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Convergence of star products: Examples & Concepts

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Introduction / Motivation

→ formal star products

Recall: M smooth manifold

A formal star product $\star$ on M is a $\mathbb{C}[[\lambda]]$-bilinear multiplication for $C^\infty(M)[[\lambda]]$

$$f \star g = \sum_{r=0}^{\infty} \lambda^r C_r(f,g)$$

such that

i.) associative

ii) $C_0(f,g) = fg$ pointwise product

iii) $f \star 1 = f = 1 \star f$

iv) C_r bidifferential operators

consequence: $C_1(f,g) - C_1(g,f) = i\{f,g\}$

is a Poisson bracket

Big result: for any Poisson bracket $\{\cdot,\cdot\}$
(Kontsevich) on M there exists a $\star$.

now: use $\star$ to model NCG

- $*$-involution $\overline{f \star g} = \overline{g} \star \overline{f}$
  where $\overline{\lambda} = \lambda$ (always $\exists$ such $\star$)

- deform calculus (not so easy)

- deform vector bundles $E \to M$
  by deforming the module $\Gamma^\infty(E)$
  into right module w.r.t $\star$

- deform PFB

$$G \circlearrowright P \xrightarrow{pr} M$$

$\mathcal{C}^\infty(P)$ $\mathcal{C}^\infty(M)$ classical right module structure

$\to$ deform into $\star$-right module

$$\bullet : \mathcal{C}^\infty(P)[[\lambda]] \times \mathcal{C}^\infty(M)[[\lambda]] \longrightarrow \mathcal{C}^\infty(P)[[\lambda]]$$

$$(F \bullet f) \bullet g = F \bullet (f \star g)$$

$$F \bullet f = F \, pr^*(f) + \cdots \text{ higher orders}$$

want $\bullet$ to be G-invariant

Result: $\exists \bullet$ unique up to equivalence
(Bordemann, Neumaier, Waldmann(?), W.)
Crelle 2010 (?)

also Thomas Weber, Chiara Esposito, René Bielawski(?)

& F. d'Andrea

at the end of the day: get some geometry
"NCG"

BUT $\to$ $\mathbb{C} \rightsquigarrow \mathbb{C}[[\lambda]]$

λ is $\hbar$ or $\ell_p, \ldots$

$\to$ $(\mathcal{C}^\infty(M)[[\lambda]], \star)$ algebras over the wrong scalars!

<u>Idea:</u> want λ to become number in $\mathbb{R}$

$\Longrightarrow$ need to talk about convergence!

known approaches:

- using integral formulas for $\star$ based on ΨDO calculus

  formal $\star$ is asymptotic expansions for $\hbar \to 0$ of these integral formulas

  $\Rightarrow$ can obtain nice top. algebras $C^*, \ldots$

  <u>But:</u> need finite dimensions for $\int$!

- generators & relations (q-deformation) some new approaches?

- still botanics ....

Roughly the hope is: start with some M, $\star$

① determine a small class of functions on M for which $\star$ „trivially" converges

e.g. a) Weyl product on $\mathbb{R}^{2n}$

$$f \star_{Weyl} g = \mu \circ e^{\frac{i\lambda}{2}\left(\frac{\partial}{\partial q^k} \otimes \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q^k}\right)} (f \otimes g)$$

for f, g polynomials $\to$ exp terminates

$\lambda \rightsquigarrow \hbar$

b) $\mathfrak{g}$ Lie algebra, $S^\bullet(\mathfrak{g})$ (morally $\mathrm{Pol}(\mathfrak{g}^*)$) with Gutt star product $\star_G$

$S^\bullet(\mathfrak{g}) \underset{PBW}{\simeq} U(\mathfrak{g})$ pull back the product

with $\hbar$ at the right places...

② Find lc topology on these functions such that $\star$ becomes continuous.

Carefull: CCR do not allow for submultiplicative seminorms best you can hope for: countable system of seminorms

Weyl, Gutt, ... on V vector space and are products $\star$ on $S(V)$ $(\cong \mathrm{Pol}(V^*))$

$\rightsquigarrow$ suppose V is lc and $\{\cdot,\cdot\}$ is continuous
or Lie alg structure

S_R - topology: $R \in \mathbb{R}$

$$v = \sum_{n=0}^{\infty} v_n \qquad v_n \in S^n(V)$$

p is a cont seminorm on V

define

$$p_R(v) = \sum_{n=0}^{\infty} n!^R \, (p^{\otimes n})(v_n)$$

all those p_R's define S_R-topology on $S(V)$

Then $\star_{\text{Weyl}}$, $\star_G$ are continuous for $R \geq 1/2$, $R \geq 1$

completions depend on R

(3) Take completions to enlarge the space of functions

$\rightsquigarrow$ get complete lc algebra $\mathcal{A}$

want topology to be coarse as possible so that completion is as large as possible

(4) investigate $\mathcal{A}$:

- are evaluations at $p \in M$ continuous?
  ($\hat{=}$ elements of $\mathcal{A}$ are functions)

- study representations, ...