The Euclidean contour rotation in quantum gravity

John Barrett

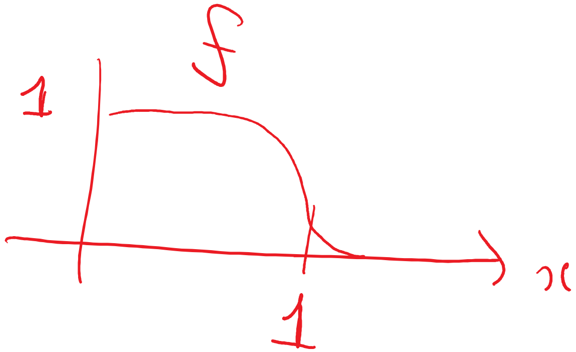
University of Nottingham

Abstract: The talk will discuss the rotation of the contour of functional integration in quantum gravity from Lorentzian geometries to Euclidean geometries. In the usual framework of metric tensors the functional integral does not have a good definition and so the formulas are necessarily heuristic. However it is hoped that these formulas will provide exact mathematical results when applied to theories that are constructed with a fundamental Planck scale cut-off.

Euclidean "Physics"

Euclidean signature ++++

- Euclidean QG (Hawking)
- Heat Kernel Expansion
- NCG & Connes-Chamseddine spectral action



$D : \mathcal{H} \to \mathcal{H}$ Dirac

$J : \mathcal{H} \to \mathcal{H}$ charge conj.

$$S_{cc} = \mathrm{tr}\left(D^2/\Lambda^2\right) + \langle J\psi, D\psi\rangle \qquad \Lambda = \text{Planck mass}$$

$$\underset{\Lambda\to\infty}{\sim} c\Lambda^4 \int dv + c'\Lambda^2 \int R\, dv + \dots + SM$$

Formula to "prove"

$$\int_{G_L,M_L} e^{iS_L(l,\phi)}\, Dl\, D\phi = \int_{G_E,M_E} e^{-S_E(e,\Phi)}\, De\, D\Phi$$

$l \in G_L$ Lorentzian geometry

$\phi \in M_L$ Matter fields

$e \in G_E$ Euclidean geometry

$\Phi \in M_E$ Euclidean matter

LHS : physics S_L

RHS : whatever emerges

- Boundary conditions

Issues

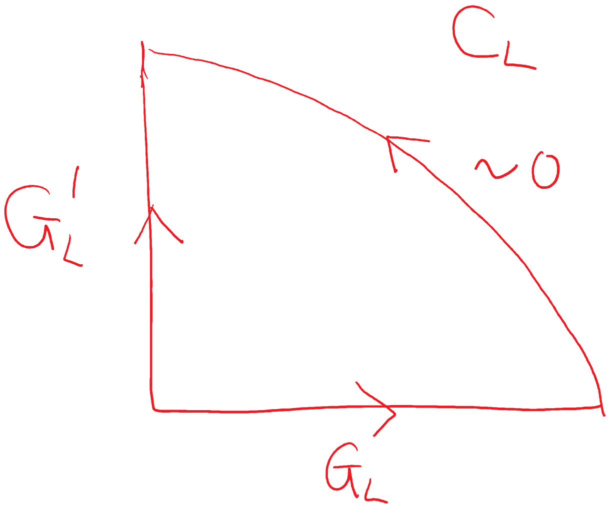
- Functional integrals heuristic
- Euclidean E-H action not bounded below
- Planck scale

$$\ell_p = \sqrt{G\hbar}$$

- cutoff ?
- NC ?
- new physics ?

Proof

- Embed $G_L \subset C_L$ Complex Lorentzian geometries
- Rotate to $G_L' \subset C_L$ Imaginary Lorentzian contour
- Euclideanisation $G_L' \cong G_E$



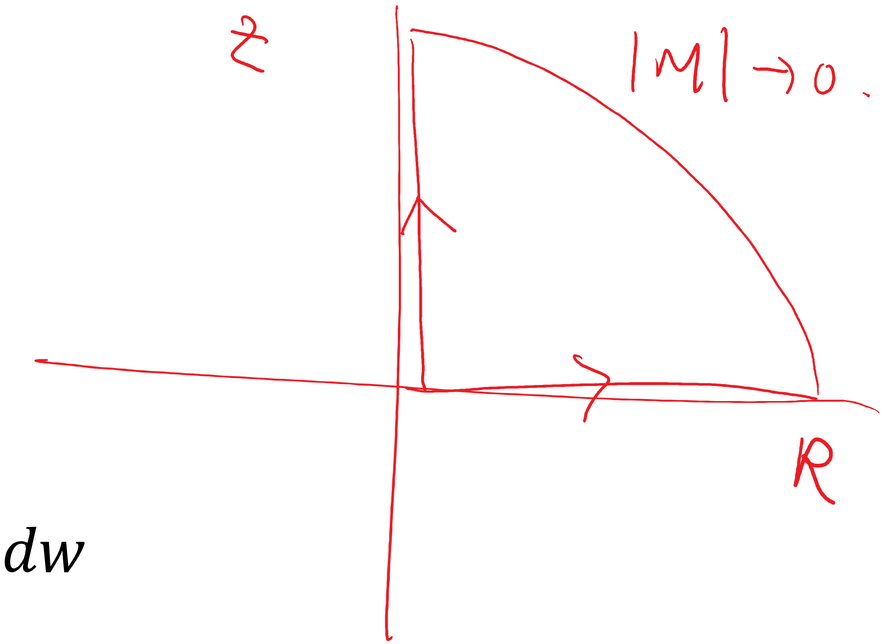
$$\int_{G_L} e^{iS_L} \mathcal{D}g \overset{rot}{=} \int_{G_L'} e^{iS_L} \mathcal{D}g \overset{Euc}{=} \int_{G_E} e^{-S_E} \mathcal{D}g$$

- Include matter fields. Scalars unchanged.

Contour rotation

in $\mathbb{C}$

$$\lim_{R\to\infty} \int_0^R e^{ikx} M(x)\, dx \ = i \int_0^\infty e^{-ky} M(iy) dy$$

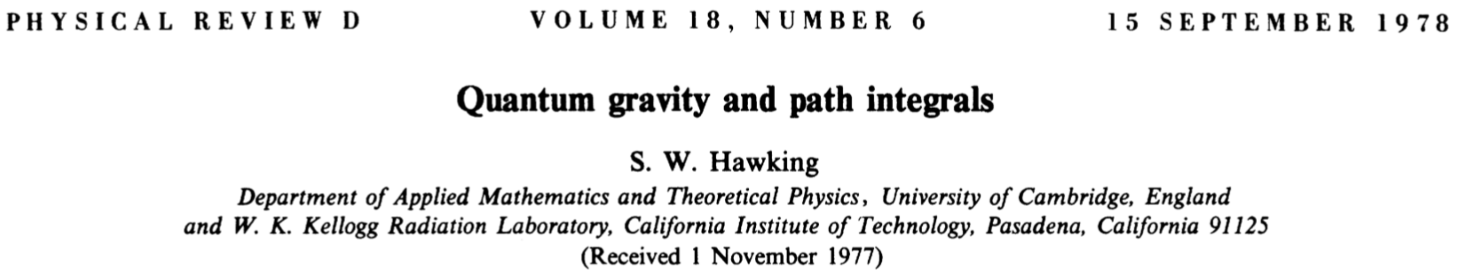


Locally,

$$\int_0^R e^{iw(x)}\, dx = \int_{w(0)}^{w(R)} e^{iw} \frac{1}{w'(x(w))}\, dw$$

Multiple integrals – same R.

History

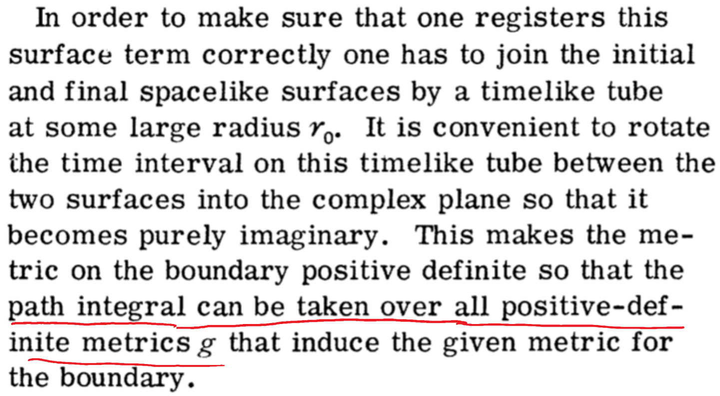


PHYSICAL REVIEW D VOLUME 18, NUMBER 6 15 SEPTEMBER 1978

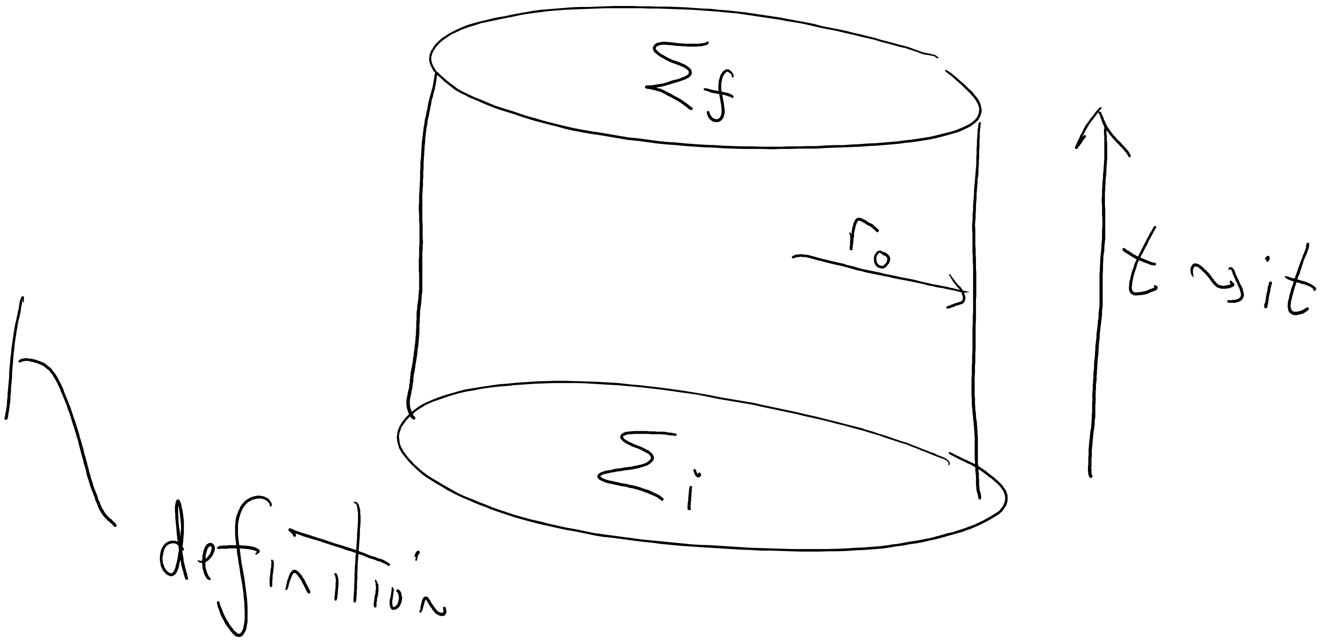
Quantum gravity and path integrals

S. W. Hawking
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England
and W. K. Kellogg Radiation Laboratory, California Institute of Technology, Pasadena, California 91125

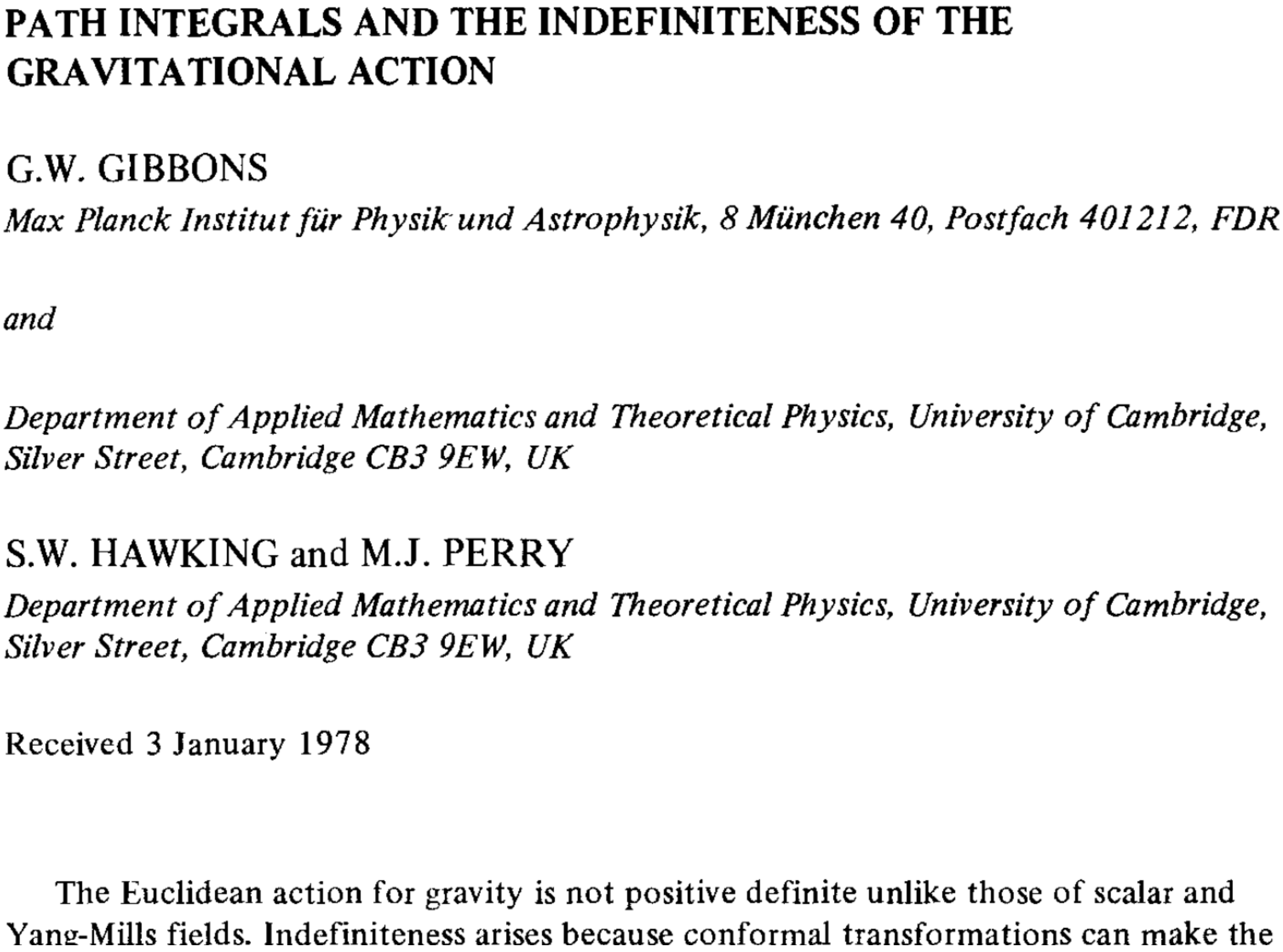
(Received 1 November 1977)



In order to make sure that one registers this surface term correctly one has to join the initial and final spacelike surfaces by a timelike tube at some large radius r_0. It is convenient to rotate the time interval on this timelike tube between the two surfaces into the complex plane so that it becomes purely imaginary. This makes the metric on the boundary positive definite so that the path integral can be taken over all positive-definite metrics g that induce the given metric for the boundary.



More history



PATH INTEGRALS AND THE INDEFINITENESS OF THE GRAVITATIONAL ACTION

G.W. GIBBONS
Max Planck Institut für Physik und Astrophysik, 8 München 40, Postfach 401212, FDR

and

Department of Applied Mathematics and Theoretical Physics, University of Cambridge,
Silver Street, Cambridge CB3 9EW, UK

S.W. HAWKING and M.J. PERRY
Department of Applied Mathematics and Theoretical Physics, University of Cambridge,
Silver Street, Cambridge CB3 9EW, UK

Received 3 January 1978

The Euclidean action for gravity is not positive definite unlike those of scalar and Yang-Mills fields. Indefiniteness arises because conformal transformations can make the

The Euclidean action for gravity is not positive definite unlike those of scalar and Yang-Mills fields. Indefiniteness arises because conformal transformations can make the action arbitrarily negative. In order to make the path integral converge one has to take the contour of integration for the conformal factor to be parallel to the imaginary axis.

... unbounded below

$$\hat{I}[g] = \frac{-1}{16\pi G}\int \underline{R(g)^{1/2}\,\mathrm{d}^4x} - \frac{1}{8\pi G}\int [K](h)^{1/2}\,\mathrm{d}^3x \tag{1.3}$$

is not positive semi-definite. (The minus sign comes from the direction of the Wick rotation, which has to be chosen to be consistent with that for the matter fields.) Under conformal transformations of the metric $\underline{\widetilde{g}_{ab} = \Omega^2 g_{ab}}$, R transforms as

$$\widetilde{R} = \Omega^{-2}R - 6\Omega^{-3}\Box\Omega\,, \tag{1.4}$$

and

$$\widetilde{K} = \Omega^{-1}K + 3\Omega^{-2}\Omega_{,a}n^a\,, \tag{1.5}$$

where n^a is the unit outward normal to the boundary ∂M. Thus

$$\hat{I}[\widetilde{g}] = \frac{-1}{16\pi G}\int_{\mathrm{M}} \underline{\Omega^2R + 6\Omega_{,a}\Omega^{,a}(g)^{1/2}}\,\mathrm{d}^4x - \frac{1}{8\pi G}\int_{\partial\mathrm{M}} [\Omega^2K](h)^{1/2}\,\mathrm{d}^4x\,. \tag{1.6}$$

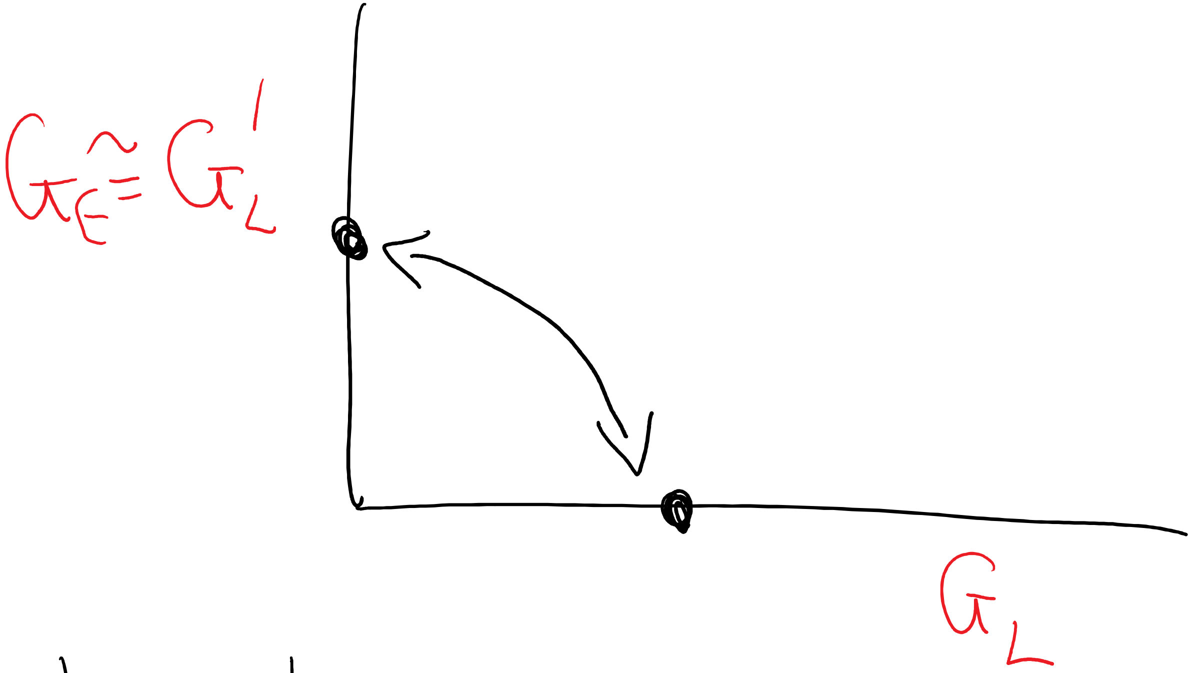
One sees that $\hat{I}$ may be as negative, as one wants, by choosing a rapidly varying conformal factor Ω.

But: Planck scale limits frequency $\Rightarrow$ S_E bounded below

Recall $S_{cc} \geq 0$!

This is not

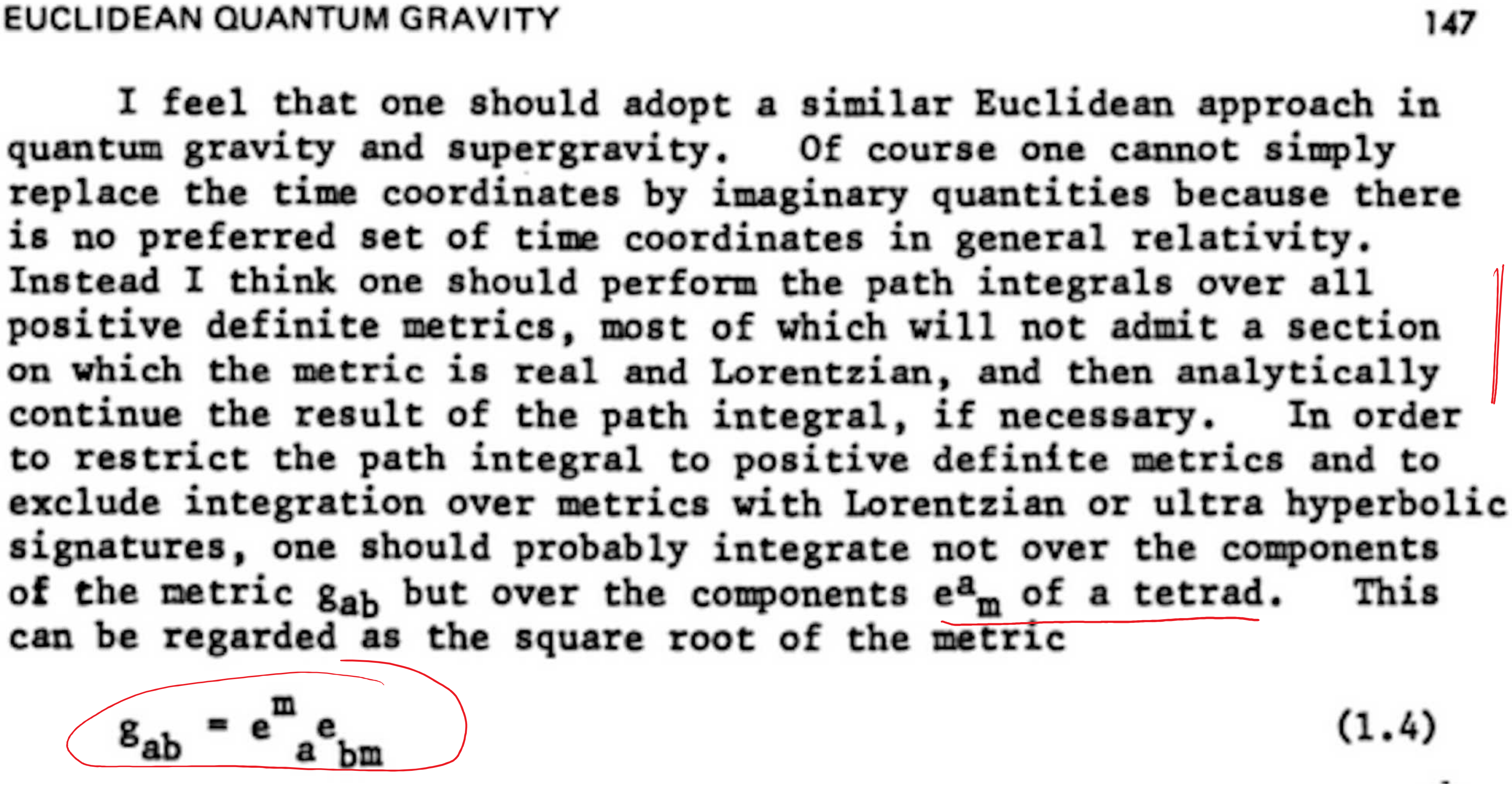
Wick Rotation



Metrics physically different!

Even more history

Hawking, Cargese lectures 1978



EUCLIDEAN QUANTUM GRAVITY 147

I feel that one should adopt a similar Euclidean approach in quantum gravity and supergravity. Of course one cannot simply replace the time coordinates by imaginary quantities because there is no preferred set of time coordinates in general relativity. Instead I think one should perform the path integrals over all positive definite metrics, most of which will not admit a section on which the metric is real and Lorentzian, and then analytically continue the result of the path integral, if necessary. In order to restrict the path integral to positive definite metrics and to exclude integration over metrics with Lorentzian or ultra hyperbolic signatures, one should probably integrate not over the components of the metric g_{ab} but over the components $e^a{}_m$ of a tetrad. This can be regarded as the square root of the metric

$$\boxed{g_{ab} = e^m{}_a e_{bm}} \qquad (1.4)$$

Frame field

4-manifold	M
Vector fields	l_0, l_1, l_2, l_3
Spin connection	∇
Minkowski metric	$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

→ Functions σ^c_{ab}: $\nabla_{l_a} l_b = \sigma^c_{ab} l_c$

c^c_{ab}: $[l_a, l_b] = c^c_{ab} l_c$

non-degenerate case

Curvature $R(l_a, l_b)l_c = (l_a(\sigma^e_{bc}) + \sigma^d_{bc}\sigma^e_{ad} - l_b(\sigma^e_{ac}) - \sigma^d_{ac}\sigma^e_{bd} - c^d_{ab}\sigma^e_{dc})l_e$

Imaginary contour

On G_L', fields $l_0 = ie_o,\ l_1 = e_1,\ l_2 = e_2,\ l_3 = e_3,$ e real

$$\sigma^a_{bc} = i^n \omega^a_{bc},\ n = \delta_{b0} + \delta_{c0} - \delta_{d0}, \qquad \omega \text{ real}$$

Actions

$$S_L = \int -2\Lambda + \frac{R}{16\pi G}\, dV \quad + \quad S_L^{scalar}$$

$$S_E = \int 2\Lambda - \frac{R_E}{16\pi G}\, dV_E \quad + \quad S_E^{scalar}$$

Euclideanisation (c.f. Samuel 2015, D'Andrea, Kurkov, Lizzi 2016)

On G'_L : $\quad iS_L = S_E$

Spinors

$\psi \in C^4$

Euclidean inner product $(\psi', \psi)_E$

$(\psi, \psi) \geq 0$

Lorentzian $(\psi', \psi) = (\gamma^0 \psi', \psi)_E$

Gamma matrices $\gamma^a \gamma^b + \gamma^b \gamma^a = -2\eta^{ab}$

Chirality γ, $\gamma^2 = 1$

$\gamma = \pm 1$ L/R

Charge conjugation $J: C^4 \to C^4$ antilinear

$J^2 = 1$

$J\gamma = -\gamma J$

Spinors on manifold

$$\langle \psi', \psi \rangle = \int_M (\psi', \psi)\, dV$$

$$D = i\gamma^a \nabla_{l_a}$$

Euclidean spinors

$\psi(x)$ same

but conjugate fields tricky

γ same

$J_E = \gamma^0 J$

$D_E = \gamma^a_E \nabla_{E e_a}$

no i!

$D_E^* = D_E$

using $(\, , \,)_E$

Euclideanisation - chiral

On G'_L: $l_0 = ie_0$, etc.

Calculation: $D = iD_E$

$$\langle \psi, D\psi \rangle = \int (J\bar{\psi}, D\psi)\, dV \qquad \det$$

$$= \int (\gamma^0 J \bar{\psi}, i D_E \psi_E)_E \cdot (-i)\, dV_E$$

$$= \int (J_E \bar{\psi}, D_E \psi)\, dV_E$$

Connes formula