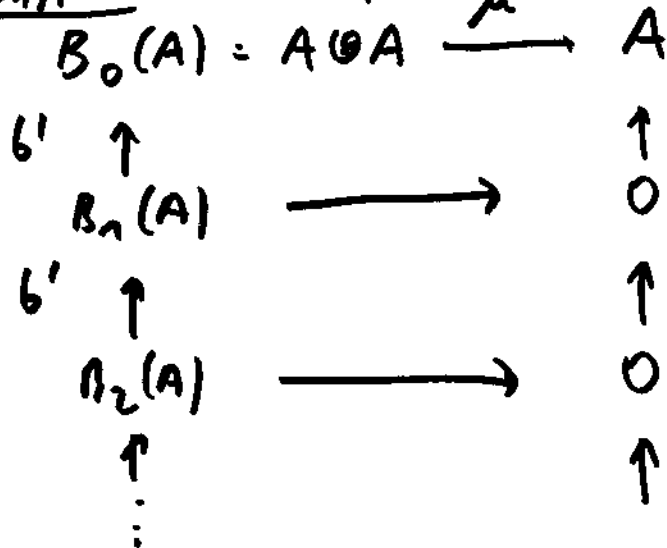
A-bimodule

$$a(a_0 \otimes \dots \otimes a_{n+1}) = aa_0 \otimes \dots \otimes a_{n+1}$$

$$(a_0 \otimes \dots \otimes a_{n+1})a = a_0 \otimes \dots \otimes a_{n+1}a$$

$$b' := \sum_{i=0}^{n-1} (-1)^i \partial_i, \quad b'b' = 0$$

Proposition The map of complexes



is a homotopy chain equivalence. (Resolution)



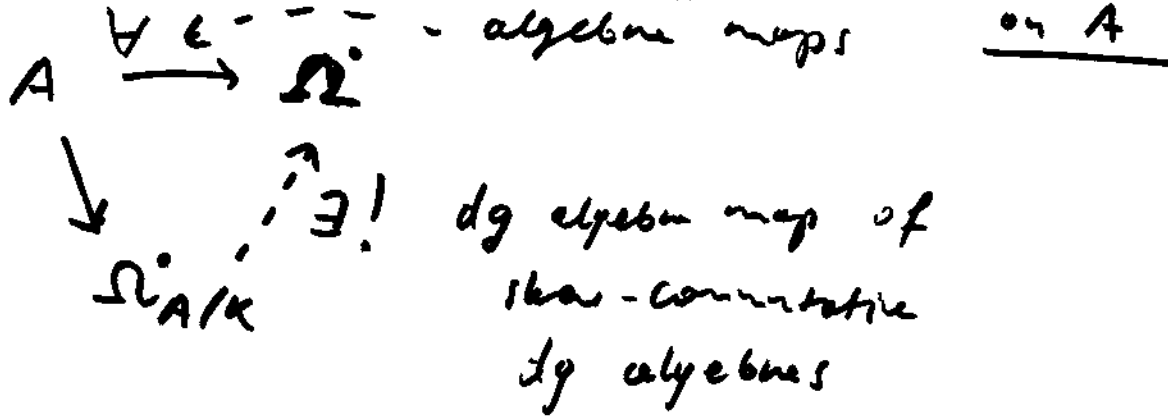
# Kähler differentials

$A$  a commutative  $k$ -algebra

$$\Omega_{A/k}^1 = \frac{(a, da)}{(d(aa') - a da' - a' da, d(x))} \quad \text{--- } k\text{-algebra}$$

→ make it a skew-commutative dg  $k$ -algebra

Universal skew-commutative differential  $k$ -algebra



universal algebra map

1) elements:  $a_0 da_1 \dots da_n$  span a  $k$ -module

2)  $d(a_0 da_1 \dots da_n) = da_0 \dots da_n$  well defined  $\Omega_{A/k}^n$ .

def.  $A$  is smooth over  $k$  if  $\Omega_{A/k}^1$  is

a direct summand in  $A$

free  $A$ -module of finite rank

and  $A$  is finitely generated as a  $k$ -algebra

and flat as a  $k$ -module.

Motivation:  $A = \mathbb{C}[x]$  - polynomial functions on a complex alg. variety  $X \subset \mathbb{C}^1 \Rightarrow A$ -smooth /  $\mathbb{C}$   $X$ -manifold.

$$H^n(\Omega_{A/k}^1, d) =: H_{DR}^n(A).$$

De Rham cohomology

Motivation:  $A = \mathbb{C}[x]$  - polynomial functions on a complex algebraic manifold  $X \subset \mathbb{C}^1 \Rightarrow H_{DR}^n(A) = H^n(X, \mathbb{C})$ .

Hochschild - Kostant - Rosenberg Theorem  
for Hochschild homology commutative

If  $A$  is smooth over  $k \supseteq \mathbb{Q}$  and  $M$  is an  $A$ -module, then

$$H_n(A, M) \xrightarrow{\cong} M \otimes_A \Omega_{A/k}^n .$$

$$m \otimes a_1 \otimes \dots \otimes a_n \mapsto m \otimes da_1 \dots da_n$$

$$\frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn} \sigma \cdot m \otimes \frac{a_{\sigma(1)}}{d\sigma(1)} \otimes \dots \otimes \frac{a_{\sigma(n)}}{d\sigma(n)} \longleftarrow m \otimes da_1 \dots da_n$$

well defined

and inverse one to each other.

# Relative Hochschild homology $HH_n(A, I)$ (13)

Def.  $I \triangleleft A$

$$HH_n(A, I) := H_n(\ker(C_*(A, A) \rightarrow C_*(A/I, A/I)))$$

Proposition Long exact sequence

$$\dots \rightarrow HH_n(A, I) \rightarrow HH_n(A) \rightarrow HH_n(A/I) \rightarrow HH_{n-1}(A, I) \rightarrow \dots$$

Morita invariance

$$M_r(M) := M_r(k) \otimes M$$

$$M_\infty(M) := \varinjlim_r M_r(M)$$

$$\text{Tr} : M_r(M) \otimes M_r(A)^{\otimes n} \rightarrow M \otimes A^{\otimes n}$$

$$(m_{ij}) \otimes (a_{1,ij}) \otimes \dots \otimes (a_{n,ij}) \mapsto$$

$$\sum_{1 \leq i_0, \dots, i_n \leq r} m_{i_0, i_1} \otimes a_{1, i_1, i_2} \otimes \dots \otimes a_{n-1, i_{n-1}, i_n} \otimes a_{n, i_n, i_0}$$

Dennis trace map

Proposition The Dennis trace map gives a morphism of complexes

$$C_n(M_r(A), M_r(M)) \rightarrow C_n(A, M)$$

Proof.  $Tr \circ \partial_i = \partial_i \circ Tr \quad \square$

Def<sup>n</sup>  $i : M \hookrightarrow M_r(M)$   
 $m \mapsto \begin{pmatrix} m & & 0 \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$ .

Theorem  $Tr \circ i = id$  on  $C_n(A, M)$   
 $i \circ Tr \sim id$  on  $C_n(M_r(A), M_r(M))$   
(chain homotopy equivalence)

Proof. (Loday)

Corollary  $Tr$  and  $i$  define isomorphisms inverse to each other

$$Tr_n : H_n(M_r(A), M_r(M)) \rightarrow H_n(A, M)$$
  
$$i_n : H_n(A, M) \rightarrow H_n(M_r(A), M_r(M)).$$

# Morita equivalence

$A, B$   $R$ -algebras

$P$  is an  $(B, A)$ -bimodule if

$$B \otimes P \xrightarrow{\lambda} P, \quad P \otimes A \xrightarrow{\rho} P$$

& commutative diagrams as above for  $A$ -bimodules.

Def.  $A \sim B$  (Morita equivalent)

if there exist  $P$  a  $(B, A)$ -bimodule and  $Q$  an  $(A, B)$ -bimodule s.t.

$$P \otimes_A Q \cong B \quad \text{as } B\text{-bimodules}$$

$$Q \otimes_B P \cong A \quad \text{as } A\text{-bimodules}$$

Example  $A = M_r(B) \sim B$

$$P = B^r \quad (\text{rows } (b_1, \dots, b_r))$$

$$Q = B^r \quad (\text{columns } \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix})$$

Exercise show that  $P \otimes_{M_r(B)} Q \cong B$

$$Q \otimes_B P \cong M_r(B).$$

Theorem  $A \sim B$  via  $P, Q$  as above

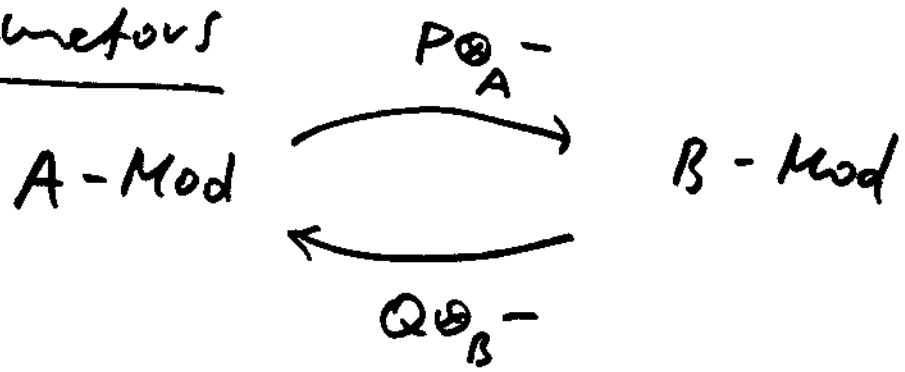
$N$   $R$ -bimodule

$$M := Q \otimes_B N \otimes_B P \quad A\text{-bimodule}$$

$$\Rightarrow H_n(A, M) \cong H_n(B, N).$$

Proof (Loday)

More-advanced understanding of Morita equivalence needs categories and functors



A pair

$(P, Q)$  defines equivalences inverse to each other.

Theorem (Morita) Every equivalence between  $A\text{-Mod}$  and  $B\text{-Mod}$  comes from some pair  $(P, Q)$ .

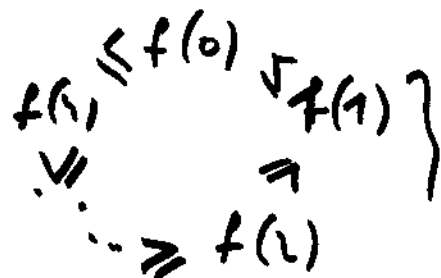
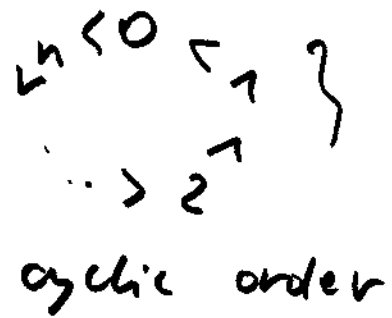
# Cyclic homology

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## Cyclic category $\Lambda$

Objects:  $[n] = \{0, 1, \dots, n\}$

$\text{Mor}_\Lambda([m], [n]) = \{f: [m] \rightarrow [n]\}$



Face and degeneracy maps as for  $\Delta$

## Def. (cyclic maps)

$$\tau_n: [n] \rightarrow [n] \quad \tau_n = \begin{cases} x-1 & x \geq 1 \\ n & x = 0 \end{cases}$$

(counter clock wise rotation)  
by one

## Proposition $\Lambda$ admits a presentation

by  $\delta_i$ 's,  $\sigma_j$ 's,  $\tau_n$ 's satisfying relations as for  $\Delta$  and  $\tau_n^{n+1} = \text{id}_{[n]}$ ,

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1}, \quad 1 \leq i \leq n$$

$$\tau_n \delta_0 = \delta_n,$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \quad 1 \leq i \leq n$$

$$\tau_n \sigma_0 = \sigma_n \tau_{n+1}^2.$$

Def. A cyclic module is a

functor  $\Lambda^{op} \rightarrow R\text{-Mod}$ .

$$\delta_i \mapsto \partial_i$$

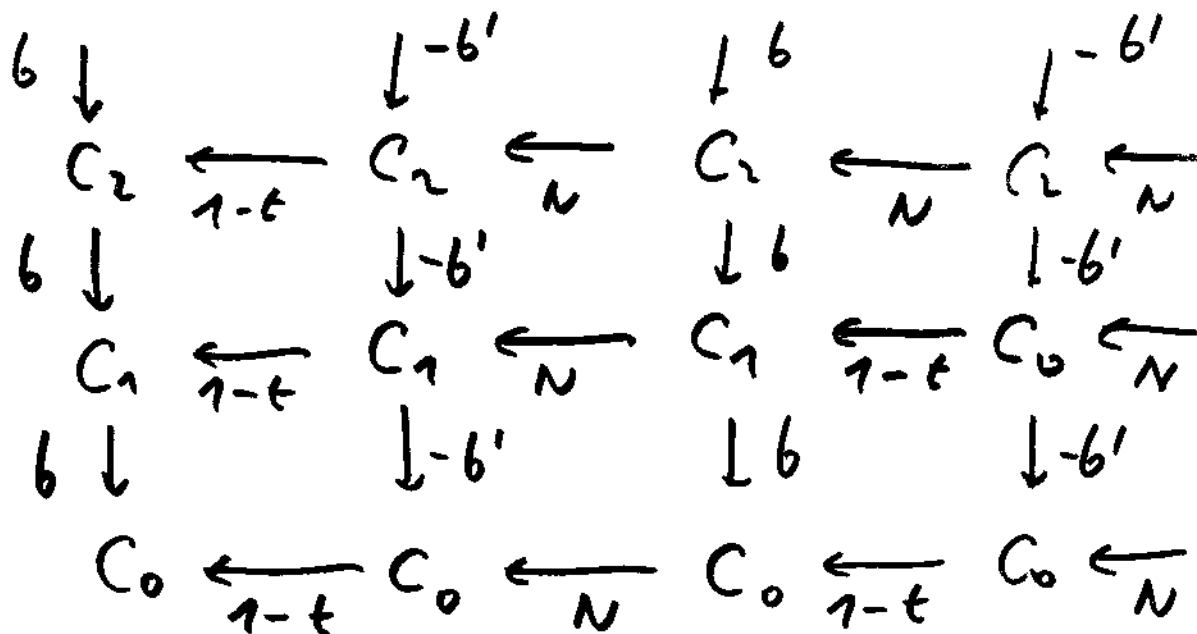
$$\sigma_j \mapsto s_j$$

$$\tau_n \mapsto t_n$$

and opposite order  
of composition

Def 1 Let  $((C_n), \partial_i, s_j, t_n)$  be a cyclic module. The associated cyclic bicomplex

$(CC_0, b, b', t, N)$  is



$$b = \sum_{i=0}^n (-1)^i \partial_i, \quad b' = \sum_{i=0}^{n-1} (-1)^i \partial_i, \quad N = \sum_{i=0}^n t_n^i$$

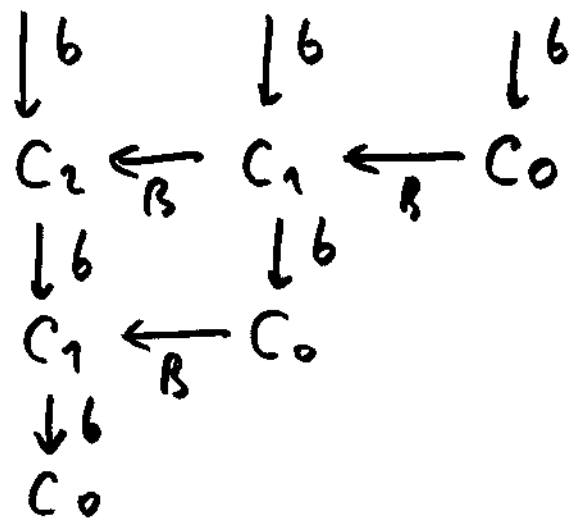
$$t = t_n.$$

Def. 2 (Connes) The complex  $(C^\lambda, b)$

$$C_n^\lambda = C_n / (1 - t_n)$$

$$\dots \xrightarrow{b} C_1^\lambda \xrightarrow{b} C_0^\lambda \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

Def. 3 Mixed complex BC



$$B := (1-t)S_N, \quad s := (-1)^n t_n S_n = C_n \rightarrow C_{n+1}$$

Theorem  $k \supset \mathbb{Q} \Rightarrow$

$$H_n(\text{Tot}(CC..)) \cong H_n(C^\lambda) \cong H_n(\text{Tot}(BC))$$

Common name:  $HC_n(C)$   
(homologie cyclique)

Proof (Loday)

Example.  $A$  a  $k$ -algebra

$$[n] = \begin{pmatrix} \mathbb{Z} & & & \\ & \mathbb{Z} & & \\ & & \ddots & \\ & & & \mathbb{Z} \end{pmatrix} \rightsquigarrow \left( \begin{array}{c} a_n \oplus a_0 \oplus a_n \\ \oplus \\ \dots \oplus a_n \end{array} \right)$$

$\partial_i$ 's,  $\delta_j$ 's as for  $C_\bullet(A, A)$

$$t_n \left( \begin{array}{c} a_n \oplus a_0 \oplus a_n \\ \oplus \\ \dots \oplus a_n \end{array} \right) = \begin{pmatrix} n \\ \vdots \\ 1 \end{pmatrix} \left( \begin{array}{c} a_n \oplus a_n \oplus a_0 \\ \oplus \\ \dots \oplus a_n \end{array} \right)$$

clockwise rotation

(remember: cyclic module if  $\alpha$ )  
contravariant functor

Exercise show that  $(C_\bullet(A, A), \partial_i, \delta_j, t_n)$   
is a cyclic module.

$$HC_n(C_\bullet(A, A)) = : HC_n(A)$$

cyclic homology of a  $k$ -algebra  $A$ .

Theorem (Connes) There is a long exact sequence (Connes' Periodicity)

$$\dots \rightarrow HC_{n-2}(A) \xrightarrow{B} HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} \dots$$

where  $S$  is induced by

$$C_n(A, A) \rightarrow C_{n-2}(A, A), \quad n \geq 2$$

$$x \mapsto \frac{-1}{n(n-1)} \sum_{0 \leq i < j \leq n} (-1)^{i+j} \partial_i \partial_j(x)$$

(Periodicity map)

$I$  is induced by the projection

$$C_n(A, A) \rightarrow C_n(A, A) / (1-t_n),$$

$B = (1-t)S$  can be computed as

$$C_n(A, A) \rightarrow C_{n+1}(A, A)$$

$$B(a_0, \dots, a_n) = \sum_{i=0}^{n-1} \left( (-1)^{ni} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1}) \right. \\ \left. - (-1)^{n(i-1)} (a_{i-1}, 1, a_i, \dots, a_n, a_0, \dots, a_{i-2}) \right)$$

Exercise  $HC_n(k) = \begin{cases} k & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$

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Exercise  $HC_0(A) \cong HH_0(A) \cong A/[A, A]$

Proposition  $A$ -commutative  $\Rightarrow$   
 $HC_1(A) \cong \Omega_{A/k}^1 / dA$ .

Kind of  
Hochschild-Koiter-Rosenberg Theorem  
for cyclic homology:

$A$  smooth over  $k$

$$HC_{2n}(A) \cong \Omega_{A/k}^{2n} / d\Omega_{A/k}^{2n-1} \oplus H_{DR}^{2n-2}(A) \oplus \dots \oplus H_{DR}^0(A)$$

$$HC_{2n+1}(A) \cong \Omega_{A/k}^{2n+1} / d\Omega_{A/k}^{2n} \oplus H_{DR}^{2n-1}(A) \oplus \dots \oplus H_{DR}^1(A).$$