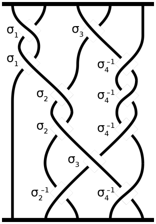
A New Look at Symmetries in Noncommutative Field Theory

Richard Szabo

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES



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Outline

- Introduction/Motivation
- Gauge symmetry and L_∞-algebras
- Braided gauge symmetry and braided L_∞-algebras
- Example: Braided noncommutative gravity

with M. Dimitrijević Ćirić, G. Giotopoulos & V. Radovanović

[arXiv:2103.08939]

Introduction

- In this talk, by a **noncommutative field theory** I will mean a field theory that is a deformation of a classical field theory via a star-product on the algebras of functions, differential forms, . . .
- Of particular interest (e.g. in string theory) are noncommutative gauge theories — after over 20 years of intensive work, there are still many open general problems in the construction of these theories

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- Problems with star-gauge transformations:

$$\delta^\star_\lambda A \;=\; \mathrm{d}\lambda + [\lambda \stackrel{\star}{,} A] \;=\; \mathrm{d}\lambda + \lambda \star A - A \star \lambda$$

  In general, closure of gauge algebra is obstructed:

$$(\delta^\star_{\lambda_1}\,\delta^\star_{\lambda_2} - \delta^\star_{\lambda_2}\,\delta^\star_{\lambda_1})A \;\neq\; \delta^\star_{[\lambda_1 \stackrel{\star}{,} \lambda_2]} A$$

- Failure of Leibniz rule: $\mathrm{d}(f \star g) \;\neq\; \mathrm{d}f \star g + f \star \mathrm{d}g$

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- **L_∞-algebras** offer a natural arena for systematic constructions of noncommutative gauge theories that deal with these issues — so far not understood beyond "semi-classical (Poisson) level"
  (Blumenhagen, Brunner, Kupriyanov & Lüst '18; Kupriyanov & Sz '21)

L_∞-Algebras in Physics & Mathematics

- Higher spin gauge theories with field-dependent gauge parameters:
(Berends, Burgers & van Dam '85)

$$(\delta_\alpha\,\delta_\beta - \delta_\beta\,\delta_\alpha)\Phi \;=\; \delta_{C(\alpha,\beta,\Phi)}\Phi$$

- "Generalized" gauge symmetries of closed string field theory involve higher brackets: (Zwiebach '92)

$$\delta_\alpha\Phi \;=\; \sum_n \ell_n(\alpha,\Phi^{n-1})$$

- Dual to differential graded (commutative) algebras (Lada & Stasheff '92)
- Deformation theory: Kontsevich's Formality Theorem based on L_∞-quasi-isomorphims of differential graded Lie algebras
- Any classical field theory with "generalized" gauge symmetries is determined by an L_∞-algebra, due to duality with BV–BRST
(Hohm & Zwiebach '17; Jurčo, Raspollini, Sämann & Wolf '18)

Goals & Disclaimers

- Twisted diffeomorphism symmetry does not fit (nicely) into L_∞-algebra picture $\Longrightarrow$ deform L_∞-algebra to make it compatible

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- **Disclaimers:**
  - I do not claim notion of 'braided gauge symmetry' is new — kinematical aspects of this idea have appeared before (Brzezinkski & Majid '92; . . .) — ideas and techniques borrowed from twisted noncommutative gravity
  - I do not know anything yet about corresponding QFTs — they should be related to Oeckl's 'braided QFT' (Oeckl '99; Sasai & Sasakura '07)
  - I'll only discuss diffeomorphism-invariant field theories here for simplicity — Yang-Mills theory, scalar field theories, . . . also fit
  - Physical realizations? To be looked into . . .

What is a Gauge Symmetry?

- Consider the example of Chern-Simons theory on a 3D manifold M: Let $\mathfrak{g}$ be a quadratic Lie algebra with pairing $\mathrm{Tr}_{\mathfrak{g}}$, then the Chern-Simons action for a gauge field $A \in \Omega^1(M, \mathfrak{g})$ is

$$S = \int_M \mathrm{Tr}_{\mathfrak{g}}\Big(\frac{1}{2}\, A \wedge \mathrm{d}A + \frac{1}{3!}\, A \wedge [A, A]_{\mathfrak{g}}\Big)$$

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- The Euler-Lagrange equations $\delta S = 0$ (for arbitrary variations δA) are $F_A = 0$, where $F_A = \mathrm{d}A + \frac{1}{2}\, [A, A]_{\mathfrak{g}} \in \Omega^2(M, \mathfrak{g})$ is the curvature of the connection A, which is covariant: $\delta_\lambda F_A = [\lambda, F_A]_{\mathfrak{g}}$

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- Space of physical states: Moduli space of classical solutions (flat connections) modulo gauge transformations

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- An equivalent perspective on gauge redundancies: gauge transformations $\delta_\lambda A$ are special cases of general field variations δA:

$$\delta_\lambda S \ = \ \int_M \mathrm{Tr}_{\mathfrak{g}}\big(\delta_\lambda A \wedge F_A\big) \ = \ -\int_M \mathrm{Tr}_{\mathfrak{g}}\big(\lambda \ \mathrm{d}^A F_A\big)$$

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- Noether identities exhibit interdependence of degrees of freedom due to gauge symmetries

What is a Gauge Symmetry?

- To describe the classical moduli space of Chern-Simons theory, we relied on 3 ingredients:
  - The graded vector space $V = \Omega^\bullet(M, \mathfrak{g}) = V_0 \oplus V_1 \oplus V_2 \oplus V_3$, where $V_p = \Omega^p(M, \mathfrak{g})$ ($p = 0$ are gauge parameters, $p = 1$ are fields, $p = 2$ are field equations, $p = 3$ are Noether identities)

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  - The 'brackets' $\ell_1 = \mathrm{d}$, $\ell_2 = [-,-]_{\mathfrak{g}}$ on V; ℓ_1 makes V into a cochain complex, which is a derivation of ℓ_2, while ℓ_2 is a graded Lie bracket on V (antisymmetric and satisfies Jacobi identity)

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- Chern-Simons gauge theory is organised by a **(cyclic) differential graded Lie algebra**
- This is the prototypical example of a more general statement: Any classical field theory with "generalized" gauge symmetries is organised by a (cyclic) L_∞-algebra

What is an L_∞-Algebra?

- Graded vector space: $V = \cdots \oplus V_{-1} \oplus V_0 \oplus V_1 \oplus \cdots$, with graded exterior algebra $\Lambda_V = \wedge^\bullet(V[1])$ viewed as a free cocommutative coalgebra
- $L : \Lambda_V \longrightarrow \Lambda_V$ coderivation of degree $|L| = 1$, with $L^2 = 0$

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- Write $L^2 = 0$ in 'components' $L = \{\ell_n\}$ where $\ell_n : \wedge^n(V[1]) \longrightarrow V[1]$ with $|\ell_n| = 1$, or restoring original grading $\ell_n : \wedge^n V \longrightarrow V$ with $|\ell_n| = 2 - n$:

$$\ell_1(\ell_1(v)) = 0 \qquad (V, \ell_1) \text{ is a cochain complex}$$

$$\ell_1(\ell_2(v,w)) = \ell_2(\ell_1(v), w) \pm \ell_2(v, \ell_1(w)) \qquad \ell_1 \text{ is a derivation of } \ell_2$$

$$\ell_2(v, \ell_2(w,u)) + \text{cyclic} = (\ell_1 \circ \ell_3 \pm \ell_3 \circ \ell_1)(v,w,u) \qquad \text{Jacobi up to homotopy}$$

plus "higher homotopy Jacobi identities"

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- L_∞-algebras are generalizations of differential graded Lie algebras

Cyclic L_∞-Algebras

- Cyclic pairing $\langle -, - \rangle : V \times V \longrightarrow \mathbb{R}$ is non-degenerate, graded symmetric, bilinear and satisfies cyclicity:

$$\langle v_0, \ell_n(v_1, v_2, \dots, v_n) \rangle \;=\; \pm \langle v_n, \ell_n(v_0, v_1, \dots, v_{n-1}) \rangle$$

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- Cyclic pairing is dually a graded symplectic 2-form $\omega \in \Omega^2(V[1])$ which is Q-invariant

L_∞-Algebras of Classical Field Theories

- BV formalism constructs a dg algebra $(C^\infty_\bullet(V[1]), Q_{\rm BV})$ on graded vector space V of BV fields (ghosts, fields and antifields)
- Translate coordinate functions ξ to elements of vector spaces, then action of $Q_{\rm BV}$ is a polynomial in ghosts, fields and antifields and their derivatives, dual to sum over all brackets ℓ_n on V:

$$Q_{\rm BV}\xi \ = \ \ell_1(\xi) + \tfrac{1}{2}\,\ell_2(\xi,\xi) + \cdots$$

- BV symplectic form (inducing antibracket) of degree -1 on V induces cyclic pairing of degree -3

$\cdots$	V_0	V_1	V_2	V_3	$\cdots$
$\cdots$	gauge par.	fields	field eqs.	Noether ids.	$\cdots$

- V_{-k} encode 'higher gauge transformations' (ghosts-for-ghosts, etc.) for reducible symmetries

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- Noether ids: $\mathcal{I}_A F_A \;=\; \ell_1(F_A) + \ell_2(F_A, A) + \cdots \;\equiv\; 0$ (*off-shell*)

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$$C(\lambda_1, \lambda_2; A) = \ell_2(\lambda_1, \lambda_2) + \ell_3(\lambda_1, \lambda_2, A) + \cdots$$

- Field equations: $F_A = \ell_1(A) - \frac{1}{2}\,\ell_2(A, A) + \cdots$

- Gauge covariance: $\delta_\lambda F_A = \ell_2(\lambda, F_A) + \ell_3(\lambda, F_A, A) + \cdots$

- Noether ids: $\mathcal{I}_A F_A = \ell_1(F_A) + \ell_2(F_A, A) + \cdots \equiv 0$ (*off-shell*)

- Action: $S = \frac{1}{2}\,\langle A, \ell_1(A)\rangle - \frac{1}{3!}\,\langle A, \ell_2(A, A)\rangle + \cdots$

$$\delta S = \langle \delta A, F_A\rangle \quad , \quad \delta_\lambda S = \langle \delta_\lambda A, F_A\rangle = -\langle \lambda, \mathcal{I}_A F_A\rangle$$

L_∞-Algebras of Classical Field Theories

- Gauge transformations of fields $A \in V_1$ by $\lambda \in V_0$:

$$\delta_\lambda A \;=\; \ell_1(\lambda) + \ell_2(\lambda, A) + \cdots$$

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- Moduli space $=$ field equations / gauge transformations

Conventional Star-Gauge Symmetry

- Consider noncomm. field theory defined with the Moyal-Weyl star-product, for a constant Poisson bivector θ on $M = \mathbb{R}^d$:

$$f \star g \ = \ \sum_{n=0}^{\infty} \left(\frac{\mathrm{i}}{2}\right)^n \frac{1}{n!} \, \theta^{\mu_1\nu_1} \cdots \theta^{\mu_n\nu_n} \, \partial_{\mu_1} \cdots \partial_{\mu_n} f \, \partial_{\nu_1} \cdots \partial_{\nu_n} g$$

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- **Problem:** These gauge variations do not close on $\mathfrak{g}$: $[\delta^\star_{\lambda_1}, \delta^\star_{\lambda_2}] \ = \ \delta^\star_{[\lambda_1 \stackrel{\star}{,} \lambda_2]_{\mathfrak{g}}}$, but star-commutator does not close:

$$[\lambda_1 \stackrel{\star}{,} \lambda_2]_{\mathfrak{g}} \ := \ \lambda_1 \star \lambda_2 - \lambda_2 \star \lambda_1 \ \notin \ \Omega^0(M, \mathfrak{g})$$

(Exception: $\mathfrak{g} = \mathfrak{u}(N)$ in fundamental representation)

Closing Star-Gauge Transformations

- Enveloping alg-valued gauge symm: Closure takes place in universal enveloping algebra $U\mathfrak{g}$, so extend $\lambda \in \Omega^0(M, U\mathfrak{g})$, $A \in \Omega^1(M, U\mathfrak{g})$

  (Jurčo, Schraml, Schupp & Wess '00; Aschieri & Castellani '09)

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- Seiberg-Witten map: Noncommutative gauge orbits induced by classical gauge orbits: $\hat{A}(A + \delta_\lambda A) = \hat{A}(A) + \delta^\star_{\hat{\lambda}(\lambda,A)} \hat{A}(A)$; no new degrees of freedom, new interactions appear

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- Gravity? If ξ_1, ξ_2 are vector fields, then $[\xi_1 \overset{\star}{,} \xi_2]$ is not a vector field

  No analog of Seiberg-Witten map for deformed diffeomorphisms, naturally defined using Drinfel'd twist techniques (Aschieri *et al.* '05)

Drinfel'd Twist Deformation Quantization

- Let $\mathcal{F} = \mathrm{f}^\alpha \otimes \mathrm{f}_\alpha \in U\Gamma(TM) \otimes U\Gamma(TM)$ be a Drinfel'd twist;
  e.g. Moyal-Weyl twist $\mathcal{F} = \exp\big(-\frac{\mathrm{i}}{2}\,\theta^{\mu\nu}\,\partial_\mu \otimes \partial_\nu\big)$

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$$a \star b \;=\; \cdot\,\mathcal{F}^{-1}(a \otimes b) \;=\; \bar{\mathrm{f}}^\alpha(a) \cdot \bar{\mathrm{f}}_\alpha(b)$$

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- If $\mathcal{A}$ is commutative, then $\mathcal{A}_\star$ is braided-commutative:

$$a \star b \ = \ \bar{\mathrm{R}}^\alpha(b) \star \bar{\mathrm{R}}_\alpha(a)$$

$\mathcal{R} \ = \ \mathcal{F}^{-2} \ = \ \mathrm{R}^\alpha \otimes \mathrm{R}_\alpha \ = \$ triangular $\mathcal{R}$-matrix

Braided Gauge Symmetry

- Braided Lie algebra $\Omega^0_\star(M,\mathfrak{g})$: $[\lambda_1,\lambda_2]^\star_{\mathfrak{g}} \ := \ [-,-]_{\mathfrak{g}} \circ \mathcal{F}^{-1}(\lambda_1 \otimes \lambda_2)$

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$$[\lambda_1,\lambda_2]^\star_{\mathfrak{g}} = -[\bar{\mathrm{R}}^\alpha \lambda_2, \bar{\mathrm{R}}_\alpha \lambda_1]_\star$$

$$[\lambda_1,[\lambda_2,\lambda_3]^\star_{\mathfrak{g}}]^\star_{\mathfrak{g}} = [[\lambda_1,\lambda_2]^\star_{\mathfrak{g}},\lambda_3]^\star_{\mathfrak{g}} + [\bar{\mathrm{R}}^\alpha(\lambda_2),[\bar{\mathrm{R}}_\alpha(\lambda_1),\lambda_3]^\star_{\mathfrak{g}}]^\star_{\mathfrak{g}}$$

(Woronowicz '89; Majid '93; ...)

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- For matrix $\mathfrak{g}$: $[\lambda_1,\lambda_2]^\star_{\mathfrak{g}} \ = \ \lambda_1 \star \lambda_2 - \bar{\mathrm{R}}^\alpha(\lambda_2) \star \bar{\mathrm{R}}_\alpha(\lambda_1) \ \neq \ [\lambda_1 \overset{\star}{,} \lambda_2]_{\mathfrak{g}}$

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- Braided gauge fields, matter fields $A \in \Omega^1_\star(M,\mathfrak{g})$, $\phi \in \Omega^p_\star(M,W)$ transform in left/right braided representations:

$$\delta^{\star\mathrm{L}}_\lambda \phi = -\lambda \star \phi \quad , \quad \delta^{\star\mathrm{L}}_\lambda A = \mathrm{d}\lambda - [\lambda, A]^\star_{\mathfrak{g}}$$

$$\delta^{\star\mathrm{R}}_\lambda \phi = -\bar{\mathrm{R}}^\alpha(\lambda) \star \bar{\mathrm{R}}_\alpha(\phi) \quad , \quad \delta^{\star\mathrm{R}}_\lambda A = \mathrm{d}\lambda + [A, \lambda]^\star_{\mathfrak{g}}$$

Star-gauge transformations don't see left/right distinction
— we'll only consider left ones from now on

Braided Gauge Symmetry

- Braided gauge transformations satisfy braided Leibniz rule:

$$\delta^\star_\lambda(\phi \otimes A) \;=\; \delta^\star_\lambda \phi \otimes A + \bar{\mathrm{R}}^\alpha \phi \otimes \delta^\star_{\bar{\mathrm{R}}_\alpha \lambda} A$$

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- They close a braided Lie algebra:

$$\left[\delta^\star_{\lambda_1}, \delta^\star_{\lambda_2}\right]^\star \;:=\; \delta^\star_{\lambda_1} \circ \delta^\star_{\lambda_2} - \delta^\star_{\bar{\mathrm{R}}^\alpha \lambda_2} \circ \delta^\star_{\bar{\mathrm{R}}_\alpha \lambda_1} \;=\; \delta^\star_{[\lambda_1, \lambda_2]^\star_{\mathfrak{g}}}$$

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- Braided left/right covariant derivatives:

$$\mathrm{d}^A_{\star\mathrm{L}}\phi \;:=\; \mathrm{d}\phi + A \wedge_\star \phi \qquad , \qquad \mathrm{d}^A_{\star\mathrm{R}}\phi \;:=\; \mathrm{d}\phi + \bar{\mathrm{R}}^\alpha(A) \wedge_\star \bar{\mathrm{R}}_\alpha(\phi)$$

Braided covariance: $\delta^\star_\lambda\big(\mathrm{d}^A_{\star\mathrm{L,R}}\phi\big) \;=\; -\lambda \star \big(\mathrm{d}^A_{\star\mathrm{L,R}}\phi\big)$

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Braided covariance: $\delta^\star_\lambda\big(\mathrm{d}^A_{\star\mathrm{L,R}}\phi\big) \;=\; -\lambda \star \big(\mathrm{d}^A_{\star\mathrm{L,R}}\phi\big)$

- Braided curvature:

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- Braided diffeomorphisms $\Gamma_\star(TM)$:

$$\mathcal{L}^\star_\xi T \ := \ \mathcal{L}_{\bar{\mathrm{f}}^\alpha \xi}(\bar{\mathrm{f}}_\alpha T) \quad , \quad \left[\mathcal{L}^\star_{\xi_1}, \mathcal{L}^\star_{\xi_2}\right]^\star \ = \ \mathcal{L}^\star_{[\xi_1, \xi_2]^\star}$$

Braided Chern-Simons Theory

$$S^\star = \int_M \mathrm{Tr}_{\mathfrak{g}}\Big(\frac{1}{2}\,A\wedge_\star \mathrm{d}A + \frac{1}{3!}\,A\wedge_\star [A,A]^\star_{\mathfrak{g}}\Big)$$

- Invariant under braided gauge transformations from $\Omega^0_\star(M,\mathfrak{g})$
  No extra degrees of freedom introduced

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  There is no "moduli space" of classical solutions

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- Bianchi identities are modified:
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- Braided Noether identity *off-shell*: justifies interpretation of local braided symmetries as "gauge"

Braided L_∞-Algebras

- If $(V, \{\ell_n\})$ is a classical L_∞-algebra in the category of $U\Gamma(TM)$-modules, then $(V, \{\ell_n^\star\})$ is a braided L_∞-algebra in the category of $U_{\mathcal{F}}\Gamma(TM)$-modules, where

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- Braided graded antisymmetry:

$$\ell_n^\star(\dots, v, v', \dots) \ = \ -(-1)^{|v|\,|v'|} \ \ell_n^\star(\dots, \bar{\mathrm{R}}^\alpha(v'), \bar{\mathrm{R}}_\alpha(v), \dots)$$

$+$ braided homotopy Jacobi identities (unchanged for $n = 1, 2$)

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$$\ell_n^\star(v_1 \wedge \cdots \wedge v_n) \ := \ \ell_n(v_1 \wedge_\star \cdots \wedge_\star v_n)$$

- Braided graded antisymmetry:

$$\ell_n^\star(\dots, v, v', \dots) \ = \ -(-1)^{|v|\,|v'|} \ \ell_n^\star(\dots, \bar{\mathrm{R}}^\alpha(v'), \bar{\mathrm{R}}_\alpha(v), \dots)$$

$+$ braided homotopy Jacobi identities (unchanged for $n = 1, 2$)

- Cyclic pairing: $\langle -, - \rangle_\star \ := \ \langle -, - \rangle \circ \mathcal{F}^{-1}$

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- **Example:** Braided Chern-Simons theory built on dg braided Lie algebra with $V_p \ = \ \Omega^p(M,\mathfrak{g})$ and

$$\ell_1^\star \ = \ \ell_1 \ = \ \mathrm{d} \quad , \quad \ell_2^\star \ = \ [-,-]_{\mathfrak{g}}^\star$$

Braided L_∞-Algebras of Braided Field Theories

- Braided gauge transformations $\delta^\star_\lambda A \ = \ \ell^\star_1(\lambda) + \ell^\star_2(\lambda, A) + \cdots$ close a braided Lie algebra under braided commutator $[-,-]^\star$

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- No moduli space of solutions to $F^\star_A \ = \ 0$, but braided Noether ids from weighted sum over all braided homotopy relations on (A^n):

$$\begin{aligned} \mathcal{I}^\star_A F^\star_A \ &= \ \ell^\star_1(F^\star_A) + \tfrac{1}{2}\left(\ell^\star_2(F^\star_A, A) - \ell^\star_2(A, F^\star_A)\right) \\ &\quad + \tfrac{1}{3!}\,\ell^\star_1\left(\ell^\star_3(A^3)\right) + \tfrac{1}{4}\left(\ell^\star_2(\ell^\star_2(A^2), A) - \ell^\star_2(A, \ell^\star_2(A^2))\right) + \cdots \ \equiv \ 0 \end{aligned}$$

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- Action: $S \;=\; \frac{1}{2}\,\langle A, \ell^\star_1(A)\rangle_\star - \frac{1}{3!}\,\langle A, \ell^\star_2(A,A)\rangle_\star + \cdots$

$$\delta S \;=\; \langle \delta A, F^\star_A\rangle_\star \quad , \quad \delta^\star_\lambda S \;=\; -\langle \lambda, \mathcal{I}^\star_A F^\star_A\rangle_\star \;\neq\; \langle \delta^\star_\lambda A, F^\star_A\rangle_\star$$

Braided gauge variations not special directions of general field variations

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- Systematic constructions of **new** noncomm. field theories with no
  new degrees of freedom, good classical limit, and some “surprises”

Einstein-Cartan-Palatini Gravity (4d)

$$S \ = \ \int_M \ \mathrm{Tr}\Big(\frac{1}{2}\, e \wedge e \wedge R + \frac{\Lambda}{4}\, e \wedge e \wedge e \wedge e\Big)$$

- Fields: $e \ \in \ \Omega^1(M, \mathbb{R}^{1,3}) \ \ , \ \ \omega \ \in \ \Omega^1(M, \mathfrak{so}(1,3))$

  $R \ = \ \mathrm{d}\omega + \frac{1}{2}\,[\omega, \omega] \ \in \ \Omega^2(M, \mathfrak{so}(1,3)) \ , \quad \mathrm{Tr} : \wedge^4(\mathbb{R}^{1,3}) \longrightarrow \mathbb{R}$

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  For e non-degenerate, equivalent to torsion-free + Einstein equations

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- L_∞-algebra is not a dg Lie algebra ($\ell_3 \neq 0$)

(Dimitrijević Ćirić, Giotopoulos, Radovanović & Sz '20)

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- Invariant under braided semi-direct product:

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Gauge invariant with good classical limit

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- Noether ids: complicated . . . — New deformation of general relativity