PROBLEMS
MONOPOLES AND INSTANTONES

1 We define a vector field X as a linear operator on smooth functions which obeys the Leibniz rule. Please show that vector fields is local, that is if f vanishes identically on an open set U then so does $X(f)$.

2 If $\phi : M \to N$ is a map between manifolds, we define, for a given vector field on M: $\phi_*(X)$ which is an operator acting on $C^\infty(N)$:

$$\phi_*(X)(f) = X(f \circ \phi).$$

What is the target space of this map ? Can we modify the definition to get a vector field on N ?

3 Please show that the Moebius band is not a trivial real vector bundle over the circle.

4 Let ∇ be a connection on a vector bundle E seen as a map from $\Sigma(E) \to \Omega^1(M) \otimes \Sigma(E)$. Let U be a C^∞-linear map on $\Sigma(E)$. Please check that $U^{-1}\nabla U$ is also a connection and find the difference between ∇ nad $U^{-1}\nabla U$.

5 The following defines a projector on S^2.

$$p(\theta, \phi) = \frac{1}{2} \begin{pmatrix} 1 + \cos(\theta) & \sin(\theta)e^{i\phi} \\ \sin(\theta)e^{-i\phi} & 1 - \cos(\theta) \end{pmatrix},$$

How can we argue that this defines a nontrivial (complex) vector bundle ?

6 For the projector above please do compute the connection ∇_p on the sections of the bundle defined by p and the curvature.

7 Using two local trivialisation of the above bundle (one over the northern hemisphere, one over the southern) plase compute the associated connection one-form.

8 The potential of the magnetic monopole expressed locally (using local trivialisations) is:

$$A^N_\phi = g\,\frac{1 - \cos\theta}{r \sin\theta}, \quad A^S_\phi = -g\,\frac{1 + \cos\theta}{r \sin\theta}.$$

Please compute the change of the phase of the point particle of charge e transported along the equator γ,

$$\Delta\Phi = e \int_\gamma A,$$

where the integral is just the integral of a one form along the curve.

9 Taking F to be a curvature two-form we can define the characteristic polynomials in different ways, for example:

$$\det\left(\frac{iFt}{2\pi}+1\right)=\sum_k c_k(F)\,t^k,$$

and

$$\mathrm{tr}\left(e^{\frac{iFt}{2\pi}}\right)=\sum_k ch_k(F)\,t^k,$$

Please, relate c_1, c_2 and ch_1, ch_2.

10 Let F be a two-form on $\mathbb{R}^4$ such that $F=dA$. Please verify if there are any nontrivial solutions of the equation:

$$F=\pm * F,$$

where $*$ is the Hodge star.

11 Try to solve the previous problem on $\mathbb{R}^4\setminus\{x=y=z=0\}$ (4-dimensional space without a line).

12 Assume the solution of t'Hooft-Polyakov monopole in the BPS limit as:

$$\Phi^a(x)=\frac{x^a}{er^2}H(\xi),\quad A^a_j(x)=\epsilon_{ajk}\frac{x^k}{er^2}(1-K(\xi)),$$

gdzie:

$$\xi=vex,\quad K(\xi)=\frac{\xi}{\sinh\xi},\quad H(\xi)=\xi\coth\xi-1,$$

please check that these are the solutions and compute the t'Hooft tensor:

$$\mathcal{F}_{\mu\nu}=\widehat{\Phi}^aF^a_{\mu\nu}+\frac{1}{e}\epsilon_{abc}\widehat{\Phi}^a(D_\mu\widehat{\Phi}^b)(D_\nu\widehat{\Phi}^c),$$

where $\widehat{\Phi}^a=\frac{\Phi^a}{|\Phi|}$. Please compare the result to the field of the magnetic monople.

13 Let ρ^i be the matrices:

$$\rho^0=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho^1=\begin{pmatrix}i&0\\0&-i\end{pmatrix},\ \rho^2=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\ \rho^3=\begin{pmatrix}0&i\\i&0\end{pmatrix}.$$

and

$$X=\sum_{i=0}^3 x^i\rho^i,\quad dX=\sum_{i=0}^3 dx^i\rho^i.$$

Please show that

$$A(x)=\frac{1}{2}\frac{X^*dX-dX^*X}{1+|x|^2}$$

gives a selfdual (or antiselfdual) curvature F and please compute the intanton number,

$$\frac{1}{2}\int_{\mathbb{R}^4}\mathrm{tr}(F\wedge F).$$