Noncommutative renormalization Hopf algebras

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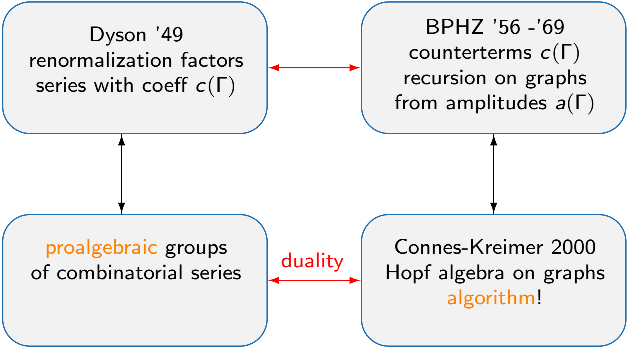
Motivation

Physics
renormalization
in pQFT

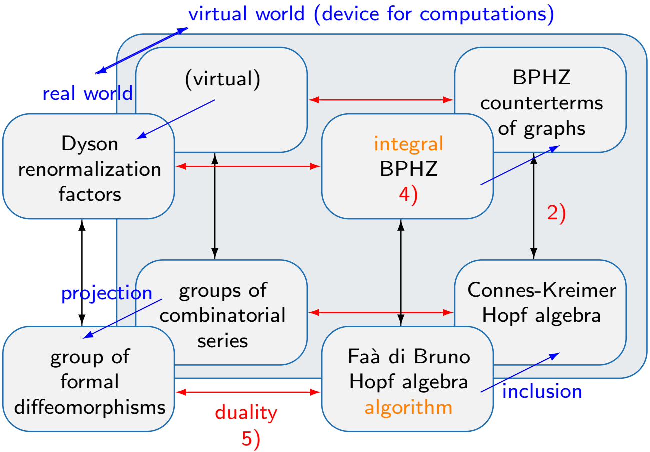
Maths
algebraic groups



Motivation and plan

Physics
1) renormalization in pQFT

Maths
3) proalgebraic groups



Aim:

4) Closed BPHZ formula for integral counterterms induced by Faà di Bruno.

5) Math duality holds iff coeff, amplitudes and Hopf algebras are commutative, but in QED and QCD amplitudes are matrices.
Extend duality to non-commutative algebras (via loops = non-assoc. groups)

1) Renormalization in QFT: QFT needs an algebra (and series)

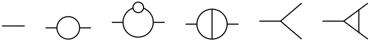
- **Lagrangian**: $\mathcal{L}(\phi; m, \lambda) = \mathcal{L}_0(\phi; m) + \lambda\, \mathcal{L}_{int}(\phi; m)$ — m mass; λ coupling (charge, flavour ...)

scalar ϕ^3: $\mathcal{L}(\phi) = \mathcal{L}_{Klein-Gordon}(\phi) - \frac{1}{3!}\lambda\, \phi^3$

QED: $\mathcal{L}(\psi, A^\mu) = \mathcal{L}_{Dirac}(\psi) + \mathcal{L}_{Maxwell}(A^\mu) - e\, \bar{\psi}\gamma^\mu A_\mu \psi$ (γ^μ Dirac 4x4 matrices)

- **Correlation functions**:

$$\begin{aligned} G^{(k)}(x_1, \dots, x_k; m, \lambda) &= \langle 0 | T \hat{\phi}(x_1) \cdots \hat{\phi}(x_k) | 0 \rangle \\ &= \sum_{E(\Gamma)=k} a(\Gamma; m, x_1, \dots, x_k)\, \hbar^{L(\Gamma)} \lambda^{V(\Gamma)} \end{aligned}$$

Feynman graphs Γ allowed by $\mathcal{L}$: e.g. ϕ^3 

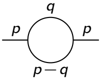
amplitude $a(\Gamma)$ from free propagator of $\mathcal{L}_0$: ϕ^3 $G_0(p) = \frac{i}{p^\mu p_\mu - m^2 + i\epsilon} \in \mathbb{C}$

QED $S_0(p) = \frac{i}{\gamma^\mu p_\mu - m + i\epsilon} \in M_4(\mathbb{C})$

- **Asymptotic series in λ** with coefficients in an algebra $A = \mathbb{C}, M_4(\mathbb{C})$... given by $\mathcal{L}_0$:

$$G_n^{(k)} = \sum_{\substack{V(\Gamma)=n \\ E(\Gamma)=k}} a(\Gamma)\, \hbar^{L(\Gamma)} \in A[\hbar] \quad \Longrightarrow \quad G^{(k)}(\lambda) = \sum_{n \geqslant 0} G_n^{(k)}\, \lambda^n \in A[\hbar][[\lambda]]$$

QFT needs renormalization

- **Divergent graphs**: 

$= \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2+m^2} \frac{1}{(p-q)^2+m^2} \simeq \int_{|q|_{min}}^{\infty} d|q| \frac{1}{|q|} = \infty$!

  need to *regularise* (by dim. reg. ϵ, cutoff Λ...) $\Longrightarrow$ $G(\lambda) \in A_\epsilon[\hbar][[\lambda]]$

  need to *renormalise* $\Longrightarrow$ **counterterm** $c(\Gamma) = -$ divergent part (scalar inside A_ε!)

- **Renormalisable theory**: can collect all counterterms $c(\Gamma)$ into few series $Z_i(\lambda)$ s.t.

  for $\begin{aligned} \phi_0 &= \phi\, Z_3(\lambda)^{1/2} \\ m_0 &= m\, Z_m(\lambda)^{1/2} Z_3(\lambda)^{-1/2} \\ \lambda_0 &= \lambda\, Z_1(\lambda) Z_3(\lambda)^{-3/2} \end{aligned}$ get $\boxed{\mathcal{L}^{ren}(\phi; m, \lambda) = \mathcal{L}(\phi_0; m_0, \lambda_0)}$

  and **Dyson's formula** [1949]: $\boxed{G^{ren}(m,\lambda) = G\big(m_0(\lambda), \lambda_0(\lambda)\big)\, Z_3(\lambda)^{-1/2}}$

- **Renormalization factors**: $Z(\lambda) = 1 + O(\lambda)$ $\Rightarrow$ invertibile series with product

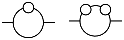
  Bare coupling: $\lambda_0(\lambda) = \lambda + O(\lambda^2)$ $\Rightarrow$ formal diffeomorphism with substitution

- **Ren. group** (formally) = $\boxed{\text{bare coupling} \ltimes \text{ren. factors}}$ contains $\big(\lambda_0(\lambda), Z_i(\lambda)\big)$

  semidirect product group $\boxed{\big(\lambda_1(\lambda_0), Z'(\lambda_0)\big) \bullet \big(\lambda_0(\lambda), Z(\lambda)\big) = \Big(\lambda_1\big(\lambda_0(\lambda)\big), Z'\big(\lambda_0(\lambda)\big) Z(\lambda)\Big)}$

  $\Longrightarrow$ acts on invertible series $G(\lambda)$ by Dyson's formula $\boxed{G^{ren} = G \bullet (\lambda_0, Z)}$

2) Renormalization Hopf algebras: need to compute counterterms!

- **BPHZ formula** ['57–'69]: recurrence on 1PI divergent subgraphs 

$$a^{ren}_{p_i}(\Gamma) = a_{p_i}(\Gamma) + c(\Gamma) + \sum_{(\gamma_k)} a_{p_i}(\Gamma_{/(\gamma_k)})\, c(\gamma_1)\cdots c(\gamma_r)$$

$$c(\Gamma) = -\mathrm{Taylor}^{div(\Gamma)}_{p^2=m^2}\Big[a_{p_i}(\Gamma) + \sum_{(\gamma_k)} a_{p_i}(\Gamma_{/(\gamma_k)})\, c(\gamma_1)\cdots c(\gamma_r)\Big]$$

p_i external momenta

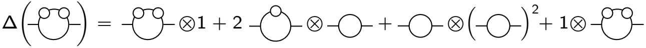
$\gamma_1, \ldots, \gamma_r \subset \Gamma$
1PI disjoint

- **Hopf algebra on Feynman graphs** [Connes-Kreimer '98-2000]:

$$H_{\mathrm{CK}} = \mathbb{C}[\mathrm{1PI}\ \Gamma] \quad \text{free commutative product}$$

$$\Delta_{\mathrm{CK}}(\Gamma) = \Gamma\otimes 1 + 1\otimes\Gamma + \sum_{(\gamma_k)} \Gamma_{/(\gamma_k)}\otimes\gamma_1\cdots\gamma_r$$

$$S(\Gamma) = -\Big[\Gamma + \sum_{(\gamma_k)} \Gamma_{/(\gamma_k)} S(\gamma_1)\cdots S(\gamma_r)\Big]$$

e.g. 

$\Longrightarrow$ amplitudes = **algebra maps** $\boxed{a_{p_i}, a^{ren}_{p_i} : H_{\mathrm{CK}} \to A_\epsilon[\hbar]}$ related to coproduct Δ

counterterms = **algebra map** $\boxed{c : H_{\mathrm{CK}} \to \mathbb{C}_\epsilon \subset A_\epsilon[\hbar]}$ related to antipode S

- **Why Hopf algebras?** Hopf algebra = linearisation of a group $\Rightarrow$ algorithms!

3) Proalgebraic groups: $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$, $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ and $\mathrm{Spin}(n)$...

- **Coordinate ring**: for $\mathbb{S}^1$: $\mathbb{R}[x,y]/\langle x^2+y^2=1\rangle$ algebra of "regular" functions on $\mathbb{S}^1$
  for $SL_2(\mathbb{R})$: $\mathbb{R}[x_{11},x_{12},x_{21},x_{22}]/\langle x_{11}x_{22}-x_{12}x_{21}=1\rangle$ extra opt. $\Delta(x_{11}) = x_{11}\otimes x_{11} + x_{12}\otimes x_{21}$ etc.

- **Algebraic group**: representable functor
$$\begin{array}{rcl} G : \mathcal{C}om_{\mathbb{K}} & \longrightarrow & \mathcal{G}roups \\ A & \longmapsto & G(A) = \mathrm{Hom}_{\mathcal{C}om_{\mathbb{K}}}(H,A) \end{array}$$

H is a **Hopf algebra**

multiplication	$m : H\otimes H \to H$	associative
unit	$u : \mathbb{K} \hookrightarrow H$	$1_H = u(1_{\mathbb{K}})$
comultiplication	$\Delta : H \to H\otimes H$	coassociative
counit	$\varepsilon : H \to \mathbb{K}$	+ prop
antipode	$S : H \to H$	+ prop

H is the dual Hopf algebra of the **universal enveloping algebra** $U(\mathfrak{g})$ of $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{K}))$

- **Duality:** Algebraic groups $\equiv$ finitely generated commutative Hopf algebras

$G \Rightarrow$ $H = $ coordinate ring of $G(\mathbb{K})$ $\cong \{\ h : G(A) \to A \text{ regular}\}$ is a Hopf alg. with
$$\begin{cases} \Delta(h)(a,b) = h(a\cdot b) \\ \varepsilon(h) = h(1_G) \\ S(h)(a) = h(a^{-1}) \end{cases}$$

$H \Rightarrow$ $G(A) = \mathrm{Hom}_{\mathcal{C}om_{\mathbb{K}}}(H,A)$ is a group with
$$\begin{cases} \text{convolution} & a*b = m_A(a\otimes b)\Delta_H \\ \text{unit} & 1 = u_A\,\varepsilon_H \\ \text{inversion} & a^{-1} = a\,S_H \end{cases}$$

- **Proalgebraic group** if H is not finitely generated e.g. infinite matrices
$$SUT_\infty = \begin{pmatrix} 1 & a & b & \cdots \\ 0 & 1 & c & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \cdots \end{pmatrix}$$

Groups of series with coefficients in a commutative algebra A

- **Invertible series**: $\mathrm{Inv}(A) = \left\{ a(\lambda) = \sum_{n\geqslant 0} a_n \lambda^n \mid a_0 = 1,\, a_n \in A \right\} \quad (a\,b)(\lambda) = a(\lambda)\, b(\lambda)$

  Hopf algebra: set $x_n(a) = a_n$ $(x_0 = 1)$ $\quad H_{\mathrm{inv}} = \mathbb{K}[x_n \mid n \geqslant 1] \quad \Delta_{\mathrm{inv}}(x_n) = \sum_{0\leqslant m\leqslant n} x_m \otimes x_{n-m}$

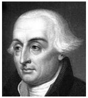
- **Diffeomorphisms**: $\mathrm{Diff}(A) = \left\{ a(\lambda) = \sum_{n\geqslant 0} a_n \lambda^{n+1} \mid a_0 = 1,\, a_n \in A \right\} \quad (a \circ b)(\lambda) = a\big(b(\lambda)\big)$

  Faà di Bruno Hopf algebra [Faà di Bruno 1855, Lagrange 1770, Joni-Rota 1979]:

  $$H_{\mathrm{FdB}} = \mathbb{K}[x_n \mid n \geqslant 1] \qquad (x_0 = 1)$$

  $$\Delta_{\mathrm{FdB}}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{(p_i)} \frac{(m+1)!}{p_0!\,p_1!\cdots p_n!}\, x_1^{p_1} \cdots x_n^{p_n}$$

  $S(x_n)$ Lagrange inversion


  $$p_0 + \cdots + p_n = m+1$$
  $$p_1 + 2p_2 + \cdots + np_n = n-m$$

- **Diffeographisms** [Connes-Kreimer 2000]:

  $$\mathrm{Diff}_{\mathrm{CK}}(A) := \mathrm{Hom}_{Com_{\mathbb{K}}}(H_{\mathrm{CK}}, A) = \left\{ a(\lambda) = \sum_{\Gamma} a_\Gamma\, \lambda^\Gamma \mid a_\Gamma \in A \right\}$$

  $$(a \bullet b)(\lambda) = \sum_{\Gamma} \Big(a_\Gamma + b_\Gamma + \sum_{(\gamma_k)} a_{\Gamma/(\gamma_k)}\, b_{\Gamma_1} \cdots b_{\Gamma_r} \Big)\, \lambda^\Gamma$$

  "virtual" series!

  "λ^Γ" symbol

- **Real $\leftrightarrow$ Virtual:** inclusion $\quad H_{\mathrm{FdB}} \hookrightarrow H_{\mathrm{CK}}, \quad x_n \mapsto \sum_{V(\Gamma)=2n+1} \Gamma/\mathrm{sym}(\Gamma)$ $\quad$ projection $\quad \mathrm{Diff}_{\mathrm{CK}}(A) \twoheadrightarrow \mathrm{Diff}(A), \quad \lambda^\Gamma \mapsto \lambda^{V(\Gamma)}$

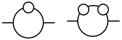
4) Integral BPHZ formulas (for massless renormalization)

- **Aim**: compute $$\boxed{Z_3(\lambda) = 1 + \sum_{E(\Gamma)=2} \frac{c(\Gamma)}{\mathrm{sym}(\Gamma)} \lambda^{V(\Gamma)}}$$

- **BPHZ on graphs**: $$\boxed{c(\Gamma) = -\mathrm{Taylor}^{div(\Gamma)}_{p^2=m^2}\Big[a_{p_i}(\Gamma) + \sum_{(\gamma_k)} a_{p_i}(\Gamma_{/(\gamma_k)})\ c(\gamma_1)\cdots c(\gamma_r)\Big]}$$

  $\Longrightarrow$ takes place in Connes-Kreimer Hopf algebra $H_{\mathrm{CK}} = \mathbb{C}[\text{1PI } \Gamma]$

  $a_{p_i}, c : H_{\mathrm{CK}} \to A_\epsilon[\hbar]$ are algebra maps, i.e. $a_{p_i}, c \in \mathrm{Diff}_{\mathrm{CK}}(A_\epsilon[\hbar])$

  recursion over 1PI divergent subgraphs  gives many Taylor expansions!

- **Remarks**:
  - Taylor expansion only affects the amplitudes $a_{p_i}(\Gamma)$
  - the recursion is linear in the amplitudes
  - in the massless case $a_{p_i}(\Gamma_{/(\gamma_k)}) = a_{p_i}(\Gamma) \prod q_j^2 / \prod a_{q_j}(\gamma_k)$
  - can sum up graphs Γ with $V(\Gamma) = 2n$: the whole formula is compatible!

$\Longrightarrow$ finally compute $$\boxed{Z_3(\lambda) = 1 + \sum_{n \geqslant 1} c(x_n)\, \lambda^{2n}}$$ with $$\boxed{x_n = \sum_{\substack{E(\Gamma)=2 \\ V(\Gamma)=2n}} \Gamma/\mathrm{sym}(\Gamma)}$$

- **integral BPHZ**: $$\boxed{c(x_n) = -\mathrm{Taylor}^{\omega}_{p^2=m^2}\Big[\text{polynomial in } a_{p_i}(x_m)\ (m \leqslant n) \text{ and } c(x_k)\ (k < n)\Big]}$$

  from Lagrange inversion formula in Faà di Bruno Hopf algebra H_{FdB}!

[Ref. Brouder-F. 2001 for QED in the massive case, work in progress for scalar theories]

5) Extension to non-commutative coefficients: facts and problems

- Counterterms for $\lambda_0(\lambda) \Rightarrow$ scalar-valued character $\lambda_0 : H_{\mathrm{FdB}} \to \mathbb{C}_\epsilon \subset A_\epsilon[\hbar]$
  group action ruled by the functor $\mathrm{Diff}: \mathcal{C}om_{\mathbb{K}} \to \mathcal{G}roups \Rightarrow$ same procedure for all QFTs!

- For **fermions** and **gauge theories** need non commutative coefficient algebra $A_\epsilon[\hbar]$
  $\Rightarrow$ the functor $\mathrm{Diff}: \mathcal{C}om_{\mathbb{K}} \to \mathcal{G}roups$ does not apply!

- There is a description by commutative renormalization Hopf algebras [Van Suijlekom 2007]
  but it is not functorial in A ($\bullet \neq$ convolution of Δ_{FdB}, need a Hopf alg. for each theory)!

- QED also given by **non-commutative FdB Hopf algebra** [Brouder-F-Krattenthaler 2006]:

$$H^{\mathrm{nc}}_{\mathrm{FdB}} = \mathbb{K}\langle x_n \mid n \geqslant 1 \rangle \qquad (x_0 = 1)$$

$$\Delta^{\mathrm{nc}}_{\mathrm{FdB}}(x_n) = \sum_{m=0}^{n} x_m \otimes \sum_{(k)} x_{k_0} \cdots x_{k_m}$$


$k_0 + k_1 + \cdots + k_m = n - m$

- Can we extend Diff to a functor $\mathrm{Diff}: \mathcal{A}s_{\mathbb{K}} \to \mathcal{G}roups$ on associative (non-com.) algebras?

  NO! If H and A are non-commutative, the convolution product

$$a * b = m_A\ (a \otimes b)\ \Delta_H \qquad \text{in} \qquad \mathrm{Hom}_{\mathcal{A}s_{\mathbb{K}}}(H, A)$$

  is not well defined because $m_A : A \otimes A \to A$ is not an algebra morphism! (old problem)

Groups of series with coefficients in a non-commutative algebra A

- **Idea**: in $As_{\mathbb{K}}$ replace the algebra $A \otimes B$ with internal product $(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb'$

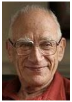
  by **free product algebra** $$A \amalg B = \mathbb{K} \oplus \bigoplus_{n \geqslant 1} \Big[\underbrace{A \otimes B \otimes A \otimes \cdots}_{n} \oplus \underbrace{B \otimes A \otimes B \otimes \cdots}_{n} \Big]$$

  with internal **concatenation** $(a \otimes b) \cdot (a' \otimes b') = a \otimes b \otimes a' \otimes b'$

  $\Rightarrow$ $m_A : A \otimes A \to A$ lifted to **folding map** $\mu_A : A \amalg A \to A$ which is an algebra map!

- **Cogroup in $As_{\mathbb{K}}$** [Kan 1958, Eckmann-Hilton 1962] $=$ associative algebra H with

comultiplication	$\Delta^{\amalg} : H \to H \amalg H$	coass.
counit	$\varepsilon : H \to \mathbb{K}$	+ prop.
antipode	$S : H \to H$	+ prop.




- **Thm.** [Kan 1958] Get a proalgebraic group on non-commutative algebras

$$G(A) := \mathrm{Hom}_{As_{\mathbb{K}}}(H, A) \quad \text{with} \quad a * b = \mu_A \; (a \amalg b) \; \Delta_H^{\amalg}$$

- **Group of invertible series:** [Brouder-F-Krattenthaler 2006]

$$\mathrm{Inv}(A) \quad \Leftrightarrow \quad H = \mathbb{K}\langle x_1, x_2, \ldots \rangle, \quad \Delta^{\amalg}(x_n) = \sum x_m \otimes x_{n-m}$$

$\Rightarrow$ good algorithmic model for **renormalization factors** $Z(\lambda)$ in QFT!

When groups fail: use loops!

- **Problem**: if A is not commutative, $\mathrm{Diff}(A)$ is not a group because the composition is not associative:

$$\big(a \circ (b \circ c) - (a \circ b) \circ c\big)(\lambda) = \big(a_1 b_1 c_1 - a_1 c_1 b_1\big)\ \lambda^4 + O(\lambda^5) \neq 0$$

- **Loop** [Moufang 1935] = set Q with

	multiplication	$a \cdot b$		(not nec. assoc.)
	unit	1		+ prop.
	left and right divisions	$a \backslash b$	a/b	+ prop.
$\Rightarrow$	left and right inverse of a	$1/a$	$a \backslash 1$	+ prop.

so that $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x = a \backslash b$, $y = b/a \in Q$

- **Associative loops** are groups, with

$$1/a = a \backslash 1 = a^{-1} \qquad a \backslash b = a^{-1} \cdot b \qquad a/b = a \cdot b^{-1}$$

- **Smallest smooth loop** which is not a group: $\mathbb{S}^7 = \{\text{unit octonions}\}$ (used for 2-qbits!)

- **Parallel transport along small geodesics** gives a local smooth loop structure to any manifold M [Sabinin 1977, 1981, 1986]. Flat connection $\Rightarrow$ global loop.

Loops of series with coefficients in a non-commutative algebra A

- **Coloop in $As_{\mathbb{K}}$** [F-Shestakov 2019] $=$ algebra H with

	comultiplication	$\Delta^{\amalg} : H \to H \amalg H$	(not nec. coass.)
	counit	$\varepsilon : H \to \mathbb{K}$	+ prop.
	codivisions	$\delta_l, \delta_r : H \to H \amalg H$	+ prop.
$\Rightarrow$	antipodes	$S_l, S_r : H \to H$	+ prop.

- **Thm.** Get a proalgebraic loop on non-commutative algebras

$$Q(A) := \mathrm{Hom}_{As_{\mathbb{K}}}(H, A) \quad \text{with} \quad a * b = \mu_A \, (a \amalg b) \, \Delta^{\amalg}_H$$

- **Loop of formal diffeomorphisms** [F-Shestakov 2019]:

$$\mathrm{Diff}(A) \quad \Leftrightarrow \quad \begin{array}{l} H = \mathbb{K}\langle x_1, x_2, \ldots \rangle \qquad \Delta^{\amalg}(x_n) = \Delta^{\mathrm{nc}}_{\mathrm{FdB}}(x_n) \\ \delta_r(x_n) = \text{non-commutative Lagrange} \\ \delta_l(x_n) = \text{new explicit formula (very complicated)} \end{array}$$

- **Thm.** In $\mathrm{Diff}(A)$ **inverse is unique** and $a/b(\lambda) = a \circ b^{-1}(\lambda)$ (while $a \backslash b(\lambda) \neq a^{-1} \circ b(\lambda)$!).

$\Rightarrow$ Dyson renormalization formulas make sense! cf. Birkhoff decomp. $G = G^{ren} \bullet (\lambda_0, Z)^{-1}$

$\Rightarrow$ good algorithmic model for **coupling renormalization** $\lambda_0(\lambda)$ in QFT!

Conclusion and perspectives

Conclusion:

- In pQFT, renormalization factors act on the Lagrangian as a semidirect product of invertible series by formal diffeomorphismes, with coefficients given by scalar counterterms of divergent Feynman graphs. **This action is functorial** in the coefficient algebra fixed by the QFT and gives **the same procedure for any scalar QFT**, ruled by **renormalization Hopf algebras**.
- Using **Faà di Bruno** Hopf algebra, BPHZ recursion can be summed up to 1PI div. graphs with $2n$ or $2n+1$ vertices and gives an **integral formula for counterterms at order** λ^{2n+1}.
- The RG action can be **extended in a functorial way to non-scalar QFTs**, if we renounce to associativity in RG ("transitivity") i.e. if we modify the assumptions on flow equations. This is possible in maths, because the running of couplings is ruled by diffeomorphisms, which form a non-associative loop with unique inversion and right division equal to the product by the inverse.

Perspectives:

- Compute integral formula for countertermes also in massive renormalization (BPHZ in Diff).
- Develop software to compute with free product instead of tensor product.
- Measure the impact of a non-associative renormalization group in Wilson's approach: replace usual flow of ODE/PDE by flow in smooth loops (cf. [Lev Sabinin 1999, book on smooth loops]).
- Can one test associativity of the RG **with experiments**?

Thank you for the attention!

Free product is necessary!

In the loop $\mathrm{Diff}(A)$, we have $1/a = a\backslash 1 =: a^{-1}$ and also $a/b = a \circ b^{-1}$ but

$$a\backslash b \neq a^{-1} \circ b \ !$$

In the series $a\backslash b$, the coefficient

$$\begin{aligned}(a\backslash b)_3 = b_3 &- \big(2a_1 b_2 + a_1 b_1^2\big) + \big(5a_1^2 b_1 + a_1 b_1 a_1 - 3a_2 b_1\big)\\ &- \big(5a_1^3 - 2a_1 a_2 - 3a_2 a_1 + a_3\big)\end{aligned}$$

contains the term $\boxed{a_1 b_1 a_1}$ which can not be represented in the form

$$x(a) \otimes y(b) \in H^{\mathrm{nc}}_{\mathrm{FdB}} \otimes H^{\mathrm{nc}}_{\mathrm{FdB}},$$

while it clearly belongs to

$$H^{\amalg}_{\mathrm{FdB}} \amalg H^{\amalg}_{\mathrm{FdB}}.$$

This **justifies the need to replace** $\otimes$ **by** $\amalg$ in the coproduct!