The Infrared Divergence Phenomena and High-Energy Processes*

D. R. Yennie†

School of Physics, University of Minnesota, Minneapolis, Minnesota

S. C. Frautschi‡

Department of Physics, University of California, Berkeley, California

AND

H. Suura

Department of Physics, Nihon University, Tokyo, Japan

A general treatment of the infrared divergence problem in quantum electrodynamics is given. The main feature of this treatment is the separation of the infrared divergences as multiplicative factors, which are treated to all orders of perturbation theory, and the conversion of the residual perturbation expansion into one which has no infrared divergence, and hence no need for an infrared cutoff. In the infrared factors, which are exponential in form, the infrared divergences arising from the real and virtual photons cancel out in the usual way. These factors can then be expressed solely in terms of the momenta of the initial and final charged particles and an integral over the region of phase space available to the undetected photons; they do not depend upon the specific details of the interaction. Electron scattering from a static potential is treated in considerable detail, and several other examples are discussed briefly. As an important byproduct of the general treatment, it is found that when the infrared contributions are separated in a particular way, they dominate the radiative corrections at high energies and together with certain "magnetic terms" and vacuum polarization corrections seem to give all the contributions proportional to $\ln (E/m)$. All of these corrections can be easily estimated (in most cases) simply from a knowledge of the external momenta of the charged particles; this then provides a very powerful and accurate way of estimating radiative corrections to high-energy processes.

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1. INTRODUCTION

The essential idea for the understanding of the infrared divergence problem was first brought out by Bloch and Nordsieck in their famous paper, published a little over two decades ago (1). In brief, this idea is that in any practical experiment involving charged particles it is impossible to specify completely the final state of the system. Because individual photons can be emitted with arbitrarily small energies, there will always be a possibility that some photons will escape detection. In fact, they showed that the probability that only a finite number of photons will escape detection is precisely zero; this is due to the infrared divergence associated with the soft virtual photons. On the other hand, when the cross section is summed over all possible final states compatible with the detection arrangement, including all possibilities for the undetected photons, a nonvanishing result is obtained. In fact, they showed that the observed cross section is very nearly the cross section that would be obtained if all radiative corrections are ignored. This is the well-known cancellation between the real and virtual infrared divergences. It is the intention of the present paper to give a modern field theoretical treatment of the infrared divergence phenomena. No attempt will be made to give a complete historical review; for such a review and bibliography, the reader is referred to the excellent paper by Jauch and Rohrlich (2).

For purposes of orientation, let us recall briefly the semi-classical arguments which predict the infrared divergence phenomena. For example, suppose an electron in motion is deflected due to its interaction with a potential. The Lorentz-contracted proper field of the electron will be altered by the collision, and the change in the proper field will be emitted as electromagnetic radiation. For sufficiently long wavelengths ($ka \ll 1$, where $k$ is the wave number and $a$ is a dimension of the scattering region), the radiation can be calculated without knowledge of the details of the trajectory in the scattering region. It depends only upon the initial and final momenta of the electron and the direction in which the radiation is observed (assuming the electron suffers no time delay in the scattering region). As is well known, the energy emitted per unit frequency is independent of frequency in this limit. Making the transcription to the photon description, it is clear that the number of photons emitted per unit frequency range is inversely proportional to the frequency; i.e., the photon spectrum is of the form $dk/k$, which diverges as $k \rightarrow 0$. This is the infrared divergence for real photons.

The angular distribution can also be understood by the semiclassical argument. In the extreme relativistic limit, the proper fields will be Lorentz contracted in a small region near the plane perpendicular to the direction of motion of the charge and moving along with the charge. This leads to a strong peaking of the radiation parallel either to the incident direction or the final direction of motion. These features appear in the classical amplitude for emission of radiation, which
is proportional to \( I \)

\[
e^{-\left(\frac{e \cdot p'}{k \cdot p} - \frac{e \cdot p}{k \cdot p'}\right)}.
\]  

\( (1.1) \)

This peaks very strongly for \( k \) parallel either to \( p \) or to \( p' \), where these are, respectively, the initial and final momenta. Assuming that the scattering has taken place, the probability that a photon of momentum \( k \) is also emitted is obtained by squaring (1.1) and multiplying the result by \( [(2\pi)^2]^{-1} \):

\[
P(k) \, dk = \frac{\alpha}{(2\pi)^2} \left( \frac{e \cdot p}{E - p \cdot \hat{k}} - \frac{e \cdot p'}{E' - p' \cdot \hat{k}} \right)^2 \, d\Omega \frac{dk}{k} \quad (1.2)
\]

This exhibits the typical infrared behavior \((dk/k)\) and the strongly peaked angular dependence about the directions of \( p \) and \( p' \). The probability for photon emission into the energy range \( dk \), integrated over photon direction, is

\[
k^2 \, dk \int P(k) \, d\Omega = \alpha A \frac{dk}{k} \quad (1.3)
\]

where \( A \) is defined in (2.32) and at high energies is given approximately by

\[
A \approx \frac{2}{\pi} \left[ \ln \left( \frac{2(E^2 - p \cdot p')}{m^2} \right) - 1 \right] \quad (1.4)
\]

Thus the probability of photon emission increases logarithmically with the energy, it is also more important for larger scattering angles; this is to be expected from the classical discussion since for a small angle deflection the proper field of the electron can more easily adjust to the transition.

It is reasonable to expect that a correct quantum-mechanical treatment of the problem will lead to the same qualitative features that have been predicted by the semiclassical argument. Since we are interested in the very long wavelength limit, the process is governed by the behavior of the scattering particle in large regions of space. Also emission and absorption of soft photons does not appreciably disturb the motion of the electron; this means that soft photons are emitted and absorbed independently, resulting in a Poisson distribution for both real and virtual photons.

This has led several authors to give a treatment of the infrared divergence problem based upon an approximation in which the charge-current distribution of the scattering particle is treated purely classically \((2, 3)\). In this method the motion of the particle without radiative corrections is calculated, and the dynamical effects of the radiative corrections are ignored. The most refined treatment of this type has been given by Lomon and collaborators \((4)\); they have developed a formalism in which the dynamical effects of radiative corrections due

\[1\] The four-dimensional dot product is given by \( a \cdot b = ab_a - a \cdot b \). In particular \( k \cdot p = k(E - p \cdot \hat{k}) \).
to hard photons can also be included. In these semiclassical methods there is always an arbitrary separation between the hard and the soft photons; the soft photon contribution is treated exactly to all orders with the neglect of dynamical corrections to the motion of the particle. In Lomon's treatment, the effect of the hard photons is incorporated in the calculation of the basic cross sections and is calculated to some order of perturbation theory. The place at which the separation occurs is then chosen to minimize the error made. As a typical result of this approach, we quote from Lomon's paper (4) with some change of notation,

$$\sigma(\Delta E,E,\theta) \approx \left( \frac{\Delta E}{e\epsilon} \right)^{a_1} \sigma_n(\epsilon,E,\theta),$$

(1.5)

where $\epsilon$ is the separation energy, $\Delta E$ is the energy resolution of the detector, $\sigma_n$ is the cross section up to $n$th order in the perturbation series with $\epsilon$ as the lower cutoff on the photon energy, $C$ is Euler's constant, and $\theta$ is the scattering angle. We have omitted from (1.5) a factor $F(\alpha A)$, which is very close to unity in practical situations. This factor, which is given in (2.45) below, was first evaluated in terms of simple functions by Lomon and Shaw. Their calculation is reproduced in Section 2.

A completely quantum mechanical treatment of the infrared divergence problem is somewhat more difficult. Cancellation of the real and virtual infrared divergences in lowest order in $\alpha$ has been verified in many calculations of particular processes. Some examples in the literature are radiative corrections to the following processes: Coulomb scattering of electrons to lowest order in the potential (5); Compton scattering (6); Coulomb scattering of electrons to second and higher orders in the potential (7, 8); electron-electron scattering (9, 10); wide angle pair production (11); and bremsstrahlung (12, 13). A general proof of the cancellation to all orders has been given by Jauch and Rohrlich (2); the present paper is in a sense a refinement of their work. The main elements in this type of proof will now be described briefly. It is first shown that the infrared divergence is associated with soft photons (real or virtual) which are emitted or absorbed from external charged lines. This is physically plausible since a long wavelength photon should sense only the large scale features of the charge-current distribution, and emission from an interior line somehow corresponds to emission from the current in a small region of space. It is not surprising then that the matrix element for emission of a soft photon is given simply by the classical

\[ E. L. Lomon and A. Shaw (private communication). \]
\[ The results of these references disagree with those of Ref. 11, to which they are related. They have been found to be in error (private communication from P. I. Fomin to S. D. Drell); and after corrections, they agree with Ref. 11. \]
\[ Reference 13 disagrees with Ref. 12, and is not in a form for easy comparison with Ref. 11. \]
expression (1.1) times the matrix element which describes the basic process. The cross section for emission of a real soft photon is then given by (1.2) times the cross section for the basic process. Similarly the matrix element for emission and reabsorption of a soft virtual photon is approximately given by a simple factor depending only upon the external lines times the basic matrix element. It is this extraction of infrared factors which permits a complete treatment of the soft photon contributions, and in particular allows one to demonstrate the complete cancellation of infrared divergences. Before the final cancellation it is necessary to use some sort of an infrared cutoff. Conventionally, this may be either a small photon mass or a minimum photon energy. The final result is the same, of course. We shall use the photon mass method in the present work.

There are two ways in which we believe the present approach is an improvement over Ref. 2. The first improvement is a somewhat better argument about the extraction of the infrared divergence factors, in which the overlapping infrared divergences are not ignored. While this new argument is perhaps not strong enough to be considered a proof, we hope it removes any reasonable doubts about the factorization of the infrared divergence. This new argument is equivalent to, but more transparent than, one given earlier by Yennie and Suura (14). The second improvement is really an extension of Ref. 2 to give a systematic treatment of the noninfrared contributions by means of a perturbation expansion. It may also be remarked that the particular infrared separation that will be made is gauge-invariant for both the real and virtual photons; for the virtual photons it is also covariant (if we use a photon mass for the infrared cutoff). Also, the infrared factors are completely renormalized so that the renormalization of the residual perturbation expansion proceeds in the usual way. Another treatment of the cancellation of the infrared divergence has been given by Nakanishi (15). This treatment seems more rigorous than the present treatment, but it is restricted to the total cross section, not the differential cross section with energy resolution.

Our major aim in the present paper is to give a complete treatment of the infrared divergence phenomena within the framework of modern quantum electrodynamics, following the general approach we have just outlined. The end result is a method of calculation in which the infrared factors are given explicitly for any process in terms of the four-momenta of the external charged lines and do not involve the details of the process. The infrared cutoff no longer appears in this final result. On the other hand, the residual perturbation expansion is given in terms of integrals in which no infrared divergence occurs and no infrared cutoff is necessary. Although our emphasis will be on the generality and completeness of the method, some results of practical value will be obtained. In many practical problems the radiative corrections come from the regions in which

\footnote{See Note added in proof at end of paper.
the energy denominators are small. Thus it should be possible to use our results to estimate the bulk of the radiative corrections to any process; these will generally involve “double logarithm” terms of the form $\alpha \ln (E/m) \ln(\Delta E/E)$ or $\alpha [\ln (E/m)]^2$ times the basic cross section. The contribution of these terms can be very sensitive to the experimental arrangement; a very careful treatment of the actual experimental conditions is therefore very important. Examples of such calculations are given in (10, 11); examples of calculations which do not appear to be easily applicable to experiment because they do not take cognizance of the experimental situation are contained in (6, 9).

It may seem paradoxical that the infrared contributions dominate the radiative corrections at high energies. It is possible to understand this result intuitively in terms of the semiclassical discussion given previously. In the scattering of a high-energy charged particle by a potential (for example), the Lorentz-contracted proper field surrounding the particle must change in a very short time at great distances from the particle. Since the total electromagnetic field cannot change in such a rapid manner, compensating radiation fields are produced; these fields correspond to real photon emission. Since scattering without photon emission is impossible, there must be a negative radiative reaction associated with the virtual photons. This arises due to the lack of good overlap between the initial and final Lorentz-contracted proper fields of the particle and again depends mainly on the large distance properties of the electromagnetic field. Thus, both real and virtual photons are governed by a $dk/k$ spectrum for some range of $k$ values; this spectrum must break off when the wavelength becomes so small that it becomes comparable with the dimensions of the region in which the main deflection occurs ($ka \sim 1$) or the quantum-mechanical properties of the particle become important ($k \sim E$). It should be pointed out that $a$ is not necessarily the range of the potential; in wide angle scattering from a Coulomb potential (infinite range), for example, the main deflection takes place within a very small distance. For a quantum-mechanical discussion of the conditions under which the $dk/k$ spectrum holds, the reader is referred to the papers by Lomon (4); he reaches the conclusion that it is a good approximation for $k \ll E$.

The complete radiative corrections to order $\alpha$ are thus given approximately by $\alpha A \ln (\Delta E/E)$ times the basic cross section. The factor $A$ contains a logarithm due to the strong peaking of the angular integration over $k$, which in turn reflects the strong Lorentz-contraction of the proper fields. The factor $\ln (\Delta E/E)$ comes from the integrals of $dk/k$ and takes into account the facts that the contributions of real and virtual photons of energy less than $\Delta E$ cancel each other and also that the virtual photon spectrum must be cut off somehow for $k \gtrsim E$. Thus the infrared terms contribute a product of two logarithms and hence dominate the radiative corrections. With the use of a more refined treatment of the infrared terms at their high-energy limits, some of the single logarithm con-
tributions to the radiative corrections are also obtained. Together with certain "magnetic terms," which will not be discussed in detail here, these seem to give all the contributions of logarithmic order, and hence a very good estimate of the radiative corrections for any process.

The arguments regarding the extraction of infrared factors are contained in the Appendix. While no attempt is made to be completely rigorous, the importance of a correct treatment of overlapping infrared divergences is emphasized; and it is shown to be highly plausible that when the infrared factors are extracted, the remainder has no pathological properties. In Section 2, a thorough discussion is given of electron scattering in an external potential. This example, which is relatively simple, yet important, illustrates the main features which are present in most problems. In Section 3 several other examples are treated in a briefer manner. Section 4 contains a discussion of various considerations which enter a general treatment of the infrared problem. A general proof of the infrared cancellation is given, and some further examples are discussed briefly. In Section 5 some purely theoretical implications of the infrared phenomena are discussed. Here, our knowledge of the infrared factor to all orders gives some limited insight into theoretical problems in the high-energy limit. In this limit the infrared factor may become especially large, and we have much more detailed information on it than on any other contribution. Finally, in Section 6 are given the summary and discussion.

A few words about terminology may be helpful. In this paper the word "infrared" refers to a part of the matrix element (or cross section) in which the infrared dependence is factored out in a particular way. "Noninfrared" refers to the residual dependence. "Infrared-photon" is not synonymous with "soft-photon," since the infrared contribution is defined even for very large photon energies. We might define a photon as "soft" if the infrared contribution gives a good approximation to its behavior. According to a discussion at the end of Appendix A, a photon is then "soft" if its momentum is small compared to a typical momentum transfer in the process. Thus in wide-angle electron scattering at high energies, a photon is soft even though it removes several percent of the electron's energy; but in small angle scattering, it becomes "hard" when its momentum is comparable to or larger than the momentum transfer \( (2p \sin \theta; 2 \approx p\theta) \).

2. RADIATIVE CORRECTIONS TO ELECTRON SCATTERING

In this section we shall treat the virtual and real photon corrections to electron potential scattering. The infrared contributions will be factored out explicitly, identified with a well-defined set of diagrams, and discussed in some detail. We assume initially that the scattering center takes on momentum but no energy, with the consequences for undetected real photons that their total energy is just the energy loss of the scattered electron, and their total momentum \( K \) has the
bounds $0 \leq |K| \leq \epsilon$. Electron scattering obeying these assumptions is experimentally interesting and relatively simple to discuss; in later sections the treatment will be generalized to more complicated problems.

(a) **Virtual Photon Radiative Corrections**

Consider a process in which a certain number of photons are produced while the electron scatters from a state of momentum $p$ to one of momentum $p'$. We hold the final state fixed and consider the radiative corrections to the lowest order matrix element. Let $M_n(p, p')$ be the contribution to the matrix element corresponding to all diagrams in which there are $n$ virtual photons, which are distinguished from the potential interactions in $M_0$. Real photon variables are suppressed. The complete matrix element is then

$$M(p, p') = \sum_{n=0}^{\infty} M_n(p, p').$$  \hspace{1cm} (2.1)

Since there are $n$ photons, we expect $M_n$ to have an infrared divergence of $n$th order. In fact, it is intuitively clear that $M_n$ will be a polynomial of degree $n$ in the logarithm of the infrared cutoff. The objective of our discussion of virtual photons will be to show that the $M_n$ have the structure

$$M_0 = m_0,$$  \hspace{1cm} (2.2a)

$$M_1 = m_0 \alpha B + m_1,$$  \hspace{1cm} (2.2b)

$$M_2 = m_0 \left( \frac{\alpha B}{2} \right)^2 + m_1 \alpha B + m_2,$$  \hspace{1cm} (2.2c)

$$M_n = \sum_{r=0}^{n} m_{n-r} \frac{(\alpha B)^r}{r!},$$  \hspace{1cm} (2.2d)

where $m_j$ is an infrared divergenceless function (independent of $n$) of order $\alpha^j$ relative to $M_0$, and each factor $\alpha B$ contains the infrared contribution from one virtual photon. From (2.1), (2.2) it follows immediately that

$$M = \exp(\alpha B) \sum_{n=0}^{\infty} m_n$$  \hspace{1cm} (2.3)

with all the infrared terms appearing in the exponential.

In order to prove (2.2) we begin with the definition

$$M_n = \frac{1}{n!} \int \cdots \int \prod_{i=1}^{n} \frac{dk_i}{k_i^2 \lambda^2} \rho_n(k_1, \cdots k_n).$$  \hspace{1cm} (2.4)

We use a photon mass $\lambda$; the equivalence of this treatment to the treatment with a minimum photon momentum will emerge later. The factor $1/n!$ is intro-
Fig. 1. A representation of the set of basic diagrams containing all possible potential interactions and any set of real photons and \((n - 1)\) virtual photons.

Fig. 2. A representation of the possible ways an additional virtual photon can be inserted into the diagrams of Fig. 1.
duced into (2.4) by symmetrization of the \( n \) virtual photons in \( \rho_n \). This symmetrization will play an essential role in obtaining the symmetric form (2.2).

Now consider \( \rho_n \) as a function of \( k_n \). Figure 1 represents the set of basic diagrams associated with the first \( (n - 1) \) photons and an arbitrary number of potential interactions. Figure 2 represents the possible ways the \( n \)th photon can be inserted into various basic diagrams. Diagrams in which both ends of the \( n \)th virtual photon terminate on external charge lines (Fig. 2a, b, c) are infrared in \( k_n \), as indicated in the Introduction. The remaining diagrams in which at least one end of the \( n \)th photon terminates on an internal line (Fig. 2d, e, f), are finite as \( k_n \to 0 \) if all other virtual photon momenta \( k_i \) are nonzero. As \( k_n \to 0 \) and \( k_i \to 0 \) simultaneously, overlapping divergences in \( k_n \) and \( k_i \) arise. In Appendix A we demonstrate that all overlapping infrared divergences cancel when individually nongauge-invariant terms are combined into gauge invariant expressions. The only remaining divergence corresponds to Fig. 2a, b, c with \( k_n \) set equal to zero inside the basic diagram. This allows us to write

\[
\rho_n(k_1 \cdots k_n) = S(k_n)\rho_{n-1}(k_1 \cdots k_{n-1}) + \beta^{(1)}_n(k_1 \cdots k_{n-1} ; k_n),
\]

(2.5)

where \( S(k_n) \) contains the \( k_n \) infrared contribution from Fig. 2. The integral of the remainder \( \beta \) has no infrared divergence in \( k_n \), and its infrared divergence in the other \( k_i \)'s has not been made worse by the separation. We shall often refer to this property by saying "\( \beta \) is noninfrared in \( k_n \)."

Iteration of (2.5) yields

\[
\rho_n(k_1 \cdots k_n) = S(k_n)S(k_{n-1})\rho_{n-2}(k_1 \cdots k_{n-2}) + \\
S(k_n)\beta^{(1)}_{n-1}(k_1 \cdots k_{n-2} ; k_{n-1}) + \\
S(k_{n-1})\beta^{(1)}_{n-2}(k_1 \cdots k_{n-2} ; k_n) + \\
\frac{1}{2} - S(k_{n-1})\beta^{(1)}_{n-1}(k_1 \cdots k_{n-2} ; k_n) + \beta^{(1)}_n(k_1 \cdots k_{n-1} ; k_n). \tag{2.6}
\]

The symmetry of \( \rho_n \) in \( k_{n-1} \) and \( k_n \) implies invariance of the bracketed quantity at the end of (2.6) under interchange of \( k_{n-1} \) and \( k_n \). Since the bracketed quantity is noninfrared in \( k_n \) for all values of the other \( k_i \), it must also be noninfrared in \( k_{n-1} \) for all values of the other \( k_i \). We represent this property by the notation

\[
\{ - S(k_{n-1})\beta^{(1)}_{n-1}(k_1 \cdots k_{n-2} ; k_n) + \beta^{(1)}_n(k_1 \cdots k_{n-1} ; k_n) \} = \beta^{(2)}_n(k_1 \cdots k_{n-2} ; k_{n-1} ; k_n). \tag{2.7}
\]

Equations (2.6) and (2.7) have the characteristic, expected from our discussion of long wavelength photons, that the infrared contributions from \( k_n \) and \( k_{n-1} \) are mutually independent. Repeated interations of (2.5) and exploitation of the
symmetry of $\rho_n$ now lead to the relation

$$
\rho_n(k_1 \cdots k_n) = S(k_1) \cdots S(k_n)\beta_0 \\
+ \sum_{i=1}^{n} S(k_1) \cdots S(k_{i-1})S(k_{i+1}) \cdots S(k_n)\beta_1(k_i) \\
+ \cdots + \sum_{i=1}^{n} S(k_i)\beta_{n-1}(k_1 \cdots k_{i-1}, k_{i+1} \cdots k_n) + \beta_n(k_1 \cdots k_n).
$$

(\beta_i is just $\beta_i^{(i)}$ with the superscript dropped.)

Equation (2.8) can be expressed as a sum over all permutations of $k_i$ and $k_j$:

$$
\rho_n(k_1 \cdots k_n) = \sum_{\text{perm}} \sum_{r=0}^{n} \frac{1}{r!(n - r)!} \prod_{i=1}^{r} S(k_i)\beta_{n-r}(k_{r+1} \cdots k_n).
$$

(2.9)

The $\beta_{n-r}$ are noninfrared. Insertion of (2.9) into (2.4) yields

$$
M_n = \sum_{r=0}^{n} \frac{1}{r!(n - r)!} \left( \int \frac{d^4k}{k^2 - \lambda^2} \right)^r \int \prod_{i=1}^{r-1} \frac{d^4k_i}{k_i^2} \beta_{n-r}(k_{r+1} \cdots k_n).
$$

(2.10)

(We shall drop $\lambda$ whenever it is not needed to prevent an infrared divergence.) Finally, the definitions

$$
\alpha B[p, p' (\epsilon)] = \int \frac{d^4k}{k^2 - \lambda^2} S(k)
$$

(2.11)

and

$$
m_r[p, p' (\epsilon)] = \frac{1}{r!} \int \prod_{i=1}^{r} \frac{d^4k_i}{k_i^2} \beta_r(k_1 \cdots k_r)
$$

(2.12)

convert (2.10) into the desired result (2.2). (Note that $B$ and $m_r$ depend on $\epsilon$ through the energy conservation relation $E' = E - \epsilon$).

(b) Real Photon Radiative Corrections

Equation (2.3) leads to a cross section proportional to $\exp(2\alpha B)$. The differential cross section for emission of $n$ undetected real photons with total energy $\epsilon$, symmetrized in the real photons, then has the form

$$
\frac{d\sigma_n}{d\epsilon} = \exp(2\alpha B) \frac{1}{n!} \int \prod_{m=1}^{n} \frac{d^3k_m}{(k_m^2 + \lambda^2)^{1/2}} \delta \left( \epsilon - \sum_{i=1}^{n} k_i \right) \\
\times \tilde{\beta}_n(p, p', k_1 \cdots k_n).
$$

(2.13)

The quantity $\tilde{\beta}_n$ which plays a role similar to that of $\rho_n$ in the treatment of virtual photons, is given by the absolute square of $\sum m_r$. It is defined for any $E' = E$.
\[ \sum k_n \text{, which need not coincide with } E' = E - \epsilon. \text{ However, the } \delta \text{ function ensures that } \tilde{\rho}_n \text{ contributes to (2.13) only when } E' - E - \epsilon. \text{ The sum over all possible undetected photons provides the complete differential cross section} \]

\[ \frac{d\sigma}{d\epsilon} = \lim_{\lambda \to 0} \sum_{n=0}^{\infty} \frac{d\sigma_n}{d\epsilon}. \quad (2.14) \]

Infrared terms can be factored out of \( \tilde{\rho}_n \) by the same methods employed for \( \rho_n \) (Eqs. (2.5) to (2.9)) because \( \tilde{\rho}_n \) is symmetric in the real photons and overlapping infrared divergences cancel in the same manner for both real and virtual photons. With the overlapping divergences cancelled out, photons must terminate exclusively on external charge lines (Fig. 3a, b) in order to contribute.

Fig. 3. A representation of the possible ways an additional real photon can be inserted into the basic diagrams of Fig. 1.
infrared terms. We obtain a relation analogous to (2.8):

\[ \bar{p}_n(k_1 \cdots k_n) = \tilde{S}(k_1) \cdots \tilde{S}(k_n) \beta_0 \]

\[ + \sum_{i=1}^{n} \tilde{S}(k_1) \cdots \tilde{S}(k_{i-1}) \tilde{S}(k_{i+1}) \cdots \tilde{S}(k_n) \beta_i(k_i) \]

\[ + \cdots + \sum_{i=1}^{n} \tilde{S}(k_1) \beta_{n-1}(k_1, \cdots k_{i-1}k_{i+1} \cdots k_n) + \tilde{\beta}_n(k_1 \cdots k_n) \] (2.15)

As before, \( \tilde{S} \) contains an infrared divergence (Fig. 3a, b) and \( \beta_i \) has none. We must evaluate \( \tilde{S} \) at \( E' = E - \epsilon \). On the other hand the cross section \( \beta_i (\cdots k_m \cdots) \) is defined on the energy shell \( E' = E - \sum k_m \), that is, \( \tilde{S} \) is defined only at \( E' = E \), \( \beta_i(k_i) \) is defined at any \( E' = E - k_i \), etc. Of course, because of \( \delta(\epsilon - \sum k) \prod \tilde{S}(k_i) \beta_i (\cdots k_m \cdots) \) contributes to (2.13) only when

\[ \epsilon = \sum k_i + \sum k_m \].

The energy-conserving \( \delta \) function in (2.13) is conveniently represented by\(^5\)

\[ \delta \left( \epsilon - \sum_{m=1}^{n} k_m \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \exp \left[ iy \left( \epsilon - \sum_{m=1}^{n} k_m \right) \right]. \] (2.16)

We can now substitute (2.16), (2.15), and (2.13) into (2.11). After some simple manipulations we obtain

\[ \frac{d\sigma}{d\epsilon} = \lim_{\lambda \to 0} \frac{2\alpha B}{2\pi} \int_{-\infty}^{\infty} dy e^{iyy} \]

\[ \times \exp \left[ \int_{\epsilon - \epsilon}^0 \frac{dk}{(k^2 + \lambda^2)^{1/2}} \tilde{S}[k, p, p'(\epsilon)]e^{-iky} \right] \]

\[ \times \left\{ \beta_0 + \frac{1}{n!} \sum_{m=1}^{\infty} \frac{1}{n!} \int \frac{dk_m}{k_m} e^{-iky} \beta_n(p, p', k_1 \cdots k_n) \right\}. \] (2.17)

Equation (2.17) is perhaps not as complicated as it looks; the first exponential represents the sum over virtual infrared photons, the second exponential containing \( \tilde{S} \) represents the similar sum over real infrared photons, the sum over \( n \) refers explicitly to noninfrared real photons, and \( \beta_n \) contains implicitly a similar sum over noninfrared virtual photons.

The real infrared photons in the third exponential of (2.17) are still kinematically connected with the other real photons by the factor \( e^{-iky} \) which ensures that \( \sum k = \epsilon \). In order to make the infrared photons kinematically independent, we write

\[ \int_{\epsilon - \epsilon}^{\epsilon} \frac{dk}{(k^2 + \lambda^2)^{1/2}} \tilde{S} e^{-iky} = 2\alpha B + D, \] (2.18)

\(^5\) A similar computational trick was employed previously by Jauch and Rohrlich (2, 3) and by Yennie and Suura (14).
where
\[ 2\alpha\beta[p, p'(\epsilon)] = \int^{k < \epsilon} \frac{d^3k}{(k^2 + \lambda^2)^{1/2}} S \tag{2.19} \]
and
\[ D[p, p'(\epsilon), y] = \int^{k < \epsilon} \frac{d^3k}{k} S(e^{-ik} - 1). \tag{2.20} \]

The term \(2\alpha\beta\) is independent of \(y\), while the integrand of \(D\) is well behaved as \(k \to 0\). The \(\exp(2\alpha\beta)\) violates the condition \(\sum k = \epsilon\), but \(\exp(D)\) just restores this condition and thereby maintains energy conservation.

Equations (2.18) to (2.20), together with the definition
\[ \frac{d\sigma}{d\epsilon} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \ e^{iy\mu + \nu} \left\{ \hat{B}_0 + \sum_{\beta=1}^{n} \int \frac{d^3k}{k} e^{-ik}\beta_{\mu}\right\} \tag{2.21} \]
for the noninfrared part of (2.17), now allow us to write (2.17) in the important form
\[ \frac{d\sigma}{d\epsilon} = \exp \left\{ \lim_{\lambda \to 0} \frac{1}{2\alpha}(B + \beta) \right\} \frac{d\tilde{\sigma}}{d\epsilon}. \tag{2.22} \]

The factor \(d\sigma/d\epsilon\) is independent of the soft photon limit (i.e., it is finite as \(\lambda \to 0\)), whereas \(B\) and \(\beta\) represent known infrared contributions from the lowest order radiative correction. Thus the cancellation of infrared terms to all orders is assured by the well-known cancellation in lowest order (5–13); the equivalence to all orders of treatments involving a minimum photon momentum to calculations with a photon mass follows from the well-known equivalence in lowest order.

(c) DETAILS OF INFRARED FACTORS

As we have seen, \(B\) arises from the diagrams in Fig. 2(a), (b), (c), and \(\beta\) from Fig. 3 (a), (b). These diagrams have the special characteristic that the photon which contributes to \(B\) or \(\beta\) is associated with external charge lines, and is entirely independent of the internal details of the diagram. According to Appendix A, \(B\) and \(\beta\) can be represented by the gauge invariant expressions
\[ B = \frac{i}{(2\pi)^3} \int \frac{d^4k}{k^2 - \lambda^2} \left( \frac{2p^\mu_k - k^\mu}{2p^\mu_k - k^2} - \frac{2p^\mu_k - k^\mu}{2p^\mu_k - k^2} \right)^2 \tag{2.23} \]
and
\[ \beta = \frac{-1}{8\pi^2} \int \frac{d^4k}{(k^2 + \lambda^2)^{3/2}} \left( \frac{p^\mu_k}{k^\mu} - \frac{p^\mu_k}{k^\mu} \right)^2. \tag{2.24} \]
In (2.24) we have let \(2p \cdot k - \lambda^2 \to 2p \cdot k\) and \(2p_\mu - k_\mu \to 2p_\mu\), etc., because the omitted terms are found to vanish as \(\lambda \to 0\). Calculation reveals that the infrared contribution to \(B\) arises from the poles in

\[
\frac{1}{k^2 - \lambda^2 + i\epsilon} = \text{P.V.} \frac{1}{k^2 - \lambda^2} - i\pi \delta(k^2 - \lambda^2),
\]

where the virtual photon in \(B\) becomes effectively real. These poles contribute the amount

\[
\frac{1}{8\pi^2} \int \frac{d^3k}{(k^2 + \lambda^2)^{1/2}} \left( \frac{2p'_\mu - k_\mu}{2p' \cdot k - \lambda^2} - \frac{2p_\mu - k_\mu}{2p \cdot k - \lambda^2} \right)^2
\]

(2.26)

to \(B\). As \(k \to 0\), the diverging integrands of (2.26) and \(\bar{B}\) cancel. The contributions from poles in the electron propagators \((2p \cdot k - k^2)^{-1}\) and \((2p' \cdot k - k^2)^{-1}\) are finite as \(\lambda^2 \to 0\) in the present case (in Section 4(b) we shall discuss cases where these contributions diverge).

When \(p' \to p\), \(\bar{B}\) vanishes in accordance with the classical result that an unaccelerated particle radiates no energy. The similar vanishing of \(B\) as \(p' \to p\) reflects the cancellation of wave function and vertex renormalizations, which ensures the uniqueness of the physical electric charge. The integral in our representation of \(B\) has also been chosen to converge as \(k^2 \to \infty\), so \(B\) requires no ultraviolet cutoff.

At high energies and small \(\epsilon\), \(B\), and \(\bar{B}\) have the approximate forms (see Appendix C)

\[
B = -\frac{1}{2\pi} \left[ \ln \frac{2p \cdot p'}{m^2} \left( \ln \frac{m^2}{\lambda^2} + \frac{1}{2} \ln \frac{2p \cdot p'}{m^2} - \frac{1}{2} \right) - \ln \frac{m^2}{\lambda^2} \right]
\]

(2.27)

and

\[
\bar{B} = \frac{1}{2\pi} \left[ \ln \frac{2p \cdot p'}{m^2} \left( \ln \frac{m^2}{\lambda^2} + \frac{1}{2} \ln \frac{2p \cdot p'}{m^2} - \ln \frac{EE'}{\epsilon^2} \right) - \ln \frac{m^2}{\lambda^2} + \ln \frac{EE'}{\epsilon^2} \right].
\]

(2.28)

If we use a minimum photon momentum \(k_{\text{min}}\) instead of the photon mass \(\lambda\), (2.27) and (2.28) are individually quite changed:

\[
B = -\frac{1}{2\pi} \left[ \ln \frac{2p \cdot p'}{m^2} \left( \ln \frac{EE'}{k_{\text{min}}^2} - \frac{1}{2} \right) - \ln \frac{EE'}{k_{\text{min}}^2} \right]
\]

(2.29)

and

\[
\bar{B} = \frac{1}{2\pi} \left( \ln \frac{2p \cdot p'}{m^2} - 1 \right) \ln \frac{\epsilon^2}{k_{\text{min}}^2}.
\]

(2.30)
However, the sum $B(\lambda) + \bar{B}(\lambda)$ is the same as $B(k_{\text{min}}) + \bar{B}(k_{\text{min}})$:

$$2\alpha(B + \bar{B}) = -\frac{\alpha A}{2} \ln \frac{EE'}{e^2} + \frac{\alpha}{2\pi} \ln \frac{2p \cdot p'}{m^2}, \quad (2.31)$$

where

$$\alpha A = -\frac{k^2\alpha}{4\pi^2} \int d\Omega \left( \frac{p_\mu'}{p' \cdot k} - \frac{p_\mu}{p \cdot k} \right)^2.$$

(Note that (2.31) is not the same as the incorrect expression $B(\lambda) + B(k_{\text{min}})$ which happens to be easier to calculate.)

The sum of $B$ and $\bar{B}$ also reduces to a simple form in the limit of small momentum transfers

$$|q^2| = |(p' - p)^2| \ll m^2, \quad (2.33)$$

where

$$2\alpha(B + \bar{B}) = \frac{2\alpha q^2}{3\pi m^2} \ln \frac{E}{\epsilon}. \quad (2.34)$$

The arguments of Appendix A, and of Section 2 up to this point, also apply to potential scattering of charged bosons. This is not surprising, because the infrared factor is classical. The explicit equations (2.23) and (2.24) for $B$ and $\bar{B}$ are easily seen to represent the effects of convection currents, which are independent of the spin of the scattered particles. These equations, as well as (2.25)-(2.31), therefore hold for bosons. On account of the factor $\ln 2p \cdot p'/m^2$, however, $\exp [2\alpha(B + \bar{B})]$ is more important for electrons than heavier particles at present experimental energies.

(d) DETAILS OF NONINFRARED VIRTUAL PHOTON TERMS

We now discuss the virtual photon remainders (2.2),

$$m_1 = M_1 - \alpha BM_0 \quad (2.35)$$

$$m_2 = M_2 - \alpha BM_1 + \frac{(\alpha B)^2}{2!} M_0 \quad (2.36)$$

Note that the separation of $M_1$ into $\alpha BM_0$ and $m_1$ is not unique. Recoil terms such as $k^2$ in $(k^2 - 2p \cdot k)^{-1}$ do not affect the infrared divergence, but are retained in $R$ to make the integral (2.23) converge naturally as $k \to \infty$. Recoil currents such as $k_\nu$ in $(2p_\mu - k_\nu)$ do not contribute to the infrared divergence, but they have been retained in (2.23) to make $B$ gauge invariant.
What are the important contributions to $m_1$? In general, $m_1$ would be very complicated, so we shall give an explicit discussion only for the lowest order in the external potential. We observe that we have not completely evaluated the diagram in Fig. 2a for an electron. The “incoming” part of Fig. 2a can be represented by

$$\ldots \frac{(p - k_i + m)\epsilon_i}{k_i^2 - 2p \cdot k_i} u_p = \ldots \frac{1}{k_i^2 - 2p \cdot k_i} \frac{(2p - k_i) \cdot \epsilon_i - \frac{1}{2}[k_i, \epsilon_i]}{2 \epsilon_i} u_p. \quad (2.37)$$

The first term on the right is the “convection” current we have used in $B$, and the second term is a finite “magnetic” effect which would not arise on a spin zero charge line.

Straightforward calculation of the “magnetic” terms at high energies yields the contribution

$$\frac{\alpha M_0}{2\pi} \ln \frac{2p \cdot p'}{m^2} + O(\alpha M_0) \quad (2.38)$$

to $m_1$. Another sort of contribution is given by vacuum polarization of the potential. For example, if the potential acts on the electron once, vacuum polarization adds

$$\frac{\alpha M_0}{3\pi} \ln \frac{2p \cdot p'}{m^2} + O(\alpha M_0) \quad (2.39)$$

to $m_1$. The infrared factor $2\alpha (B + \bar{B})$ Eq. (2.31), plus twice the sum of (2.38) and (2.39) (twice because $M_1$ interferes with $M_0$; this has already been taken into account in $2\alpha (B + \bar{B})$), give the Schwinger correction of order $\alpha$ to electron-potential scattering (5).

When the external potential acts more than once on the electron, the calculation can still be arranged in such a way that the magnetic term (2.38) appears as one of the contributions. It is also quite likely that (2.39) serves as a good approximation to the total contribution from vacuum polarization in high-energy electron scattering from nuclei. This is because in this process most of the momentum transfer takes place at a single interaction. This one large momentum transfer contributes (2.39) and the vacuum polarization of the many additional interactions with low momentum transfer may be neglected. The remaining contributions to $m_1$ are not easily estimated. An elaboration of the methods of Appendix A indicates that these contributions are not of order $\alpha M_0 \ln^2 (E/m)$, but might possibly be of order $\alpha M_0 \ln (E/m)$. This conclusion is in agreement with general arguments of Suura (8), and the explicit calculation of radiative corrections to second-order Coulomb scattering by Newton (7) and Chrichton (7). In some recent calculations of various processes (6, 9–11), not even the single logarithm occurs.
To summarize, direct calculation yields

$$m_1 = \frac{5\alpha M_0}{6\pi} \ln \frac{2p \cdot p'}{m^2} + O(\alpha M_0)$$  \hspace{1cm} (2.40)$$

for a single potential interaction. If the potential acts more than once, one may possibly obtain additional single logarithm terms.

The results of this section can be compared with the closely related radiative corrections computed for $\gamma + P \rightarrow e^+ + e^- + P$ (11). Wide angle electron scattering, with associated production of bremsstrahlung at wide angles with respect to the initial and final electrons (12) corresponds to pair production at wide angles. The order $\alpha \ln (E/m) \ln (E'/\epsilon)$ and $\alpha \ln (E/m)$ corrections computed for pair production in this case (Eq. (29) of Ref. 11) agree exactly with the leading order corrections we have found: Eqs. (2.31) and (2.40). If only one of the pair is detected, the other may emerge parallel to the incident photon. In this case the electron propagator may become nearly real, and additional single logarithm terms arise from the integration over electron angles (Eqs. (17)-(22) of Ref. 11).

(e) DETAILS OF NONINFRARED REAL PHOTON TERMS

Having discussed the infrared terms $B$ and $\tilde{B}$ in (2.22), and the noninfrared virtual photon corrections contained implicitly in $\beta_n$ (2.21), we now turn to the noninfrared real photon corrections which appear explicitly in (2.21). The expansion of the real photon correction in $n$ is an expansion in the number of noninfrared real photons (photons with the spectrum $dk$ rather than $dk/k$ as $k \rightarrow 0$). Therefore the $n$th order contains $n$ powers $\alpha$ from noninfrared real photons, together with any number of $\alpha$'s from infrared $(dk/k)$ photons and noninfrared virtual photons. At least one photon must be emitted to make up the energy $\epsilon$ lost by the electron. The $n = 0$ term contains the $dk/k$ contribution from this one photon and the $n = 1$ term contains the $dk$, $k$ $dk$, $\cdots$ contributions from one photon, so the $n = 0$ and $n = 1$ terms both start with one power of $\alpha$.

We begin by considering the $n = 0$ term in (2.21):

$$\frac{d\sigma_0}{d\epsilon} = \beta_0 I,$$  \hspace{1cm} (2.41)$$

with $I$ defined as

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \, e^{iyn+\rho}.$$  \hspace{1cm} (2.42)$$

If all radiative corrections were ignored, $\beta_0$ would be just the elastic scattering cross section. After (2.15) we pointed out that $\beta_0$ is calculated on the energy shell, where $E' = E$. On account of (2.19), (2.20), (2.24) and (2.32), $D$ can be
written

\[ D = \alpha A \int_0^\infty \frac{dk}{k} (e^{-iyk} - 1). \quad (2.43) \]

\( I \) can be evaluated in terms of tabulated functions with the result

\[ I = \frac{\alpha A}{\varepsilon} F(\alpha A), \quad (2.44) \]

where

\[ F(\alpha A) = \frac{e^{-\alpha A \varepsilon}}{\Gamma(1 + \alpha A)} \]

\[ = 1 - \frac{\pi^2 (\alpha A)^2}{12} \quad \text{and} \quad C \text{ is Euler's constant.} \]

As the first step in deriving (2.44)-(2.45), we change \( D \) (2.43) to \( D' \):

\[ D' = D + \alpha A \int_\varepsilon^\infty \frac{dk}{k} e^{-iyk}. \quad (2.46) \]

Substitution of \( D' \) for \( D \) in \( I \) (2.42) has no effect on \( I \) because

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \, e^{iy(e^{-K})} = 0 \quad (K > \varepsilon) \quad (2.47) \]

as can be seen by closing the contour in the upper half plane. Evaluation of the \( \sin x/x \) and \( \cos x/x \) integrations in \( D' \) yields

\[ D' = \alpha A [-C - \ln (\varepsilon y) - i\pi/2] \quad (2.48) \]

The change of variable \( r = -iy\varepsilon \) in (2.48) now leads to

\[ I = \frac{e^{-\alpha A (\varepsilon + i|x|^2)}}{2\pi\varepsilon (-i)^{1/\alpha A}} \int_{i\infty}^{i\infty} dr e^{-r} \cdot \quad (2.49) \]

We shall evaluate (2.49) for \( 0 < \alpha A < 1 \), which is the usual physical case. The result can be analytically continued to the entire physical range \( \alpha A > 0 \). By closing the contour of integration to the right and taking the \( r^{-\alpha A} \) branch cut along the positive real axis, the integral in (2.49) can be expressed in terms of standard \( \Gamma \) functions. The relation

\[ \Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin \pi z} \quad (2.50) \]

then leads to (2.44) and (2.45).

\(^6\) An integral representation of \( F \) was given in Eq. (9) of Ref. 14.
The behavior of $F(\alpha A)$ for $\alpha A \gg 1$ is of no experimental interest, but does have some bearing on the theoretical behavior of quantum electrodynamics in the high-energy limit. According to (2.45), $F$ is of order $[(\alpha A)]^{-1} \sim \exp \left[-\alpha A \ln \alpha A\right]$ in this limit, thus providing a strongly convergent factor for the theory. Let us seek a physical explanation of this convergent factor; as we shall see, the effect can be interpreted in terms of a competition between the soft photons for the given amount of energy $\epsilon$. We recall that the probability of emitting a $dk/k$ photon, $\alpha A \; dk/k$, is proportional to $\alpha A$. When $\alpha A \gg 1$, then the average number of energetic $dk/k$ photons exceeds one, and the average energy available to each one will be much smaller than $\epsilon$. Our results show that the differential cross section for soft photons to carry off the energy $\epsilon$ is proportional to $(\epsilon)^{aA-1}$. For small $\alpha A$, the total energy carried off by the soft photons tends to be small; for large $\alpha A$, it tends to be large. Now consider one soft photon in competition with all the others. The differential cross section for that one photon to take an energy $k$ will contain a factor like $(\epsilon - k)^{aA-1}$, associated with the total energy carried off by the remaining photons. For small $\alpha A$, this will not seriously inhibit the set of values over which $k$ ranges; but for large $\alpha A$, it will cut down the cross section strongly for those values of $k$ which exceed $\epsilon / (\alpha A - 1)$. In this way competition effectively replaces $\epsilon$ by $\epsilon / \alpha A$ as the approximate upper limit of the energy each photon can independently take off at large $\alpha A$, so in the cross section $\exp \left(-\alpha A \ln E'/\epsilon\right)$ is modified by a factor $\exp \left(-\alpha A \ln \alpha A\right)$ which is approximately $F$. The damping due to $F$ at ultrahigh energies is an effect which will multiply each term (since $\exp \left(-\alpha A \ln E'/\epsilon\right)$ multiplies each term) in any process involving charged external particles. Of course, there are many other factors multiplying $F$ which we cannot analyze, so no definite conclusion about the convergence of the theory can be reached.

While discussing virtual photon corrections (Section 2d) we pointed out that the separation $M_1 = m_1 + \alpha B M_0$ was not unique. The separation of the contribution of real photons into an “infrared factor,” $2a \tilde{B}$ and “noninfrared factors,” $\tilde{\beta}_n$, is not unique either, because the upper limit of the “infrared photon” energy is arbitrary. The choice we have made for the upper energy limit, $k \leq \epsilon$, has the consequence that $F(\alpha A)$ (Eq. (2.45)) has no term of order $\alpha A$. In general, other choices for the upper limit in $2a \tilde{B}$ would give $F$ a finite $\alpha A$ term which would just compensate for the change in $2a \tilde{B}$.

Next we turn to the $n = 1$ term in (2.21). This term is given by

$$
\frac{d\sigma_1}{d\epsilon} = \int_{k_1}^{k_1' \leq \epsilon} \frac{d^3k_1}{k_1} \beta_1(k_1) \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy e^{iy(k_1 + k_1')}.
$$

(2.51)

The cross section $\beta_1$ contains the $dk$, $dk$, $\cdots$ terms from emission of one real photon. As we have mentioned after (2.15), in $\beta_1(k_1)$, $E'$ is set equal to $E - k_1$.
rather than \( E - \epsilon \). It is convenient to define \( G_1 \):

\[
G_1(k_1) = k_1 \int d\Omega \beta_1(k_1).
\]

The factor \( G_1(k_1) \) is finite at \( k_1 = 0 \).

The integration over \( y \) in (2.51) is handled in the same fashion as Eqs. (2.46) to (2.50), except that the appropriate substitution after (2.48) is now \( r = -iy(\epsilon - k) \) instead of \( r = -iy \epsilon \). We obtain, in place of (2.42),

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dy \ e^{iy(\epsilon - k_1) + \rho} = \left( \frac{\epsilon}{\epsilon - k_1} \right)^{1 - \alpha A} I(\alpha A, \epsilon)
\]

so that

\[
\frac{d\sigma_1}{d\epsilon} = \frac{\alpha A}{\epsilon} F(\alpha A) \int_0^\epsilon dk_1 G_1(k_1) \left( \frac{\epsilon}{\epsilon - k_1} \right)^{1 - \alpha A}.
\]

It can be shown that \( \alpha A \geq 0 \), which ensures the convergence of the integral. In (2.54), the energy loss \( \epsilon \) of the electron appears to be shared between a bundle of \( dk/k \) photons (from (2.53)) with total energy \( \epsilon - k_1 \) and one "\( dk \) photon" (in \( G_1 \)) with energy \( k_1 \). The \( (\epsilon - k_1)^{\alpha A - 1} \) factor from \( dk/k \) photons weights the integrand in (2.54) strongly towards \( k_1 \approx \epsilon \). This motivates an expansion of \( G_1(k_1) \) about \( G_1(\epsilon) \), with the result

\[
\frac{d\sigma_1}{d\epsilon} \approx F(\alpha A) \left\{ G_1(\epsilon) - \frac{\alpha A \epsilon}{\alpha A + 1} \frac{dG_1(k = \epsilon)}{dk} + \ldots \right\}.
\]

The \( G_1(\epsilon) \) term is of order \( \alpha(A) \) contains a factor \( \alpha \), and \( F \approx 1 \) in practical cases). The remaining (derivative) terms contain the effect of infrared photons in addition to the single "\( dk \) photon" in \( G_1 \).

We shall now specialize, for the remainder of Section 2, to the case \( \epsilon \leq E/2 \). In this case the derivative terms in (2.55) are of order \( \alpha^2 \epsilon/E \) and will be neglected. The case \( \epsilon > E/2 \) requires a different treatment which will be discussed in Section 3(a).

The higher order terms \( d\sigma_n/d\epsilon \) can be analyzed in the same fashion. It will be shown in Section 3a that these terms are of the following order of magnitude.

\[
\frac{d\sigma_n}{d\epsilon} \sim \alpha^{n-1} \left( \frac{\epsilon}{E} \right)^{n-1} \frac{d\sigma_1}{d\epsilon}
\]

but our ignorance of the other factors in \( d\sigma_n/d\epsilon \) blocks any understanding of the convergence properties of \( \sum_n d\sigma_n/d\epsilon \). Each higher order above \( n = 1 \) is reduced by a factor \( \alpha \), and calculation of the higher orders would be prohibitively difficult, so in accordance with the usual spirit of perturbation theory, we shall ignore terms with \( n > 1 \) in spite of our complete ignorance of their convergence proper-
ties. Then (2.22), (2.41), (2.44) and (2.55) give the differential cross section, with the infrared factor included. For \( \epsilon < E/2 \) the cross section simplifies to

\[
\frac{d\sigma}{d\epsilon} = F(\alpha A) \exp \left[ 2\alpha(B + \not{B}) \right] \left\{ \frac{\alpha A}{\epsilon} \beta_0 + G_1(\epsilon) \right\}.
\] (2.57)

The quantity in brackets is the conventional expression for bremsstrahlung separated into the \( d\kappa/d\kappa \) contribution and the remainder \( G_1 \). Thus the differential cross section for electron scattering with energy loss \( \epsilon \lesssim E/2 \) is essentially the cross section for emitting one photon of energy \( \epsilon \), multiplied by an exponential factor representing multiple soft photon effects.

(f) Electron Scattering with Energy Resolution \( \Delta E \)

The detection of potential scattered electrons with energy \( E' = E - \epsilon \) must always involve some experimental error. An important case is the scattering of electrons with \( 0 \leq \epsilon \leq \Delta E \), \( \Delta E \) being the energy resolution of the detecting device. The cross section for this case is

\[
\sigma = \int_{-\Delta E}^{\Delta E} \left( \frac{d\sigma}{d\epsilon} \right) d\epsilon,
\] (2.58)

which has the explicit form for high-energy scattering with \( \Delta E < E/2 \) [(2.57), (2.31)]:

\[
\sigma = \int_{-\Delta E}^{\Delta E} d\epsilon F(\alpha A) \exp \left\{ -\frac{\alpha A}{2} \ln \frac{EE'}{\epsilon^2} + \frac{\alpha}{2\pi} \ln \frac{2p'p'}{m^2} \right\} \left[ \frac{\alpha A}{\epsilon} \beta_0 + G_1(\epsilon) \right].
\] (2.59)

The exponential and \( A \) depend on \( \epsilon \) via the relation \( E' = E - \epsilon \). As we pointed out after (2.15), \( p' \) is not a function of \( \epsilon \) in \( \beta_0 \). We now set \( E' = E \) everywhere in the integrand of (2.59) except the three places where \( \epsilon \) dependence is explicitly indicated; the error introduced by doing so is of the order \( \alpha \epsilon/E \). Then integration of (2.59) yields

\[
\sigma \approx \int_{-\Delta E}^{\Delta E} F(\alpha A) \exp \left\{ \alpha \ln \frac{2p'p'}{m^2} \right\} \beta_0 \exp \left( -\alpha A \ln \frac{E}{\Delta E} \right) + \int_{0}^{\Delta E} d\epsilon G_1(\epsilon) \exp \left( -\alpha A \ln \frac{E}{\epsilon} \right)
\] (2.60)

The first term in (2.60) is the product of the elastic cross section \( \beta_0 \) [which includes the finite radiative corrections discussed in Section 2(d)] and factors representing soft photon effects. We shall see that if \( \Delta E \lesssim E/2 \), the second \( (G_1) \) term in (2.60) is smaller than the first by a factor \( \sim \alpha(\Delta E/E) \ln (E/m) \).

Several types of terms are included in \( G_1(\epsilon) \): (i) the magnetic moment-interaction (see discussion after (2.37)) when the photon is emitted from an external
line; (ii) emission of photons from internal lines; and (iii) contribution of \( k \) in internal lines. The magnetic moment contribution (i) amounts to a term of relative order \( k/E \) in the differential bremsstrahlung cross section. Compared to elastic scattering, it is \( O(\alpha(\Delta E/E)\ln(E/m)) \). These estimates have to be reappraised in a process such as \( e + e \rightarrow e + e \), where an \( e \) particle is at rest \( (E = m) \) in the laboratory.\(^7\) The combined effect of (ii) and (iii) can be crudely estimated from the work of the Appendix. One obtains at most contributions of relative order \( \alpha(\Delta E/E) \ln(E/m) \). In case the scattering is nearly elastic \( G_1(\epsilon) \) can be replaced by \( G_1(0) \), which is rather simply related to \( \beta_0 \) by gauge invariance and is therefore easy to calculate (\(^{16}\)).

Emission of \( d\epsilon/k \) photons from the recoiling potential provides a further correction to our previous treatment. Recoil effects are negligible in most cases (\(^{17}\)) because of the great disparity in mass between the electron and potential, but they become significant in high-energy electron-proton scattering. Recoil effects will be discussed in Section 3(d).

To close this discussion of electron-potential scattering, a numerical estimate of the various terms contributing to the total cross section in (260) may be helpful. If \( E = 500 \text{ Mev}, \Delta E = 5 \text{ Mev}, \) and \( p \cdot p' = E' E' \), then \( F(\alpha A) = 0.99 \), the combined exponential is 0.73, the finite virtual photon corrections (2.40) to \( \beta_0 \) increase \( \beta_0 \) by about 7\%, and the \( G_1 \) term is negligible. It should be noted that the value 0.73 for the exponential includes a \( \alpha^2 \) contribution from the \( \alpha^2 \) term in the expansion. We believe that the other \( \alpha^2 \) contributions not included in the soft photon exponential are smaller by a factor \( (\ln(E/\Delta E))^{-2} \).

3. FURTHER EXAMPLES

In this section we shall give some further examples which can be handled by simple extensions of the methods described in the preceding section. The first of these will be electron scattering with large energy loss. This will involve nothing new in principle, but will be simply an elaboration of the results of Section 2. The second example will be radiative corrections to inelastic electron scattering cross sections, such as those associated with nuclear excitation. The third example will be multiple photon production with results in substantial agreement with those of Gupta (\(^{18}\)). The fourth example will be radiative corrections to electron scattering with nuclear recoil taken into account in lowest order. The final example will be a brief discussion of the bremsstrahlung produced in the scattering of nucleons by nuclei. Although this work will be described in greater detail elsewhere (\(^{19}\)), it is included here as an example of the infrared phenomenon.

\(^7\) Y. S. Tsai (private communications) and Ref. 10.
(a) **Electron Scattering with Large Energy Loss**

The complete result for electron scattering with energy loss $\epsilon$ is given by (2.21) and (2.22). Introducing the definition

$$\hat{G}(k) = \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \prod_{m=1}^{n} \frac{d^3k_m}{k_m^3} \delta(k - \sum k_m) \hat{\beta}_n(k_1, \ldots, k_n)$$

$$= G_1(k) + G_2(k) + \cdots,$$

the cross section for inelastic scattering may be expressed

$$\frac{d\sigma}{d\epsilon} = \exp \left[2\alpha(B + \vec{B})\right] \frac{\alpha A}{\epsilon} F(\alpha A) \left\{ \hat{\beta}_0 + \int_0^\epsilon dk \hat{G}(k) \left( \frac{\epsilon}{\epsilon - k} \right)^{1-\alpha A} \right\}. \quad (3.2)$$

The integral in (3.2) is seen to be the convolution of a cross section for the production of noninfrared photons of total energy $k$ with the probability for emission of an indefinite number of infrared photons whose total energy is $(\epsilon - k)$. Since $\alpha A$ is usually small compared to 1 (it is about 0.07 in the example cited at the end of Section 2), the main contribution to the integral in (3.2) comes from the region $k \approx \epsilon$. This allows us to extract the leading contribution to the integral by the following procedure

$$\frac{\alpha A}{\epsilon} \int_0^\epsilon dk \hat{G}(k) \left( \frac{\epsilon}{\epsilon - k} \right)^{1-\alpha A}$$

$$= \hat{G}(\epsilon) + \alpha A \int_0^\epsilon \frac{\hat{G}(k) - \hat{G}(\epsilon)}{(\epsilon - k)} \left( \frac{\epsilon - k}{\epsilon} \right)^{\alpha A} dk \quad (3.3)$$

$$\approx \hat{G}(\epsilon) + \alpha A \int_0^\epsilon \frac{\hat{G}(k) - \hat{G}(\epsilon)}{\epsilon - k} \frac{1}{\epsilon} \, dk.$$

The error involved in the approximate form is of order $(\alpha A)^2$ compared to $\hat{G}$. For small $k$, it is expected that $\hat{\beta}_n$ will have the form

$$\int \cdots \int d\Omega_m \hat{\beta}_n(k_1, \ldots, k_n) = \alpha^n K_n \left( \frac{E}{m} \right) \hat{\beta}_0 \prod_{i=1}^{n} \frac{1}{k_i E}. \quad (3.4)$$

The $k_i$ dependence in (3.4) is chosen to give the leading order noninfrared term and the remaining dependence is fixed dimensionally. This gives

$$G_n(k) = \frac{1}{n!(n - 1)!} \left( \frac{\alpha k}{E} \right)^{n-1} \frac{K_n}{K_1} G_1(k), \quad (3.5)$$

which leads directly to (2.56). The presence of the factor $[n! (n - 1)!]^{-1}$ makes likely the convergence of the series of real photon contributions, but of course nothing is known about the properties of $K_n$ as a function of $n$. For small $\epsilon$, the results of Section 2 given in (2.55) and (2.57) are obtained immediately from (3.3).
Now let us turn to the discussion of electron scattering with large energy loss. The quantity

$$k \int d\Omega \, \tilde{\sigma}_1(k) = \frac{\alpha A}{k} \bar{\beta}_0 + G_1(k)$$  \hspace{1cm} (3.6)$$

is simply the cross section for electron scattering with emission of a single photon of energy $k$, including the radiative corrections due to noninfrared virtual photons. This cross section has been calculated to lowest order by Racah (20) who used the Born approximation for the potential, and neglected radiative corrections. There are four types of radiative corrections to the lowest order result.

(i) Those $dk/k$ real and virtual photon corrections which are known exactly and are incorporated in $B + \bar{B}$ and $F$. (ii) Those arising from virtual noninfrared photons; they are probably of relative order $\alpha A$. (iii) Those arising from real noninfrared photons; they are contained in $G_i$ for $i \geq 2$ and are also probably of relative order $\alpha A$. (iv) The corrections represented by the second term on the right side of (3.3); these are also from the $dk/k$ part of the spectrum. They are of order $\alpha A$, but may be large if the $\tilde{G}$ is rapidly changing in the vicinity of $\epsilon$. We shall not attempt a complete calculation of the radiative corrections here, but only those associated with $dk/k$ photons.

For purposes of illustration, we shall use a simple approximation to (3.6), rather than the complete result of Racah (20). In Born approximation, the cross section for electron scattering at high energies in an external potential takes the form

$$\sigma_B(\theta, E) = W(q^2)p^2 \cos^2 \frac{1}{2} \theta \sigma, \hspace{1cm} (3.7)$$

where $q = \mathbf{p} - \mathbf{p}'$ and the function $W$ is proportional to the square of the Fourier transform of the external potential. If we neglect the magnetic terms in the amplitude for emission of a real photon, the differential cross section for scattering of the electron in the direction $\theta$, together with the emission of a photon of momentum $\mathbf{k}$ is given by

$$\bar{S}(\mathbf{k}) W[(\mathbf{p} - \mathbf{k} - \mathbf{p}')^2]p'^2 \cos^2 \frac{1}{2} \theta \delta(\mathbf{p} - \mathbf{p}' - \mathbf{k}) \, dp' \, \frac{d^2k}{k}. \hspace{1cm} (3.8)$$

Since the photon is not observed, this is to be integrated with respect to $k$. The result can be estimated by noting that the function $\bar{S}(\mathbf{k})$ peaks up very strongly when $\mathbf{k}$ is parallel either to $\mathbf{p}$ or to $\mathbf{p}'$. Thus for $\mathbf{k}$ nearly parallel to $\mathbf{p}$, we approximate

$$\bar{S}(\mathbf{k}) \approx \frac{\alpha}{4\pi^2} \left[ \frac{2 |\mathbf{p}|}{|\mathbf{k}|(k \cdot p)} - \frac{m^2}{(k \cdot p)^2} \right].$$

\footnote{For a better method of approximation in which the magnetic terms are not neglected, see Schiff (21).}
The integrals are now easily performed, and we find for the leading (logarithmic) contribution

\[
\sigma_1(\epsilon) \approx \frac{2\alpha}{\pi \epsilon} \left[ \sigma_0(\theta, E) \frac{p'^2}{p^2} \ln \frac{E'}{m} + \sigma_0(\theta, E') \ln \frac{E'}{m} \right],
\]

where

\[p' \approx E' = E - \epsilon.\]

The two terms in the bracket can be interpreted as follows. The first term represents the contribution from a photon emitted parallel to the final velocity of the electron; thus it contains potential scattering at the full momentum transfer associated with elastic scattering. The second term represents the contribution from a photon emitted parallel to the initial velocity and it contains potential scattering at the smaller momentum transfer associated with the final electron energy. A typical distribution curve is shown in Fig. 4. The rise which occurs as \(\epsilon\) approaches \(p\) comes from the second term of (3.9). This is due to the fact that,

---

**Fig. 4.** A plot of the spectrum of scattered electrons which have lost energy to bremsstrahlung. The decomposition of the spectrum into an infrared part \((\alpha A \beta_0/\epsilon)\) and a non-infrared part \((G_1(\epsilon))\) is also shown.
at a fixed scattering angle, the cross section for scattering in a Coulomb potential increases rapidly as the energy is decreased; the behavior is even more pronounced for scattering from an extended charge distribution. Also shown in the figure are the separate contributions due to the two terms of (3.6). From (3.9) we find

\[
G_1(k) \approx \frac{2\alpha}{\pi k} \left\{ \sigma_\theta(\theta, E - k) \ln \frac{E}{m} + \sigma_\theta(\theta, E) \right. \\
\left. \cdot \left[ \left( \frac{E - k}{E - \epsilon} \right)^2 \ln \left( \frac{E - k}{m} \right) - 2 \ln \frac{E}{m} \right] \right\}.
\] (3.10)

In order to estimate the integral in (3.3), we make the following additional approximations in (3.10). We neglect the \( k \) inside the logarithm; this is justifiable since the logarithm is slowly varying, and when \( k \) becomes large that term becomes small compared to the other ones. In addition, we choose \( \sigma_{\theta} \) to be a point Coulomb cross section; it is then proportional to \( E^{-2} \). With these simplifications, (3.10) becomes

\[
G_1(k) \approx \frac{2\alpha k}{\pi E^2} \left( \frac{2E - k}{E - k} \right)^2 \sigma_\theta(\theta, E) \ln \frac{E}{m}.
\] (3.11)

The integral in (3.3) may now be easily evaluated, and we find for the inelastic scattering cross section

\[
\frac{d\sigma}{d\epsilon} \approx \exp \left[ 2\alpha(B + \tilde{B}) \right] F(\alpha A) \left\{ \frac{\alpha A}{\epsilon} \sigma_\theta(\theta, E) + G_1(\epsilon)[1 + \alpha A R] \right\},
\] (3.12)

where

\[
R = -\left( \frac{E}{2E - \epsilon} \right)^2 \left\{ 1 + \frac{(E - \epsilon)^2}{E^2} + \frac{2E - \epsilon}{\epsilon} \ln \frac{E}{E - \epsilon} \right\}.
\]

For large \( \epsilon \), the correction represented by the term \( \alpha A R \) can become sizable; for example, if \( \epsilon = 0.9E \), the correction is approximately \( 3.2\alpha A \) times the single photon cross section. At typical electron scattering energies, this might amount to a 15 to 20% correction. The effect could be further enhanced by the form factor dependence (associated with the finite size of the nucleus) which we have neglected in passing from (3.10) to (3.11).

(b) **Inelastic Electron Scattering**

In this type of process the electron excites the target in some manner which we need not consider; examples would be nuclear excitation, electrodisintegration of the deuteron or meson production. Without radiative corrections such processes result in a spectrum in the energy of the scattered electrons which we represent at a given angle by the function \( \hat{\sigma}(\epsilon) \). The differential cross section
for the electron to lose an amount of energy $\epsilon'$ to the target, together with the emission of $n$ photons, is given by an expression like (2.15). Treating all the photons in the infrared limit, we have

$$\tilde{\sigma}_n \approx \prod_{i=1}^{n} S(k_i) \tilde{\sigma}(\epsilon').$$  \hspace{1cm} (3.13)

The differential cross section for the electron to lose a total energy $\epsilon$ while exciting the target to the energy $\epsilon'$ is then given by

$$\frac{d^2\sigma(\epsilon, \epsilon')}{d\epsilon \, d\epsilon'} \approx \exp \left[ 2\alpha(B + \tilde{B}) ] F(\alpha A) \tilde{\sigma}(\epsilon') \frac{\alpha A}{\epsilon} \left( \frac{\epsilon - \epsilon'}{\epsilon} \right)^{aA-1} \right].$$  \hspace{1cm} (3.14)

If the final state of the target is not observed, the energy spectrum of the inelastically scattered electrons is

$$\tilde{\sigma}_{\tilde{\sigma}}(\epsilon) \approx \exp \left[ 2\alpha(B + \tilde{B}) ] F(\alpha A) \int_0^\epsilon d\epsilon' \tilde{\sigma}(\epsilon') \frac{\alpha A}{\epsilon} \left( \frac{\epsilon - \epsilon'}{\epsilon} \right)^{aA-1} \right].$$  \hspace{1cm} (3.15)

(The observable cross section is of course the sum of (3.2) and (3.15).) As before, this is seen to be the convolution of the cross section for inelastic scattering (energy loss $\epsilon'$) with the probability of soft photon emission (of total energy $(\epsilon - \epsilon')$). The radiative correction will result in a change in the shape of the inelastic spectrum; in particular, narrow lines will be transformed into lines with the characteristic bremsstrahlung tail.

It is interesting to notice that the total area under the inelastic scattering curve is altered very little by the radiative corrections. Neglecting the dependence of $\alpha A$ upon $\epsilon$, and using (2.31), we see that the area under the corrected curve between 0 and $\tilde{\epsilon}$ is given by

$$F(\alpha A) \int_0^{\tilde{\epsilon}} d\epsilon' \tilde{\sigma}(\epsilon') \left( \frac{\epsilon - \epsilon'}{\epsilon} \right)^{aA-1} \tilde{\sigma}(\epsilon').$$

Carrying out the $\epsilon$ integration first, and noting that the main contribution comes from $\epsilon \approx \epsilon'$, we find

$$\int_0^{\tilde{\epsilon}} \tilde{\sigma}_{\tilde{\sigma}}(\epsilon) \, d\epsilon \approx F(\alpha A) \int_0^{\tilde{\epsilon}} \left( \frac{\epsilon - \epsilon'}{\sqrt{E(E - \epsilon')}} \right)^{aA} \tilde{\sigma}(\epsilon') \, d\epsilon' \approx F(\alpha A) \left( \frac{\tilde{\epsilon} - \tilde{\epsilon}'}{\sqrt{E(E - \tilde{\epsilon}')}} \right)^{aA} \int_0^{\tilde{\epsilon}} \tilde{\sigma}(\epsilon') \, d\epsilon'. $$  \hspace{1cm} (3.17)

The second form holds when $(\tilde{\epsilon} - \tilde{\epsilon}')$ is large compared to the range of values of $\epsilon'$, for which $\tilde{\sigma}(\epsilon')$ is non-negligible, and $\tilde{\epsilon}'$ represents a typical value of $\epsilon'$ in this range. If $\tilde{\epsilon}'$ is small compared to $E$, the corrected area is $\exp (\alpha A \ln \Delta E/E) F(\alpha A)$.
times the uncorrected area, where $\Delta E = \epsilon - \epsilon'_0$ represents an energy resolution associated with the measurement of the inelastic peak. The resemblance to (2.60) is obvious.

(c) Multiple Photon Production

Multiple production of high-energy photons has been discussed by Gupta (18). The machinery we have developed in Section 2 allows us to rederive and extend Gupta’s results easily. The difference between our results and Gupta’s will be small in most cases of experimental interest, but may be significant if one wishes to use the results in a theoretical discussion of the high-energy limit of quantum electrodynamics.

We shall attempt to give a discussion applicable to cosmic ray work. Consider an electron, whose energy $E$ can only be estimated crudely, scattering into a definite angle. The scattering is accompanied by emission of photons. The detection device has the property that a photon is detected if its energy exceeds some value $K_a$, undetected if its energy is less than $K_a$. In practice $K_a$ might be the energy necessary to produce a pair. Suppose there are $N$ detected photons, with energies $k_i$ and angles $\Omega_i$.

It is convenient not to symmetrize detected photons with the undetected ones. The discussion of Section 2 concerning electron-potential scattering can be carried over with little change to multiple bremsstrahlung production. There is one change; since each undetected photon can assume energies up to $K_a$ and the total photon energy is not fixed, the function $F$ (Eq. (2.45)) does not appear in our present case. If the average number of undetected photons with energy $\sim K_a$ approached $E/K_a$ in magnitude, some function analogous to $F$ would be required because of energy conservation, but we shall not consider this experimentally unrealistic case explicitly. Keeping only the $dk/k$ contributions from undetected photons, we obtain the cross section for production of $N$ detected photons:

$$d\sigma = \frac{\prod_{i=1}^{N} dk_i d\Omega_i}{\prod_{i=1}^{N} dk_i d\Omega_i} \approx e^{-\sigma_{\text{in}} E/K_a} \frac{d\sigma}{\prod_{i=1}^{N} dk_i d\Omega_i},$$

(3.18)

where $d\sigma$ is the cross section in the absence of soft photon radiative corrections. Next let us express $d\sigma$ in terms of $dk/k$ contributions from the $N$ detected photons:

$$\prod_{i=1}^{N} dk_i d\Omega_i \approx \frac{\sigma_0}{N!} \left(\frac{-\alpha}{4\pi^2}\right)^N \prod_{i=1}^{N} k_i \left(\frac{p'_\mu}{p' \cdot k} - \frac{p_\mu}{p \cdot k}\right)^2,$$

(3.19)
where $\sigma_0$ is the elastic electron-potential scattering cross section. Successive integrations of (3.19) over all angles $\Omega_i$ and energies $K_i < K < K_b$ yield

$$\frac{d\sigma}{d\Omega_i \prod \frac{1}{k_i}} \approx \sigma_0 e^{-\alpha A \ln E/K_a} \frac{(\alpha A)^N}{N!} \prod_{i=1}^{N} \frac{1}{k_i}$$

(3.20)

and

$$\sigma \approx \sigma_0 e^{-\alpha A \ln E/K_a} \frac{(\alpha A \ln (K_b/K_a))^N}{N!}$$

(3.21)

Gupta computed $\bar{\sigma}$ rather than $\sigma$, so our results differ from his by the exponential factor.

As Gupta has pointed out, the long range nature of the Coulomb potential favors small momentum transfers, and the average momentum transfer to the electron in high-energy bremsstrahlung is only $|q^2| \sim m^2$. This average momentum transfer corresponds to electron scattering through an angle $\theta \sim m/E$. At such small angles the initial and final electron currents tend to cancel, with the result that $\bar{\sigma} \sim \frac{1}{m}$. Multiple bremsstrahlung is highly improbable. Our Eqs. (3.20) and (3.21) leave this conclusion of Gupta unchanged, because $\exp [-\alpha A \ln E/K_a]$ is approximately one at small momentum transfers.

Multiple bremsstrahlung assumes some importance relative to single bremsstrahlung only in the rare cases when the electron scatters into large angles ($\theta \gg m/E$) or loses most of its energy to bremsstrahlung, ($|q^2| \sim |(E - m)^2 - p^2| \sim mE$). In these cases our soft photon factor does decrease Gupta's result somewhat. As an extreme example, consider the cosmic ray case $E \sim 10^{13}$ ev, $K_a \sim 10^6$ ev. The maximum possible momentum transfer from a nucleus with mass $M$ is of order $\sqrt{M}E$. With $M \sim 10^{13}$ ev, we obtain $\alpha A = 0.15$ and $\exp [-\alpha A \ln E/K_a] = \exp (-2.8)$.

Values of $\alpha A$ in excess of one are reached only in the region $|q^2|/m^2 \sim \exp (\pi/2\alpha)$ where there is considerable doubt concerning the consistency of quantum electrodynamics. In this region the soft photon factor makes Gupta's result more convergent, although neither Gupta's result nor ours is really applicable in view of the many terms that have been neglected.

The discussion of bremsstrahlung in this section can easily be extended to other multiple photon processes.

(d) The Effect of Recoil on Electron Scattering

Recoil corrections to electron scattering have been considered by Foldy et al. (22), but without radiative corrections. There are two types of corrections; the kinematical ones, which are associated with the transformation from the center-of-mass coordinate system to the laboratory system and the dynamical ones,
which are associated with a modification of the cross section in the center-of-mass system due to terms of order \((E/M)\) in the Hamiltonian, where \(M\) is the mass of the recoiling nucleus. The dynamical corrections to the cross section turn out to be of relative order \((m^2/ME)\) and are quite negligible. The corrections that arise when recoil and radiative corrections are considered simultaneously will be called radiative recoil corrections. Clearly the radiative corrections to the ordinary dynamical corrections will be quite negligible; we will neglect any such terms that arise. However, the interaction of the recoiling nucleus with the quantized electromagnetic field must not be neglected; it gives an amplitude for soft photon production of order \((ZE/M)\) times that of the electron. In addition, there will be kinematical corrections due to the fact that if the electron has a given energy loss, the missing energy will be shared between the photons and the recoiling nucleus. We shall not attempt to give a complete discussion including an arbitrary number of photons, but shall consider only the one photon correction; at the end of the sub-section a few brief remarks will be made about the treatment of many soft photons.

Let us consider the dynamical radiative recoil corrections first. In the infrared limit, these are obtained by making the following substitution for the factor associated with the emission of the photon

\[
\left( \frac{p_\mu}{k \cdot p} - \frac{p'_\mu}{k' \cdot p'} \right) \rightarrow \left( \frac{p_\mu}{k \cdot p} - \frac{p'_\mu}{k' \cdot p'} \right) - Z \left( \frac{P_\mu}{k \cdot P} - \frac{P'_\mu}{k' \cdot P'} \right),
\]

where \(p, p'\) are the four-momenta of the electron before and after scattering, and \(P, P'\) are the corresponding four-momenta of the target nucleus, which has charge \(Z\). To obtain the differential cross section for electron scattering with given energy loss, it is necessary to integrate the square of (3.22), over photon angles. The contribution from the square of the first term is given by (2.52). The effect of recoil is thus given by the substitution

\[
A \rightarrow A - \frac{k^2}{4\pi^2} \int d\Omega \left( -2Z \left( \frac{p_\mu}{k \cdot p} - \frac{p'_\mu}{k' \cdot p'} \right) \left( \frac{p^{\mu}}{k \cdot P} - \frac{p'^{\mu}}{k' \cdot P'} \right) \right.
\]

\[
+ Z^2 \left( \frac{P_\mu}{k \cdot P} - \frac{P'_\mu}{k' \cdot P'} \right)^2 \right) \].
\]

All of the integrals occurring here can be expressed in terms of one basic one

\[
I(p_1, p_2) = - \frac{k^2}{4\pi^2} \int d\Omega \frac{p_1 \cdot p_2}{k \cdot p_1 k \cdot p_2}
\]

\[
= - \frac{p_1 \cdot p_2}{\pi} \int_0^1 dz \frac{1}{p_2^z},
\]

(3.24)
where
\[ p_z = z p_1 + (1 - z) p_2. \]

Explicitly I is given by
\[ I(p_1, p_2) = -\frac{p_1 \cdot p_2}{\sqrt{Q}} \ln \left\{ \frac{|2p_1 \cdot q_{12} - \sqrt{Q}|}{|2p_1 \cdot q_{12} + \sqrt{Q}|} \frac{|2p_2 \cdot q_{12} + \sqrt{Q}|}{|2p_2 \cdot q_{12} - \sqrt{Q}|} \right\}, \quad (3.25) \]

with
\[ Q = 4[(p_1 \cdot p_2)^2 - p_1^2 p_2^2], \]
\[ q_{12}^2 = p_1^2 - p_2^2. \]

In the extreme relativistic limit for the electron, the interference term takes the form
\[ A_{\text{int}} = \frac{4Z}{\pi} \ln \left( \frac{p \cdot P}{p' \cdot P'} \right). \quad (3.26) \]

(Here we have used the fact that, for \(|k| \ll |p|, p \cdot P = p' \cdot P' = p' \cdot P \).) Notice that there is no term containing \(\ln m\). This can be understood quite readily by inspection of (3.23). Ordinarily the \(\ln m\) comes from the region in which \(k\) is nearly parallel to \(p\) (or \(p'\)) making the denominator \(k \cdot p\) small. However, in this region the second factor in the interference term takes the form
\[ \left( \frac{p^\mu}{p \cdot P} - \frac{P'^\mu}{p' \cdot P'} \right) \frac{|p|}{|k|}, \]
with a relative error of order \((m^2/E^2)\); when this is multiplied by the first term, the result is zero and the near singularity of the first factor is cancelled out. The cancellation of \(\ln m\) terms is thus seen to be essentially a consequence of gauge invariance. If now we make the nonrelativistic approximation for the recoiling nucleus, the interference contribution simplifies further to
\[ A_{\text{int}} \approx -\frac{Z}{\pi} \frac{q^2}{p \cdot P'}, \approx + \frac{Z}{\pi} \frac{q^2}{EM}. \quad (3.27) \]

The contribution from the recoiling nucleus is given by
\[ A_{\text{nucl}} = \frac{2Z^2}{\pi} \left[ \frac{2P \cdot P'}{\sqrt{Q'}} \ln \left| \frac{\sqrt{Q'} - q'}{\sqrt{Q'} + q'} \right| - 1 \right], \quad (3.28) \]
where
\[ Q' = -q^2(P + P')^2. \]
In the nonrelativistic approximation this reduces to

$$A_{\text{nucl}} \approx \frac{2}{3} \frac{Z^2}{\pi} \frac{q^2}{M^2}.$$  \hspace{1cm} (3.29)

Next we consider kinematic radiative recoil corrections. If the scattered electron is extremely relativistic and one photon is produced, energy conservation is expressed by the relation

$$p' + [(p' + k)^2 + M^2]^{1/2} + k = p + (p^2 + M^2)^{1/2}$$  \hspace{1cm} (3.30)

in the center-of-mass system. For simplicity we are using a minimum photon momentum rather than a photon mass. The magnitude of the vector $k$, derived from (3.30) for fixed electron energy and photon angle, depends on its direction and is given by

$$k = \epsilon/\beta,$$  \hspace{1cm} (3.31)

where

$$\beta = [(p^2 + M^2)^{1/2} + (k \cdot p')/k] / [(p^2 + M^2)^{1/2} + p]$$  \hspace{1cm} (3.32)

for small $k$. The differential cross section for scattering with energy loss $\epsilon$ is now given by

$$\frac{d\sigma}{d\epsilon} = \frac{\sigma_{\text{el}}}{\epsilon} \left\{ -\frac{\alpha}{4\pi^2} \int d\Omega \left( \frac{p_\mu - p'_\mu}{k \cdot p} \right) \left[ q^2 \theta(\epsilon - \beta k) \right] \right\}$$  \hspace{1cm} (3.33)

If $\epsilon$ is large enough, it will exceed $(\beta k_{\text{min}})$ at all angles and the integral will reduce to $\alpha A$; the result is the lowest order approximation to (2.57) in the infrared limit. For values of $\epsilon$ less than $k_{\text{min}}$, the integrand of (3.33) is nonvanishing over a limited angular region. We can estimate the effect as follows. Approximately half of the integral comes from each of the regions, $k_{\text{min}}/p$, and $k_{\text{min}}/p'$, due to the small denominators in those regions. For $k_{\text{min}}/p$, $\beta$ is one; and for $k_{\text{min}}/p'$, it is

$$\beta = \xi = [(p^2 + M^2)^{1/2} + (p \cdot p')/p] / [(p^2 + M^2)^{1/2} + p]$$  \hspace{1cm} (3.34)

$$\approx 1 - \frac{1}{2} \frac{q^2}{ME}.$$  \hspace{1cm} (3.34)

With this approximation, (3.33) becomes

$$\frac{d\sigma}{d\epsilon} = \frac{\alpha A}{\epsilon} \frac{\sigma_{\text{el}}}{\epsilon} \left\{ \begin{array}{ll} 1 & (\epsilon > k_{\text{min}}) \\ \frac{\alpha A}{\epsilon} \sigma_{\text{el}} \frac{\xi}{2} & (k_{\text{min}} > \epsilon > \xi k_{\text{min}}) \\ 0 & (\xi k_{\text{min}} > \epsilon). \end{array} \right.$$  \hspace{1cm} (3.35)

Of course, the separation into these three regions is unphysical since $k_{\text{min}}$ does not actually exist. However, when (3.33) is integrated to determine the cross
section for scattering with energy loss less than $\Delta E$, the result will be given by the substitution

$$\ln \frac{\Delta E}{k_{\text{min}}} \rightarrow \ln \frac{\Delta E}{k_{\text{min}}} - \frac{1}{2} \ln \xi.$$  \hfill (3.36)

Taking into account the contribution of one virtual photon, we find the sum of kinetic and dynamic radiative recoil corrections to order $M^{-1}$ to be given by

$$\frac{\alpha}{2\pi} \frac{q^2}{ME} \left[ \ln \frac{2p \cdot p'}{m^2} - 1 - 2Z \ln \frac{E}{\Delta E} \right] \sigma_{\text{el}}.$$  \hfill (3.37)

The kinematic and dynamic corrections are of opposite sign and hence tend to balance each other out; either effect separately could account for a several percent correction at energies of several hundred million volts in electron-proton scattering (but then the expansion in $M^{-1}$ could no longer be employed). The signs of the two contributions can be understood intuitively. The dynamic correction tends to enhance the ordinary radiative correction, which is negative; the positive interference indicated in (3.27) arises simply from the fact that the electron and nucleus have opposite signs for their charges and are moving opposite directions. This results in an enhancement of the current for emission and absorption of photons. The qualitative explanation for the positive kinematical corrections is that it extends the bremsstrahlung spectrum as seen from the spectrum of the scattered electron. For positron-proton scattering, the two effects will interfere constructively. The correction may become quite large.

There are some subtle questions about this treatment which should be mentioned. If one carries out the analysis of the kinematical corrections in the laboratory system, the factor of $\frac{1}{2}$ does not appear in the approximate form of $\xi$ (in (3.34)). The kinematical correction appears to be twice as large in the laboratory as in the center-of-mass system! This difference must, of course, be compensated for by a corresponding change in the matrix element. This means that the separation of recoil corrections into a dynamic and a kinematical part is not invariant. To see explicitly where this correction occurs, consider (2.31). The noninvariant quantities in this equation are $E, E'$, and $\epsilon$; however, in the extreme relativistic limit for the electron, $(\epsilon/E')$ is invariant. (This is because the four-vector representing the energy-momentum loss of the electron is parallel to the four-vector representing its final total energy-momentum, if $m$ can be neglected.) Hence the noninvariant part of (2.31) is

$$-\frac{\alpha A}{2} \ln \frac{E}{E'}.$$  

In the center-of-mass system, this expression vanishes ($E = E'$). In the laboratory system, it reduces approximately to
This is just sufficient to bring the two results into agreement. Another paradoxical point is that when many soft photons are taken into consideration, Eq. (3.35) should be replaced by (2.57). The restrictions given by (3.35) will now have no effect in the limit \( k_{\text{min}} \to 0 \), since the exponential factor in (2.57) damps the function sufficiently to make it integrable without the limitation of (3.35). However, (3.36) leads to the right answer. To see how this happens, let us reconsider the many photon treatment of Section 2. We shall divide the soft real photons into two groups, those emitted roughly parallel to \( p \) and those emitted roughly parallel to \( p' \). Then in (2.18) one must make the substitution

\[
e^{-ik} \to e^{-i\xi k} \approx \frac{1}{2}[e^{-i\xi k} + e^{-i\xi}].
\]

The first term corresponds to a photon emitted parallel to \( p \); the factor \( \xi \) takes into account the fact that it will remove the energy \( \xi k \) from the final energy of the electron. The second term corresponds to photon emission parallel to \( p' \). When the effect of this substitution is followed through, it turns out to modify \( \alpha (B + \bar{B}) \) by the addition of

\[-\frac{\alpha A}{2} \ln \xi \]

giving agreement with the result obtained by our approximate methods. Still another derivation of this result will be given in Section 4c.

(e) BREMSSTRAHLUNG PRODUCED BY NUCLEONS SCATTERING FROM NUCLEI

In certain situations, the scattering amplitude for nuclear scattering is a very rapidly varying function of energy. This may result in interesting effects in the bremsstrahlung produced in the scattering process if the following conditions can be obtained. (i) The wavelength of the radiation is large compared with the dimensions of the scattering region; this permits the use of infrared approximations. (ii) The energy carried off by the photon is comparable with or larger than the energy interval over which a large change in the scattering amplitude occurs. A more complete discussion is given elsewhere (19); here we shall restrict our attention to a simple model of the process. Consider the scattering of a nucleon from an infinitely heavy nucleus (so that recoil currents may be neglected); the matrix element for scattering from a state of momentum \( q \), energy \( E \), to a state of momentum \( q' \), energy \( E' \), will be called \( T(E, q, q') \). Only elastic scattering will be discussed. The initial and final states need not be on the energy...
shell; $T$ is defined by

$$T(E, q, q') = \left\langle q' \left| V + V \frac{1}{E + i\epsilon - H} V \right| q \right\rangle,$$

(3.38)

where $q$ and $q'$ are plane wave states and $H$ is the complete Hamiltonian for the system, excluding all terms involving the quantized electromagnetic field.

Now we must pick out the terms which will lead to a bremsstrahlung spectrum of the form $(dk/k)$. It is obvious from our previous considerations that these will arise from the emission of the photon from the external lines; i.e., from the incident or scattered nucleon. In Ref. 19 it is shown that all other contributions are of relative order $(ka)$, where $a$ is a dimension of the scattering region. The amplitude for emission followed by scattering is clearly proportional to

$$\frac{e \cdot p}{k \cdot p} T(E - k, p - k, p'),$$

(3.39)

where $p$ is the momentum of the incident particle, $E$ is its energy ($= (p^2 + M^2)^{1/2}$) and $p'$ is the momentum of the scattered particle. Only $p - k$ is off the energy shell in the matrix element occurring in (3.39). However, since the dependence of $T$ on the momentum arguments is slowly varying, we may replace the argument $p - k$ by $\bar{p}$, where $\bar{p}$ is parallel to $p$ but lies on the energy shell; the error is again of relative order $(ka)$. The amplitude for scattering followed by emission may be treated in the same manner. In this case the final momentum $q' = p' + k$ is off the energy shell and may be replaced by $\bar{p}'$, which is parallel to $p'$ and on the energy shell. The sum of the two contributions is then proportional to

$$\left[ \frac{e \cdot p'}{k \cdot p}, f(\theta, E) - \frac{e \cdot p}{k \cdot p} f(\theta, E - k) \right],$$

(3.40)

where $f(\theta, E)$ is the amplitude for scattering at the angle $\theta$ of a particle of energy $E$. By assumption, the difference between $f(\theta, E)$ and $f(\theta, E - k)$ cannot be neglected. This gives rise to interesting interference effects and enables one to compare the scattering amplitudes at neighboring energies.

The main effect we have neglected is recoil (17); this may be taken into account by means of the following substitutions in (3.40)

$$\frac{e \cdot p}{k \cdot p} \rightarrow \frac{e \cdot p}{k \cdot p} + Z \frac{e \cdot P}{k \cdot P}, \quad \frac{e \cdot p'}{k \cdot p'} \rightarrow \frac{e \cdot p'}{k \cdot p'} + Z \frac{e \cdot P'}{k \cdot P'}.$$

For proton scattering from a nucleus, recoil reduces the current by about a factor of two and the bremsstrahlung cross section by about a factor of four. For neutron scattering from a nucleus, the entire current for bremsstrahlung emission is due to recoil; the bremsstrahlung cross section is comparable to that for proton scattering.
4. GENERALIZATIONS

This section has two purposes. The first of these is the presentation of a general treatment showing the cancellation of the infrared divergence for an arbitrary process. The main result of this treatment is that the infrared dependence can be extracted as exponential factors and treated completely, as in the discussion of electron scattering. However, the generalization of \( F(\alpha A) \) cannot, in general, be given in closed form. This analysis, which is mainly of theoretical interest, will be carried out in Sections 4(a) and 4(b). The second purpose is more practical; it is to discuss some of the problems that are likely to arise in typical calculations. Since the infrared dependence includes all the terms which are bilinear in \( \ln (E/m) \) and \( \ln (\Delta E/E) \) and some of the terms which are linear in these quantities as well, it gives an important part of the lowest order radiative corrections. Because the infrared contribution depends only on the external charged lines, it is therefore possible to make a very good estimate of the radiative corrections without a great deal of labor. It turns out, however, that the main difficulty in calculating such radiative corrections is that they may be very dependent on the experimental conditions. The maximum energy which a real photon may carry off can be very sensitive to its direction, depending upon the actual experimental conditions. It is therefore not possible to give a final formula in simple form which will give the infrared radiative corrections to an arbitrary process. However, we can give a guide for such calculations and illustrate it in a few special cases. This practical discussion will be presented in Section 4c.

(a) CANCELLATION OF INFRARED DIVERGENCE IN THE GENERAL CASE

In order to give a completely general discussion of the cancellation of the infrared divergences due to the real and virtual photons, it is necessary to discuss the situation where the experimental arrangement restricts the momentum, as well as the energy carried off by the unobserved photons. An example of such an experimental situation has been given in Section 3(d) and further examples will be given in Section 4(c). In these examples the magnitude and direction of the total momentum carried off by the photons affects the value of the total energy which they can carry off. Suppose initially that all the particles, as well as any observed photons, have momentum and energy completely fixed. Then the total energy and momentum available to unobserved photons will be called \( K_0 \) and \( \mathbf{K} \), respectively, with \( |\mathbf{K}| \) less then or equal to \( K_0 \) if the process is kinematically possible. Following the procedure of Section 2, energy and momentum conservation can be represented by the integral

\[
\delta_i(K - \sum k_i) = \frac{1}{(2\pi)^4} \int \cdots \int d^4x \ e^{i \mathbf{K} \cdot \mathbf{k}_i}.
\]

(4.1)
The generalization of (2.17) is then given by

\[ \sigma = \lim_{\lambda \to 0} \exp \left( 2\alpha \text{Re } B \right) \frac{1}{(2\pi)^4} \int \cdots \int \, d^4x \, e^{i\cdot k} \]

\[ \times \exp \left[ \int \frac{d^3k}{(k^2 + \lambda^2)^{1/2}} \bar{S}(q, k) e^{-ik \cdot x} \right] \times \left[ \beta_0(q) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \prod_{m=1}^{n} \frac{d^3k_m}{k_m} e^{-i k_m \cdot x_\lambda} \beta_n(q, k_1, \ldots, k_n) \right], \]  

where \( q \) stands for the set of all momenta of the observed particles and photons and the other quantities have the natural generalizations which would be expected.

To demonstrate the infrared cancellation, we use the general definition of \( \tilde{B} \) which will be given in (4.5), with \( K_m(\Omega) \) to be defined in some convenient way (but satisfying \( K_m(\Omega) \gg \lambda \) in all directions). Then we define \( D \) by

\[ D(q, x, [K_m]) = \int \frac{d^3k}{k} \bar{S}(q, k) \{ e^{-ik \cdot x} - \delta[K_m(\Omega) \cdot k] \}. \]  

\( D \) is a functional of \( K_m \), as well as a function of \( q \) and \( x \); it contains no infrared divergence. It will be shown in Section 4b that infrared divergences in \( \tilde{B} \) and \( \text{Re } B \) cancel each other, as in electron scattering. Then the differential cross section reduces to

\[ \sigma = \exp \left[ 2\alpha(\text{Re } B + \tilde{B}) \right] \frac{1}{(2\pi)^4} \int \cdots \int \, d^4x \, e^{i\cdot k + D} \]

\[ \times \left\{ \beta_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \prod_{m=1}^{n} \frac{d^3k_m}{k_m} e^{-i k_m \cdot x} \beta_n \right\} . \]  

We have thus shown the infrared cancellation in an arbitrary process. Eq. (4.4) is the generalization of (2.22).

In a realistic experimental situation, the observed particles will not have uniquely given values of the energy and momentum; in fact, it is not always convenient to observe the final states of all the particles (for example, the recoiling nucleus in electron scattering is not usually observed). It will then be necessary to integrate (4.4) over some range of values of the various momenta contained in \( q \); in turn this means integration over some range of \( K \) and \( K_0 \). It is not convenient or profitable to try to develop a general approach to this problem which will work in all situations. In any practical calculation, one would desire only to calculate the cross section consistently to some definite order of \( \alpha \). Equation (4.4) indicates how such a calculation can be carried out without encountering any infrared divergences aside from those contained in \( B \) and \( \tilde{B} \), which cancel each other and can be easily treated. In (4.4) the separation of
the real photon infrared divergence into a part $\tilde{B}$ and a part $D$ depends on the choice of a function $K_m(\Omega)$. It now seems appropriate to choose this function so that all of the infrared dependence to order $\alpha$ appears in $\tilde{B}$; that is, so that the contribution arising from $D$ vanishes to first order in $\alpha$. Assuming that the range of integration of $k$ includes the origin in energy and momentum space and contains a portion of the light cone, it is easily seen that the $K_m$ for a given direction is the maximum value of $|K|$ for which $|K| = K_0$ and $K$ points in the given direction. More complicated situations will not be discussed here.

(b) Generalized Definitions of $B$ and $\tilde{B}$

Next we give a general definition of $B$ and $\tilde{B}$. Consider an arbitrary process containing $w$ charged incoming and outgoing particles, each with charge $eZ_i$ and momentum $P_i$. The diagrams contributing to $\tilde{B}$ (or $B$), the infrared factor for real (virtual) photons can still be described by Fig. 3 (Fig. 2). The external lines in Figs. 2(a) and 3(a), (b) now represent any pair of free charged particles. Each charge may be incoming or outgoing, boson or fermion. The shaded area now represents the remaining part of the process including other external lines, and is taken to be independent of $k$. The appropriate generalization of Eq. (2.21) for $\tilde{B}$ is

$$\tilde{B} = \frac{1}{8\pi^2} \int_{\kappa < K_m} \frac{d^3k}{(k^2 + \lambda^2)^{3/2}} \left( \sum_{i=1}^{w} Z_i \theta_i p_i \right)^2,$$

where $Z_i$ has the sign of the $i$th charge and $\theta_i = + (-)$ if $i$ is outgoing (incoming). In general, $K_m$ may vary with angle, as we have indicated in Section 4(a). As in Section 2, $\tilde{B}$ is gauge invariant and vanishes when none of the charge lines receive any momentum transfer.

The corresponding generalization of $B$ (Eq. (2.23)), the infrared factor for virtual photons, to arbitrary processes is given by

$$B = \frac{i}{8\pi^2} \int \frac{d^3k}{k^2 - \lambda^2 + i\epsilon} \left( \sum_{i=1}^{w} Z_i^2 (2p_i - k)^2 \right)$$

$$- 2 \sum_{i<j}^{w} \frac{Z_i \theta_i Z_j \theta_j (2p_i \theta_i - k \cdot (2p_i \theta_i + k))}{(k^2 - 2k \cdot p_i \theta_i)(k^2 + 2k \cdot p_i \theta_i)}.$$  

With the help of the relationship

$$\sum_{i=1}^{w} Z_i \theta_i = 0,$$

Eqs. (4.5) and (4.6) may be transformed to the form of sums over pairs of external lines

$$\tilde{B} = \frac{1}{8\pi^2} \int_{0}^{\kappa < K_m} \frac{d^3k}{(k^2 + \lambda^2)^{3/2}} \sum_{i<j}^{w} Z_i \theta_i Z_j \theta_j \left( \frac{p_{i\mu}}{k \cdot p_i} - \frac{p_{j\mu}}{k \cdot p_j} \right)^2$$

$$= \sum_{i<j} \tilde{B}_{ij}$$

(4.8)
and

\[
B = -\frac{i}{8\pi^3} \int \frac{d^3k}{k^2 - \lambda^2} \sum_{i<j} Z_i \theta_i Z_j \theta_j \left( \frac{(2p_i \theta_i - k)_\mu}{k^2 - 2k \cdot p_i \theta_i} + \frac{(2p_j \theta_j + k)_\mu}{k^2 + 2k \cdot p_j \theta_j} \right)^2
\]

(4.9)

Equations (2.23) and (2.24) are seen to be special cases of these definitions. The contribution to \( B \) from the photon propagator pole, where the virtual photon lies on the mass shell, cancels the infrared part of \( B \), just as in Eqs. (2.25) and (2.26). If particles \( i \) and \( j \) in the initial or final states are identical, \( B \) and \( \tilde{B} \) will clearly be symmetric in the variables of these particles. Thus infrared factors do not change the overall symmetry of the matrix elements they multiply.

The quantities \( \tilde{B}_{ij} \) and \( B_{ij} \) are evaluated in Appendix C; neglecting non-logarithmic terms, they are

\[
\tilde{B}_{ij} \approx \frac{Z_i \theta_i Z_j \theta_j}{2\pi} \left\{ \ln \frac{k_i^2}{\lambda^2} + \ln \frac{m_i m_j}{E_i E_j} - \frac{1}{2} \frac{p_i \cdot p_j}{p_z^2} \int_{-1}^{1} \frac{dx}{p_z^2} \right\}
\]

(4.10)

\[
+ \tilde{B}_{ij}'([K_m], k_1),
\]

where

\[
\tilde{B}_{ij}'([K_m], k_1) = \frac{Z_i \theta_i Z_j \theta_j}{8\pi^2} \int_{k_1}^{K_m} \frac{d^3k}{k} \left( \frac{p_{i\mu}}{k \cdot p_i} - \frac{p_{j\mu}}{k \cdot p_j} \right)^2
\]

(4.11)

and

\[
B_{ij} = -\frac{Z_i \theta_i Z_j \theta_j}{2\pi} \left\{ \ln \frac{m_i m_j}{\lambda^2} + \frac{1}{2} \frac{p_i \cdot p_j \theta_i \theta_j}{p_z^2} \int_{-1}^{1} \frac{dx}{p_z^2} \right\}
\]

(4.12)

\[
+ \frac{1}{4} \int_{-1}^{1} \ln \frac{p_z^2}{m_i m_j} dx
\]

where \( 2p_z = (1 + x)p_i + (1 - x)p_j \) and \( 2p_z' = (1 + x)p_i - (1 - x)p_j \). The expression (4.10) is independent of \( k_1 \), which is to be chosen in some convenient way. If \( \theta_i \theta_j = -1 \), \( p_z' \) is equal to \( p_z \), \( B_{ii} \) is real, and several terms cancel when \( \tilde{B}_{ij} \) and \( B_{ij} \) are added. If \( \theta_i \theta_j = +1 \), the situation is more complicated; however, as shown in Appendix C by means of a change of variables, the same cancellations occur when \( \tilde{B}_{ij} \) and \( \text{Re} B_{ij} \) are added. With the neglect of nonlogarithmic terms, the result is
\[ \tilde{B}_{ij} + \text{Re} \tilde{B}_{ij} \approx \frac{Z_i \theta_i Z_j \theta_j}{2\pi} \left\{ \ln \frac{k_i^2}{E_i E_j} - \frac{1}{2} \frac{p_i \cdot p_j}{E_i E_j} \int_{-1}^{1} \frac{\ln \left( \frac{k_{z_1}^2}{E_{z_1}^2} \right)}{p_{z_1}^2} \, dx \right\} \]

\[ - \frac{1}{4} \ln \frac{p_{z_1}^2}{m_i m_j} \left\{ \ln \frac{2p_i \cdot p_j}{m_i m_j} - 1 \right\} \ln \frac{k_{z_1}^2}{E_{z_1}^2} + \frac{1}{2} \ln \frac{(p_i - p_j)^2}{m_i m_j} \]

\[ + \ln \frac{m_j}{m_i} \ln \frac{E_j}{E_i} - \frac{1}{2} \ln^2 \frac{E_j}{E_i} + \tilde{B}_{ij}'([K_m], k_1). \]

When \((p_i - p_j)^2 < 0\) [if \(m_i^2 > 2p_i \cdot p_j\), the result is altered by adding \(-\frac{1}{2} \ln^2 \left(\frac{2p_i \cdot p_j}{m_i^2}\right)\) to the bracket as shown in Appendix C, Case (b)]. If we set \(m_i = m_j = m\), \(Z_i = Z_j = 1\), \(\theta_i \theta_j = -1\), and \(K_m = k_1 = \epsilon\), this reduces to (2.31).

Those parts of \(B_{ij}\) which involve one incoming and one outgoing particle are always purely real. In terms involving a pair of incoming (or outgoing) particles, there is also an imaginary part which is diagrammed in Fig. 5.

**Fig. 5.** The typical diagrams contributing to the Coulomb phase factor.

Such imaginary parts give us the long-range contributions to the familiar Coulomb phase shift.\(^9\) Some of this imaginary part arises from the "ab-

\(^9\) This was first pointed out by Dalitz (23) who found the phase change arising from the second Born approximation and showed that the \((\log \lambda)^4\) term in the third Born approximation (for the Schrödinger equation) corresponds correctly to an expansion of a phase factor. More recently, Kacser (24) has given a more complete treatment of the infrared factor of the third Born approximation and has verified that it is \(-\frac{1}{2}\) times the square of the imaginary part of the second Born approximation phase change. His result includes angular dependence, which the present treatment does not.
sorptive contribution" corresponding to the situation when both charged particle propagators are simultaneously free; this "absorptive" part can be estimated by using the $-i\pi\delta(k^2 + 2p_i \cdot k)$ parts of both charged particle propagators (the mathematics is similar to Eqs. (2.25)-(2.26)). Doing so, one obtains for the absorptive $dk/k$ part of $\alpha B_{ij}$ in the $i-j$ center of mass:

$$\alpha B_{ij} \text{(abs.)} = \frac{-i\alpha Z_i Z_j p_i \cdot p_j}{\pi^3} \int \frac{d^4k}{k^2 - \lambda^2} \left[ -i\pi\delta(k^2 - 2p_j \cdot k) \right] \left[ -i\pi\delta(k^2 + 2p_j \cdot k) \right]$$

$$= \frac{-i\alpha Z_i Z_j p_i \cdot p_j}{2\pi(E_i + E_j)} \int \frac{d^3k}{k^2 + \lambda^2} \delta(-k^2 + 2p_j \cdot k)$$

$$= \frac{-i\alpha Z_i Z_j p_i \cdot p_j}{2p(E_i + E_j)} \ln \frac{2p}{\lambda},$$

(4.15)

where $|p_i| = |p_j| = p$. This is not the complete contribution to the $\text{Im } B_{ij}$. There is another part coming from the principal value parts of both charged particle propagators. This may be written

$$\alpha B_{ij} \text{(P.V.)} = \frac{-i\alpha Z_i Z_j p_i \cdot p_j}{\pi^3} \int \frac{d^4k}{k^2 - \lambda^2}$$

$$\cdot \frac{(k^2 - 2p_i \cdot k)}{[(k^2 - 2p_i \cdot k)^2 + \epsilon^2]} \frac{(k^2 + 2p_j \cdot k)}{[(k^2 + 2p_j \cdot k)^2 + \epsilon^2]}$$

(4.16)

The complete calculation of $\text{Im } B_{ij}$ is given in the Appendix. Here we shall give a rough estimate which indicates that the contribution of (4.16) is approximately the same as that of $\alpha B_{ij} \text{(abs.)}$. It is convenient to make the following change of variables in (4.16)

$$\int_{-1}^{1} d(\cos \theta) = \frac{1}{2k} \int_{-2kp - k^2}^{2kp - k^2} d(2k \cdot p - k^2),$$

where $k = |k|$. For $k^2 < 2kp$, we may approximate the integration over this new variable by extending the limits to $\pm \infty$; for $k^2 > 2kp$, we neglect the result. Integration over this new variable gives

$$\frac{\pi^2}{2(E_i + E_j)} \delta(k_0),$$

which leads to the same result as (4.15), and then to

$$\alpha \text{ Im } B_{ij} \approx -\frac{i\alpha Z_i Z_j p_i \cdot p_j}{p(E_i + E_j)} \ln \frac{2p}{\lambda}.$$  

(4.17)
This agrees with the result in the Appendix, to the given order of approximation. Electron-potential scattering is given by the limit $E_j \gg E_i$:

$$\alpha \text{ Im } B_{ij} \approx -\frac{i\alpha Z_j e}{\nu_i} \ln \frac{2p}{\lambda}. \quad (4.18)$$

If we sum over all such contributions for two outgoing charges, we obtain the divergent part of the familiar Coulomb wave function for two charged particles, exp $[\alpha \text{ Im } B_{ij}]$. An extension of our procedure is necessary in order to obtain the finite angle-dependent part of the Coulomb phase shift. The following plausibility argument implies that the argument of (4.18) should be multiplied by $\sin \theta/2$. The upper limit of the integration with respect to $|k|$ should not be $2p$ when the momentum transfer is small, since the soft photon would then have a greater momentum than the ones in the basic interaction. If we replace the upper limit $2p$ by the momentum transfer $2p \sin \theta/2$, the desired result is obtained.

More generally, each pair of initial and final state charges contributes a Coulomb phase factor. Rather than cancelling, these factors change the phase of the wave functions by an amount which approaches infinity as $\lambda \to 0$ (corresponding to the $\ln 2k\tau$ phase in the Coulomb wave functions in the idealized case of no screening). Such infinite phase factors have no experimental consequences.

(d) Guide for Practical Calculations

The "long-range" character of infrared radiative corrections allows us to give expressions for these corrections which apply to any process independently of its "short-range" details. The virtual states involved in infrared terms are nearly real, dominating the radiative corrections in many cases. Therefore it is both possible and desirable to give a general guide for the calculation of the infrared contribution to a general radiative correction. A complete calculation would require the evaluation of the noninfrared terms, which are represented by $\beta_{ij}$ and $\bar{\beta}_{ij}$. It is our belief that it is preferable to perform the calculation according to the pattern suggested here rather than to calculate $\beta_{ij}$ and $\bar{\beta}_{ij}$ with the arbitrary cutoff included, and then to cancel the infrared divergence later.

The fractional infrared contribution to a cross section is given by

$$2\alpha (\text{Re } B + \bar{B}) = 2\alpha \sum_{i<j} (\text{Re } B_{ij} + \bar{B}_{ij}), \quad (4.19)$$

where $\text{Re } B_{ij} + \bar{B}_{ij}$ is given by (4.13) or (4.14). These expressions are invariant (but not covariant) and hence may be evaluated in any reference frame. The difficult part of any calculation reduces to finding $K_m(\Omega)$ from the experimental conditions and then to evaluating $\tilde{B}_{ij}(\Omega, K_m, k_i)$. This requires a separate and careful discussion of each experimental arrangement.
There exist only a few examples of calculations in the literature where the
dependence on $K_n(\Omega)$ is important and has been taken into account. We shall
now give a brief discussion of these examples as well as some new ones.

(i) **Electron Scattering including nuclear recoil, excitation, and particle emission.**
These have been discussed in Sections 3(b) and 3(d). The energy available to un-
detected photons is the electron energy loss, less the energy which goes to nuclear
recoil, excitation and particle emission. Whenever the nuclear recoil energy can
be neglected, undetected photons may assume any momentum $K$ up to their total
energy $K_n$, and $K_n$ is independent of the direction of $K$. But if nuclear recoil has
to be considered explicitly, then the relationships of $K$ to nuclear momentum,
nuclear momentum to nuclear kinetic energy, and nuclear kinetic energy to $K_n$
have the consequence that $K_n$ depends on the direction of $K$, as we have seen in
Section 3(d).

In the problem treated in Section 3(d), electron scattering with nuclear re-
coil taken into account, energy-momentum conservation is expressed by

$$ (p + P - p' - P' - k)_{\mu} = 0 $$

(4.20)

for single photon emission. In the special Lorentz frame defined by

$$ \vec{P}' + \vec{k} = \vec{p} + \vec{P} - \vec{p}' = 0, $$

(4.21)

the photon energy is isotropic and is given by

$$ (M^2 + k^2)^{1/2} + k = [(p + P - p')^2]^{1/2}. $$

(4.22)

Now suppose that in the center-of-mass system the electron scatters with
energy loss $\leq \Delta E$. We choose the Lorentz frame defined by (4.21) with $P'$
taken to be the minimum momentum of the scattered electron compatible
with the given energy resolution. Then in (4.11), $K_n(\Omega)$ is isotropic and we
may take $k_1 = K_n$ thus eliminating $\vec{B}'([K_n], k_1)$. From (4.22), we find for
small $\Delta E$

$$ k_1 = \Delta E \left( p + \sqrt{p^2 + M^2} \right) / M. $$

(4.23)

In order to calculate the quantity $\vec{B}_{ij} + \text{Re } B_{ij}$ associated with the incident
and scattered electron, we also need $\vec{p}_0$ and $\vec{p}_0'$. These are given by

$$ \vec{p}_0 = p \cdot (p + P - p') / M $$
$$ = \frac{p(p^2 + M^2)^{1/2} + p \cdot p'}{M}, $$

$$ \vec{p}_0' = p' \cdot (p + P - p') / M $$
$$ = \frac{p(p^2 + M^2)^{1/2} + p^2}{M}, $$

$$ = \frac{p(p^2 + M^2)^{1/2} + p^2}{M}, $$
so that

\[
\frac{\vec{k}_i^2}{\vec{p}_i \vec{p}_f} = \frac{(\Delta E)^2}{E^2} \left( \frac{(p^2 + M^2)^{1/2} + p}{p' \cdot p / p} \right)
\]

in agreement with (3.34) and (3.36).

The net result contributed by the set of quantities $\vec{B}_{i,j} + \text{Re} B_{i,j}$ where $i$ refers to an electron line and $j$ to a nucleus line, goes through in a similar manner. The approximate form (4.14) is no longer correct since $M^2$ is not small compared to $p \cdot p'$. It also turns out that the correct result can be obtained by setting $E_x$ equal to the electron's energy in Eq. (4.13), since the main contribution comes from regions of $x$ for which this is true. The result is the same as that already given in Section 3(d).

(ii) **Pair production** $(\gamma + P \rightarrow e^+ + e^- + P)$ with the pair detected in coincidence (11). The determination of $K_m$ for a given incident photon energy in this case is very similar to electron-proton scattering, for in both processes all light particles in the final state are detected and a heavy particle is available to balance momentum. In practice one must usually integrate over a broad spectrum of incident photon energies to obtain a comparison with experiment, and $K_m$ becomes quite large unless the detected pair energy lies near the upper limit of this broad spectrum.

(iii) **Compton scattering** $(\gamma + e \rightarrow \gamma + e)$ (6) or **electron-electron scattering** $(e + e \rightarrow e + e)$ (9, 10), with both particles detected in coincidence. These cases contrast greatly with electron-potential scattering because of the absence of heavy particles. For example, in electron-potential scattering the center-of-mass and laboratory frames of reference are the same. But in Compton scattering (initial electron at rest), the infrared factors of order $\alpha \ln^2$ given by (4.13) are

\[
2\alpha (B + \vec{B}) = -\frac{\alpha}{\pi} \ln \frac{2p \cdot p'}{m^2} \ln \frac{E^2}{K_m^2}
\]

if $K_m$ is isotropic in the center of mass ($E$ is the center-of-mass electron energy), and

\[
2\alpha (B + \vec{B}) = -\frac{\alpha}{\pi} \left\{ \ln \frac{mE'}{K_m^2} \ln \frac{2E'}{m} + \frac{1}{2} \ln \frac{2E'}{m} \right\}
\]

if $K_m$ is isotropic in the laboratory (here $E'$ is the laboratory energy of the outgoing electron).19 Geometrically, one expects (4.24) and (4.25) to differ when $E \gg m$ because a sphere in the laboratory transforms into an extremely prolate ellipsoid in the center of mass. Numerically, under the condition $E' = 500$ Mev

19 Discussions with Y. S. Tsai were helpful in understanding the difference between (4.23) and (4.24). He has shown that a similar difference exists for electron-electron scattering.
Eqs. (4.24) and (4.25) have the values
\[
2\alpha(B + \tilde{B}) \approx -0.16 \quad (4.24')
\]
and
\[
2\alpha(B + \tilde{B}) \approx -0.10. \quad (4.25')
\]
The 6% difference between cross sections computed with (4.24) or (4.25) should make apparent the importance of treating $K_m$ correctly.

The mathematical origin of the difference between (4.24) and (4.25) can be traced to the $E_z$ dependence in (4.13). The approximate form (4.14) is not valid if $E_z \ll E_j$. In the present case we have $E_z = \frac{1}{2}[(1 + x)m + (1 - x)E']$ and $p_z^2 = \frac{1}{2}m[(1 + x^2)m + (1 - x^2)E]$. Although the $p_z^2$ dependence assures that most of the integral comes mainly from the regions $x = +1$ and $x = -1$, it is not true that $\ln E_z$ can be approximated by $\ln m$ and $\ln E'$ in those two regions. In fact, in the region near $x = 1$, $E_z$ varies so rapidly that the mean value of $\ln E_z$ in that region is more nearly $\ln m$ than $\ln E'$. This accounts for the difference between (4.14) and (4.25). A more detailed discussion is given in Appendix C.

In actual experiments neither of these requirements ($K_m$ isotropic in center-of-mass or laboratory system) is likely to be realistic. Tsai has considered the radiative corrections to electron-electron scattering in the center-of-mass system in a clashing beam experiment (10) (referred to as Exp. 1 by Tsai). In this experiment the two scattered electrons are detected in coincidence with virtually no energy resolution ($\Delta E \sim E$). It is clear that if a photon is emitted parallel to either final electron, $K_m$ is then of order $E$. However, if it is emitted perpendicular to the direction of the final electrons, $K_m$ is much smaller and is determined by the angular resolution ($\Delta E \sim E \Delta \theta$). Thus $K_m$ has a very strong angular dependence. It is interesting to note that when Tsai picks out the dominant terms in his cross section they correspond to the infrared approximation plus the magnetic terms plus vacuum polarization. There are no additional logarithmic terms. This is true even though the emitted photons take off a considerable fraction of the energy. Since the magnetic, and vacuum polarization corrections can easily be evaluated by straightforward extensions of the techniques presented in the present paper,\textsuperscript{11} we believe that the main labor of future calculations should be concerned with finding $K_m$ and evaluating the integral $B_{12}(K_m, k_i)$ rather than with the calculation of the matrix elements. Of course, if it is desired to go beyond the logarithmic terms a complete calculation will be necessary.

\textsuperscript{11} It is important to note in this connection that Tsai has shown (10) that the two double photon exchange diagrams taken together contain no logarithmic contribution although they do when considered separately. This can also be shown by a consideration of the regions of dominant contribution of the integrals, without detailed computation.
In the more usual experiment where initially an electron is at rest (Compton scattering or electron-electron scattering) and where the final particles are detected in coincidence with total energy resolution, $\Delta E$, other complications arise. In this case the energy of the scattered particles is very angular dependent (in the high-energy limit), so again the effect of angular resolution is very important. If the angular resolution is very good and $\Delta E \ll m$, $K_m$ will be isotropic. However, in high-energy experiments ($E \lesssim 100$ Mev) such good energy resolution is unlikely. If $\Delta E \gg m$, $K_m$ will become very anisotropic. For example, if a photon is emitted backward it is energetically impossible for it to remove an energy greater than $m/2$ (this may be seen most easily by the condition that the four-momentum of the remaining particles must be inside the light cone). If it is emitted forward there is no such restriction and $K_m \sim \Delta E$. If the angular resolution is not good and the counters are adjusted to accept only particles which exceed a minimum energy $E_m$, $\Delta E$ will vary with direction ($\Delta E = E_{el}(\theta) - E_m$), and may not have a well-defined value.

(iv) High-Energy experiments in which at least one light particle is undetected. Since an ultrarelativistic undetected particle behaves kinematically like a photon, nearly all of its energy is available to undetected photons. An example in the literature is pair production ($\gamma + P \to e^+ + e^- + P$), with the electron undetected (11). In this case $E_{el} \sim 40$ Mev considerably exceeded the energy resolution of the magnet which detected the positron, so $K_m$ for a given incident photon energy was essentially $E_p$. On account of the undetected proton, $K_m$ was nearly isotropic.

In ultrarelativistic $e^+ e^- \to e^+ e^-$, undetected photons can take nearly all the energy of an undetected electron if they are parallel to the electron, but are limited to $\sim m_e$ if they are antiparallel to the electron. In this case, then, $K_m$ is large and highly anisotropic. The dynamical effect of retardation factors strongly favors emission parallel to an electron, so the most significant values of $K_m$ are those in the direction of one of the electrons. Remarks similar to those of (iii) could be made about this process. For an example treated in detail, the reader is referred to Tsai’s discussion of Exp. II (10).

(v) Decay Processes. Our whole discussion of soft photons in scattering problems applies equally well to particle decays. In decays such as $K^+ \to 2\pi^+ + \pi^-$, etc., detection of all the charged particle tracks fixes the energy available to photons within limits set by observational errors. There are other decays, such as $\mu \to e + \nu + \bar{\nu}$, in which $K_m$ is necessarily very large.

5. SOME THEORETICAL IMPLICATIONS OF THE INFRARED DIVERGENCE PHENOMENON

It appears from the foregoing sections that, because of the infrared divergence, quantum electrodynamics is formally not a well-defined theory. However, we saw that in spite of this it is possible to extract physically meaningful results from it.
by some limiting procedure. It is likely that this formal difficulty arises because
we treat the initial and final states of motion of the charged particles as plane
wave states. In any practical situation in the laboratory, the charged particles
would remain within a bounded region during the course of the experiment; they
would therefore not easily emit photons of wavelength longer than the dimen-
sions of the laboratory. This suggests that a complete wave packet description
of an experiment, in which the charged particle starts at rest, is accelerated,
scatters (etc.), and is finally brought to rest in a detector would not require any
special cutoff. Such a description would be prohibitively difficult, and it is intuiti-
vely clear that it would not produce results which differ significantly from the
ones we have derived here. It may help reassure us to know, however, that the
theory is not inconsistent with respect to the infrared divergences and that the
cutoff procedure is really a matter of practical convenience rather than a distor-
tion of the content of the theory. On the other hand, the infrared divergence will
surely lead to difficulties in defining asymptotic states and in and out fields. We
do not intend to discuss these difficulties in the present section; rather we shall
show how results like those of the preceding sections may influence some other
theoretical considerations.

(a) KÄLLÉN'S PROOF THAT QUANTUM ELECTRODYNAMICS IS INCONSISTENT (25)

The validity of this proof has been questioned elsewhere (26). In the present
discussion, we shall indicate how infrared divergence considerations tend to make
an intermediate result in Källén's proof appear implausible. It should be stressed
in advance that the infrared divergence treatment is based on perturbation
theory and is hence not rigorous. It is also true that our result could agree with
Källén's if the so-called "spurious charge-renormalization constant" were infinite;
this would not contradict Källén's conclusion. Thus the present arguments have
intuitive value only and are not to be interpreted as a disproof of Källén's
argument.

It will be convenient to start with a review of some of the main features of
Källén's argument. In the proof it is assumed initially that the renormalization
constants, and also all physically significant matrix elements, are finite. It is
claimed that this places such strong restrictions on the matrix elements that one
is led ultimately to the conclusion that at least one of the renormalization con-
stants is infinite, in contradiction to the original assumptions. The argument pro-
ceeds through the study of the charge renormalization constant \((1 - L)^{-1/2}\),
which is given by

\[
(1 - L)^{-1} = 1 + \bar{\pi}(0),
\]

where

\[
\bar{\pi}(Q^2) = P \int_0^\infty \frac{\pi(a)}{a - Q^2} \, da
\]
and

\[ \pi(Q^2) = \langle V/3Q^2 \rangle \sum_{\mu \neq 0} \langle 0 | j_\mu | z \rangle \langle z | j_\mu | 0 \rangle. \] (5.3)

\( P \) denotes the principal value, and the sum in (5.3) goes over all states for which the vector \( Q_\mu \) has a given value. In spite of the indefinite metric of quantum electrodynamics, each state gives a positive contribution to \( \pi(a) \).

Källén considers particularly the contribution to (5.3) arising from the state of an electron-positron pair; this gives a lower bound to \( \pi(a) \). The problem then reduces to a study of the matrix element \( \langle 0 | j_\mu | q, q' \rangle \), where \( q \) and \( q' \) are, respectively, the energy-momentum vectors of the electron and positron, satisfying \( q + q' = Q \). The essential point in the proof is the assertion that as \( Q^2 \to \infty \) this matrix element approaches \( (N^2/1 - L) \) times the renormalized perturbation expression, where \( N \) is the spurious charge renormalization. This would lead to a constant contribution to \( \pi(a) \) as \( a \to \infty \) and hence would cause the divergence of the integral in (5.2) (logarithmically, as in perturbation theory), thus yielding the desired conclusion that the initial assumptions are contradicted.

On the other hand, a matrix element of the form \( \langle 0 | j_\mu | z \rangle \) has a simple physical interpretation; it is directly related to the matrix element for an external potential to induce a transition from the vacuum to the state \( z \) (to lowest order in the external potential). We can thus apply the apparatus built up in the preceding sections. In particular we expect,

\[ \langle 0 | j_\mu | q, q' \rangle = \exp(\alpha B)\langle 0 | j_\mu^{(0)} | q, q' \rangle 1 + O(\alpha). \] (5.4)

(The \( j_\mu^{(0)} \) means the matrix element in lowest order of perturbation theory.) As \( \lambda \to 0 \) (or alternatively, \( Q^2 \to \infty \)), \( B \) tends towards minus infinity; there is no other infrared divergence. Thus it seems likely that if the matrix element is not infinite, it is zero. This result has a simple physical interpretation: It is impossible for an external potential to create a pair of particles, complete with their proper fields, without creating some real photons. Equation (5.4) would be inconsistent with Källén's result unless \( N = 0 \). If \( N = 0 \), Källén's main conclusion would, of course, be correct. In an earlier criticism of Källén's proof, Johnson (27) pointed out that while \( \langle 0 | j_\mu | q, q' \rangle \) is gauge invariant in all orders, \( N \) is not. He also showed that in the lowest order of perturbation theory the matrix element does not approach \( N^2/(1 - L) \) times the Born approximations. This criticism was recently answered by Källén (28). Recently, Johnson and Zumino (29) studied the gauge dependence of \( N \) in detail. Their work indicates that in the gauge used by Källén the infrared dependence implies \( N = \infty \) rather than \( N = 0 \).

The infrared divergence in (5.4) will, of course not actually occur in (5.3) because the sum over states \( z \) will include real soft photons. In fact, we can estimate the effect of summing over states with soft photons in addition to the electron-positron pair. Suppose we permit the soft photons to take off an energy
up to the amount $K$ (in the center-of-mass coordinate system). In the problem under consideration $K$ is of course equal to $Q - 2m$. However, a complete calculation, with $Q - 2m$ as the upper limit for the total energy of the soft photons, would be rather involved. Instead for purposes of orientation, we choose $K \ll Q$. Then we may apply the methods of Section 4a with the result

$$\pi_1(Q^2, K) = F(\alpha A) \exp \left[ -\frac{1}{2} \alpha A \ln \frac{Q^2}{K^2} \right] \pi_1^{(0)}(Q^2) \{1 + O(\alpha)\}, \quad (5.5)$$

where $\pi_1^{(0)}$ is the one-pair contribution to $\pi$ in lowest order. It is clear that in the complete result the infrared suppression which occurred in (5.4) is no longer present, and it is intuitively plausible that as $K \to Q - 2m$ the exponential factor is no more important than other higher order terms which have been neglected in the one-pair contribution. It has been pointed out by Lomon and Shaw\(^{12}\) that the factor $F(\alpha A)$ in (5.5) will tend to damp out $\pi_1$ when $Q^2$ becomes very large [see Section 2(e)]. It is not clear from the present work whether this damping is sufficiently strong to bring about the convergence of (5.2).

Our intuitive discussion indicates that it may not be easy to construct a proof that $\pi(0)$ is infinite by a consideration of the one-pair contribution alone. In order to avoid the infrared divergence problem, it may be more convenient to study the matrix elements involving those states $z$ in which there are no electrons. These will not be influenced by the infrared divergence.\(^{12}\) A simple dimensional argument shows that the matrix element for the production of $n$-photons can be expressed in the form

$$(Q^2)^{(n-1)/2} \prod_{i=1}^{n} k_i^{1/2} H \left( \frac{k_1}{\sqrt{Q^2}}, \ldots, \frac{k_n}{\sqrt{Q^2}} \right). \quad (5.6)$$

If $H$ does not vanish as $(m/\sqrt{Q^2}) \to 0$, (5.6) will give a constant contribution to $\pi$ in the limit $Q^2 \to \infty$. The problem can thus be handled by studying the limit of $H$ as $m \to 0$. In the case where charged particles exist in the state $z$, the analog of (5.6) will vanish as $m \to 0$ because of the infrared divergence. However, there is no obvious reason why it should do so in the present case. The quantity $m$ enters the expression $H$ through the one or more closed electron loops which must occur in the diagrams contributing to the matrix element. There are two types of such closed loops: those which occur in vacuum polarization graphs, and closed loops with four or more "external" lines. No singularity seems to arise

\(^{12}\) The pure photon states in (5.3) were used in certain theoretical arguments by Källén (30) and Kamefuchi (31). However, they misused the results of Gupta (18) on multiple photon production. As we have seen in Section 3C, Gupta’s results are essentially the dominant infrared contribution and they apply specifically to the case where there are external charged lines. Gupta’s methods (and ours) can make no statement about pure photon processes.
when we set \( m \) equal to zero in the latter type. The former type does have a logarithmic dependence on \( m \) in the lower orders of perturbation theory. However, following the assumptions of Källén's proof, we may now invoke the condition that the renormalization constants are finite. Then \( \pi(a) \to 0 \) as \( a \to \infty \); but \( \pi \) is dimensionless and hence can depend only on \( a/m^2 \). Thus as \( m \to 0 \), \( \pi \) must vanish for all nonvanishing \( a \). The same must hold for \( \bar{\pi} \). This means that we may omit all vacuum polarization parts from internal photon lines if we multiply such lines by the factor \( (1 - L)^{-1} \). The prescription for \( H \) thus becomes: (i) omit all graphs having vacuum polarization contributions but introduce a factor \( (1 - L)^{-1} \) for each internal photon line. (ii) set \( m = 0 \) in all electron lines. If the resulting \( H \) does not vanish, the theory is inconsistent.

(b) THE INFRARED DEPENDENCE OF THE SPURIOUS CHARGE RENORMALIZATION

It is well known that as a consequence of Ward's identity \((32)\) the charge renormalizations associated with the vertex operator and the electron propagator (or wave function) cancel each other in quantum electrodynamics. Neither of these renormalizations is separately independent of the gauge of the quantized electromagnetic field; but the cancellation is gauge-independent. The gauge dependence of this spurious charge renormalization has recently been investigated thoroughly by Johnson and Zumino \((29)\). Their result may be summarized as follows. If the photon propagator undergoes the transformation

\[
\mathcal{S}_\mu^\nu(x - y) = \mathcal{S}_\mu^\nu(x - y) + \partial_\mu \partial_\nu \mathcal{M}(x - y), \tag{5.7}
\]

the spurious charge renormalization undergoes the transformation

\[
Z_1' = \exp \{-e^2 M(0)\} Z_1. \tag{5.8}
\]

It is also known that in the usual covariant gauge the spurious charge renormalization has an infrared divergence. It has been found \((33)\) that with the special gauge given by the substitution

\[
\frac{\delta_\mu^\nu}{k^2} \to \frac{\delta_\mu^\nu}{k^2} + \frac{2k_\mu k_\nu}{(k^2)^2}, \tag{5.9}
\]

the infrared divergence of \( Z_1 \) disappears in lowest order. We wish now to give arguments that with this special gauge \( Z_1 \) will not have an infrared divergence in any order. The spurious charge renormalization is given by the vertex operator evaluated on the energy shell with zero momentum transfer; this corresponds to calculating the forward scattering amplitude in lowest order in the external potential with the omission of all wave function renormalizations. According to Appendix A, Eqs. \((2.2)\) and \((2.3)\) are valid for zero momentum transfer; to pick out the spurious charge renormalization, we must now omit the contributions
given by the diagrams in Fig. 2(b), (c). In place of (2.3), we find

\[ Z^{-1}_2 a(p, p) = \exp (a B') \sum_n M_{n'}(p, p), \]

(5.10)

where

\[ B' = \frac{-8i}{(2\pi)^3} \int \frac{d^4k}{k^2 - \lambda^2} \left[ \frac{p^2 k^2}{k^2} + \frac{2(k \cdot p)^3}{k^2 - 2k \cdot p} \right]. \]

(5.11)

The treatment here is slightly different from that of Section 2 in that terms of higher order in \( k \) in the numerator have been incorporated in \( m_n' \) rather than \( B' \). In Section 2 they were retained in order to preserve gauge invariance, but that is neither necessary nor desirable here. The second term in the square bracket of (5.11) contains the contribution from the gauge change. The sum in (5.10) contains the noninfrared parts of the spurious charge renormalization, including the ultraviolet divergent part.

Direct evaluation shows that (5.11) remains finite as \( \lambda \to 0 \). This confirms that with the special choice of gauge given by (5.9) the spurious charge renormalization will have no infrared divergence. It should be emphasized that (5.9) is the form of the renormalized photon propagator in the limit \( k^2 \to 0 \) (that is, with the omission of finite radiative corrections). This is clearly only one example of a whole class of gauges having this property.

(c) The Infrared Dependence of the Vertex Function and the Electron Propagator

The electron propagator and vertex function off the mass shell have no infrared divergence, and no infrared cutoff is necessary for their evaluation. However, these functions do have a singularity as their arguments approach the mass shell; this singularity is related to the infrared divergence. Since this behavior is well known, mainly from the work of Russian physicists (34), only a brief discussion needs be given here. Because there is really no infrared divergence, there is no reasonably unambiguous procedure for separating the infrared from the noninfrared contributions. We shall proceed simply by "analytically continuing" our previous separation off the mass shell.

In place of (2.5), we then have the vertex operator\(^{13}\)

\[ \rho_n(p, p', k_1 \ldots k_n) = S''(p, p', k_n) \rho_{n-1}(p, p', k_1 \ldots k_{n-1}) + \beta^{(D)}_n(p, p', k_1 \ldots k_{n-1}; k_n), \]

(5.12)

\(^{13}\) We use here the gauge in which the renormalized photon propagator has the form \( \delta_{\mu\nu}/k^2 \) for small \( k^2 \). We are indebted to Professors Johnson and Zumino for several interesting conversations about the gauge dependence of the vertex function and electron propagator. For further details, see Refs. 29 and 36.
where

\[ S''(p, p', k) = \frac{-8i\alpha}{(2\pi)^3} \frac{p \cdot p'}{\left(k^2 - 2k \cdot p + p^2 - m^2\right)\left(k^2 - 2k \cdot p' + p'^2 - m^2\right)}. \]  

(5.13)

This leads to equations analogous to (2.8), (2.2) and the final result is

\[ \Lambda_{\nu}(p, p') = \exp \left[ \alpha B''(p, p') \right] \Lambda_{\nu}'(p, p'), \]  

(5.14)

where

\[ \alpha B''(p, p') = \int \frac{d^4k}{k^2 + i\epsilon} S''(p, p', k) \]  

\[ = -\frac{\alpha}{2\pi} \int_{-1}^{1} dx \frac{p \cdot p'}{p^2} \ln \left(\frac{p_x^2 + \Delta_x}{\Delta_x}\right), \]  

(5.15)

where

\[ 2p_x = (1 + x)p + (1 - x)p' \]  

and \[ 2\Delta_x = (1 + x)(m^2 - p^2) + (1 - x)(m^2 - p'^2). \]

This exhibits the singular behavior which occurs when both \( p^2 - m^2 \) and \( p'^2 - m^2 \) are small.

A similar discussion may be given for the proper electron self-energy part \( \Sigma(p) \). By the same type of argument, it is found that this operator takes the form

\[ \Sigma(p) = \exp \left[ \alpha B''(p, p) \right] \left[ \Sigma'(p) - S^{-1}_F(p) \right] + S^{-1}_F(p) \]  

(5.16)

so that the complete (unrenormalized) propagator may be written

\[ S'_F(p) = \left[ S^{-1}_F(p) - \Sigma'(p) \right]^{-1} \left[ 1 - \frac{p^2}{m^2} \right]^{-\alpha/x} \]  

(5.17)

displaying the singularity when \( p^2 - m^2 \) approaches zero.

Had we used the gauge introduced in Section 5(b), this singularity would not have occurred in (5.17), and (5.15) would have been changed to

\[ \frac{-8i\alpha}{(2\pi)^3} \int \frac{d^4k}{(k^2)^2} \frac{\left(k^2 p \cdot p' + 2k \cdot p \cdot p'\right)}{\left(k^2 - 2k \cdot p + p^2 - m^2\right)\left(k^2 - 2k \cdot p' + p'^2 - m^2\right)}. \]  

(5.18)

With this special gauge the infrared dependence is concentrated in the vertex operator, and even then it disappears when \( p = p' \).

6. SUMMARY AND DISCUSSION

In this paper we have tried to develop two main points. The first of these, which is mainly of theoretical interest, is the treatment of the infrared divergence
to all orders of approximation. In this treatment, we have not attempted to be absolutely rigorous, but we have emphasized the importance of a correct discussion of the problem of overlapping infrared divergences. Our own discussion (Appendix A) may seem unnecessarily elaborate, but we have not been able to find a simpler discussion which does not somehow obscure the difficulties or ignore them completely. In any case, the final result is intuitively very plausible from the classical arguments given in the Introduction. Numerically the contribution from the higher infrared corrections is not likely to be important for most experiments unless the energy becomes extremely high and the energy resolution very good. This general discussion is contained mainly in Sections 2 and 4(a), (b).

The second main point we have developed is a new practical approach to problems involving the infrared divergence phenomenon. More or less as a by-product of the discussion of the complete cancellation of infrared divergences, we have shown how the infrared divergences may conveniently be separated out as factors of noninfrared divergent quantities. We have then shown how the real and virtual infrared divergences cancel out in a way which is independent of the details of the interactions; this discussion is given in Section 4. The infrared contributions have a bilinear logarithmic dependence at high energies. One logarithm comes from the integration of $dk/k$ and the other from the angular integration; the second logarithm is associated with the strong peaking of the photon cross section in directions parallel to the motion of the charged particles. Because of this bilinear logarithmic dependence, the infrared radiative corrections dominate other corrections to a given order in $\alpha$. There is also strong evidence that the logarithms associated with the infrared dependence, together with those coming from the “magnetic terms,” and vacuum polarization, give the major (if not the only) terms proportional to the logarithm of the energy. Since the magnetic terms and vacuum polarization can be easily estimated, the methods presented in this paper should enable one to make very simple, but accurate, estimates of most processes in high energy quantum electrodynamics. For example, we believe that the Schwinger correction to electron scattering is very accurate in general, except for the nonlogarithmic terms.

With the method of calculation proposed here, the corrected cross sections are given by certain factors times the uncorrected cross section. These factors are given by the combinations $B_{ij} + Re B_{ij}$ of Section 4, augmented by the magnetic and vacuum polarization corrections; except for the vacuum polarization, these factors depend only on the external lines and in many cases the vacuum polarization can be estimated from the external lines. Since they depend only on external lines, these factors are extremely easy to calculate except for their dependence on $K_m(\Omega)$, the maximum photon energy as a function of angle. A careful treatment of $K_m(\Omega)$ is much more important than calculation of the terms, believed to be of order 1%, which cannot be factored in the manner indicated above. Thus, in
future calculations in quantum electrodynamics at high energies it should not be necessary to make detailed integrations to obtain the matrix elements (unless one is interested in 1% accuracy). The main problem will be to find $K_m(\Omega)$, as discussed in Section 4, and to carry out the integral giving the quantity $\bar{R}'([K_m], k_1)$ defined in (4.11). We cannot emphasize too strongly the importance of a proper treatment of $K_m$. As we have said previously, $K_m$ may depend critically upon the experimental conditions. So the calculation must be tailored to fit the experiment.

For the experimentalist who may not have had the patience to read through our theoretical discussions in detail, we would like to point out briefly where he may expect to find results of possible interest. For his purposes, the main result of Section 2 is our previously stated belief that the Schwinger correction should give a quite accurate estimate of radiative corrections to electron scattering, even in heavy elements; in case of high energy resolution, the higher infrared corrections given there may be important. Section 3 contains a number of practical results. Sections 3(a) and 3(b) discuss electron scattering with appreciable energy loss and give the radiative corrections to the spectrum of scattered electrons which have emitted bremsstrahlung or undergone inelastic scattering, respectively. Section 3(d) treats the effect of target recoil on the radiative corrections. In electron-proton scattering two such effects are of opposite sign and reduce the importance of the correction, but in positron-proton scattering they enhance each other and might become significant. Section 3(c) contains a discussion of the effect of time delay in nuclear scattering upon the bremsstrahlung. Finally, as we have said in the previous few paragraphs, Section 4 contains a discussion of how the bulk of the radiative corrections can be precalculated for an arbitrary process. The application of these results depends upon a critical analysis of the details of an experiment, but is otherwise quite straightforward.

In conclusion, we mention several points which are left to future discussion. For one thing, we do not claim to have presented a rigorous proof, although we have tried to make our arguments as convincing as possible. We have operated entirely within the framework of perturbation theory; one might hope to see the infrared factor extracted independently of perturbation theory, perhaps by a canonical transformation. Finally, there is a difficulty, which we have mentioned, but not discussed, in defining "in" and "out" fields in the presence of infrared divergences.

Acknowledgments

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APPENDIX A. EXTRACTION OF THE INFRARED FACTORS FROM THE MATRIX ELEMENTS

In this Appendix we shall study the matrix element of an arbitrary process in order to show how the infrared divergence associated with any one of the photons (real or virtual) may be factored out. It will be necessary to examine the overlapping divergences very carefully in order to verify that the extraction of an infrared factor for a given photon will not result in an increased infrared divergence relative to the other photons. For the purposes of the Appendix, it will prove convenient to introduce a special notation for terms which are not infrared divergent in certain photon momenta and are not worse than normal (i.e., not worse than logarithmic) in the remaining photon momenta. The notation is $K(k_i, k_j, \cdots)$ where the $k$'s that appear explicitly are the ones which do not lead to an infrared divergence upon integration. The notation may be considered as analogous to the notation for "order of;" it indicates a property, not a particular functional form. In the end, these terms can be identified with the various $\beta_i^D$ (with $i \neq 0$) occurring in Sections 2 and 4.

The method used for the extraction of the infrared divergence factors is basically the same as that given in a previous paper (14) for the treatment of real photon emission, but it has been simplified by relating it more directly to gauge invariance and has also been extended to the treatment of virtual photons. In view of the discussion of Section 2 and Appendix B, we can treat all the photons as distinguishable and insert them independently in all possible ways into the various Feynman diagrams; the overcounting of contributions is then automatically compensated for by the combinatorial factors multiplying the integrals. Consider then the Feynman diagram $G$ for some process, and the set of diagrams obtained by inserting one additional photon (real or virtual) into $G$ in all possible ways. For the moment we consider only diagrams in which there are no closed charge loops; the generalization to closed loops will be discussed at the end of this Appendix. It will be convenient to divide this set of diagrams into various subsets and consider in detail two different cases. Case (a) will consist of a subset of diagrams in which only one end of a photon line terminates on a given charge line in all possible ways; the other end of the photon line is either external or it terminates on a different charge line and is held fixed. Case (b) will consist of a subset of diagrams in which both ends of a virtual photon line terminate on the same charge line. After the results for these two special cases are obtained, it will be immediately evident how the complete result is to be constructed from them.

Case (a)

For definiteness, assume that the charge line is that of an electron, and that the photon is virtual; the result may easily be altered for other cases later. The
original electron line is indicated in Fig. 1; the shaded area represents all possible interactions of the electron: potential interactions, photons exchanged with other electrons, real photon emission and absorption, and virtual photons emitted and reabsorbed by the same electron. The additional photon line is now to be terminated on the electron line in all possible ways. Our aim will be to show that the sum of these contributions has the following form: it will be a certain infrared multiplier times the contribution represented by Fig. 1 plus a residue which will have no infrared divergence with respect to the new photon and will be no worse than normal with respect to the original photons.

Let \( q_i \) be the four-momentum transferred at the \( i \)th vertex of the electron line (the vertices are ordered in the direction of the arrow on the electron line); the matrix element corresponding to \( G \) will then contain a factor of the form

\[
\bar{u}_{p'} \Gamma(p, q_i) u_p,
\]

where

\[
p' = p + \sum q_i
\]

corresponding to this electron line. This factor is to be multiplied by factors coming from other elements of the diagram and integrated over the \( q_i \)’s, subject to the condition that \( p' \) be held fixed.

We consider first the simple case where \( p' \) is nearly parallel to \( p \); this restriction will be removed later. We adopt the convention that the four-momentum of an intermediate electron propagator is given in terms of \( p \) and the \( q_i \) transferred to the electron line before the propagator (rather than \( p' \) and the \( q_i \) transferred after the propagator). Suppose the additional photon removes the four-momentum \( k \) from the electron line; then it will be necessary to adjust the \( q_i \)'s slightly from their values in \( G \). The new values \( \tilde{q}_i \), will differ from the old ones by an amount of order \( k \) and will satisfy the relation

\[
p' = p + \sum \tilde{q}_i - k.
\]

The \( \tilde{q}_i \) will be held fixed while the additional photon is inserted in all possible ways. If the photon is real, it may be necessary to adjust \( p' \) as well, because of energy conservation.

First consider the contribution obtained by inserting the additional photon before all other interactions, Fig. 3(b); the resulting factor in the matrix element is

\[
\bar{u}_{p'} \Gamma(p - k, \tilde{q}_i)(p - k - m)^{-1}\gamma_\mu u_p
\]

\[
= \bar{u}_{p'} \Gamma(p - k, \tilde{q}_i)u_p \frac{2p_\mu - k_\mu}{k^2 - 2k \cdot p} - \bar{u}_{p'} \Gamma(p - k, \tilde{q}_i) \frac{i\gamma_\mu}{k^2 - 2k \cdot p} u_p.
\]
The singularity of order \((1/k)\) in the first term, in combination with other factors, leads to an infrared divergence. As discussed in Section 2, the form of this term has been extended to large \(k\) in a way which provides a natural high-energy cutoff in the later integration over the infrared factor; it also has a structure which insures that the separation of the infrared factor is gauge invariant. The second term, which is in the form of a magnetic moment interaction, is not infrared divergent in the \(k\)-integration and is no more infrared divergent than initially in the other variables; hence it is of the form \(K(k)\). The first term is approximated by setting \(k\) equal to zero inside the matrix \(\Gamma\), with the result

\[ \frac{2p_\mu - k_\mu}{k^2 - 2k \cdot p} \bar{u}_{\mu'} \Gamma(p, \bar{q}_i) u_{\mu}. \]  \hspace{2cm} (A-4)

The difference between this approximation and the first term of (A-3) is

\[ \frac{2p_\mu - k_\mu}{k^2 - 2k \cdot p} \bar{u}_{\mu'} \left[ \Gamma(p - k, \bar{q}_i) - \Gamma(p, \bar{q}_i) \right] u_{\mu}. \]  \hspace{2cm} (A-5)

Since the square bracket vanishes for \(k_\mu = 0\), it may appear that this term has no infrared divergence in \(k\) and should be considered part of \(K(k)\). In fact, this difference requires a very careful treatment because of the overlapping infrared divergences; it is possible for the difference to be just as divergent as either term separately. There is a simple way of interpreting (A-5), suggested by Feynman's treatment of gauge invariance (36). Let \(\Lambda_\mu(p, \bar{q}_i; k)\) be the matrix element for the emission of the additional photon from the internal part of the electron line as in Fig. 3(c). Using the operator identity

\[(p + k - m)^{-1} k(p + K - m)^{-1} = (p + k - m)^{-1} - (p + K - m)^{-1},\]

Feynman has shown that if a photon of polarization \(k_\mu\) is emitted from an electron line, the net contribution to the matrix element is zero. This result corresponds to the following matrix identity

\[ k_\mu \Lambda_\mu(p, \bar{q}_i; k) = \Gamma(p - k, \bar{q}_i) - \Gamma(p, \bar{q}_i). \]  \hspace{2cm} (A-6)

Thus the combined contribution from (A-5) and all internal photon emissions is contained in the single expression

\[ \bar{u}_{\mu'} \left( \Lambda_\mu + \frac{2p_\mu - k_\mu}{k^2 - 2k \cdot p} k_\lambda \Lambda_\lambda \right) u_{\mu}. \]  \hspace{2cm} (A-7)

This corresponds to the following substitution for the emission operator for photons emitted from internal lines

\[ \gamma_\mu \rightarrow \gamma_\mu + \frac{2p_\mu - k_\mu}{k^2 - 2k \cdot p} k = g_\mu. \]  \hspace{2cm} (A-8)
It is interesting to notice that this new emission operator is gauge invariant:

\[ k \rho f^\mu = 0. \]

Now consider a particular contribution to (A-7) arising from the insertion of the photon into a line of four-momentum \( p + Q \) where \( Q \) is the sum of the \( q_i \) preceding the line under consideration. This insertion will modify this internal propagator in the following way:

\[
(p + Q - m)^{-1} \rightarrow (p + \tilde{Q} - k - m)^{-1} 
\]

Later electron lines are also modified by the substitution \( p \rightarrow p - k \). In this form we can see why the singularity in the \( q_i \) integration may be increased by the extraction of the infrared factor for \( k \). When \( \tilde{Q} \) and \( k \) are simultaneously small, the single small denominator in \( \Gamma \) has been replaced by two small denominators in (A-7). Fortunately, this additional singularity cancels out as a result of combining the two contributions by the substitution (A-8). Rationalizing the first factor of (A-9), we find

\[
\frac{1}{(Q - k)^2 + 2(Q - k) \cdot p} \left\{ -\left( \gamma^\mu + \frac{2p^\mu - k^\mu + k^\mu}{k^2 - 2k \cdot p} \right) (p + \tilde{Q} - m)^{-1} \right\}
\]

None of the three terms in the curly bracket can cause any new difficulty when \( \tilde{Q} \) is small. In the first, one of the extra denominators is canceled. In the second the numerator vanishes when \( \tilde{Q} = 0 \). To see that the third term causes no trouble, set \( \tilde{Q} = 0 \) in the factor in front of the bracket. The \( k \) integration will then be perfectly convergent at \( k = 0 \), showing that this factor is not really singular at \( \tilde{Q} = 0 \).

Adding to (A-4) the contribution from the diagram in which the photon is emitted after all the other interactions, we obtain the expression

\[
\left( \frac{2p^\mu - k^\mu}{k^2 - 2k \cdot p} + \frac{2p^\prime\mu + k^\mu}{k^2 + 2k \cdot p^\prime} \right) a_\rho \cdot \Gamma(p, \tilde{q}_i) u_\rho. \quad (A-11)
\]

This is the main result of the discussion for Case (a) specialized to the situation \( p' \approx p \). If we like, we may replace \( \tilde{q}_i \) by \( q_i \) in the spinor matrix element; the correction is again of the form \( K(k) \).

Special care is required if the \( G \) in the electron line contains self energy parts on the external lines. Some of the \( Q \)'s will then be identically zero. It is not hard to show that these parts will produce the same wave function renormalization in
our final result, (A-11), as in the original matrix element. We skip the details of the proof.

We should now discuss how the restriction to small energy-momentum transfer can be removed. If we have such a large transfer, we assume that at least one of the $q_i$'s is large. It is now convenient to work from both ends toward the middle of the electron line where these hard momentum transfers take place. The electron lines before the first hard interaction are labeled with $p$ and the $q_i$'s transferred up to the given point, and the electron lines after the last hard interaction are labeled with $p'$ and the $q_i$'s transferred after the given point. If the additional photon is added to the initial external line, $k$ will then appear only in the lines before the first hard interaction (the hard interactions can absorb the change in $k$ without introducing any divergence). If we set $k$ equal to zero in these lines, we again obtain a result like (A-4); the error made introduces no new infrared divergence if we take into account all other ways of inserting the additional photon before the first hard interaction. The insertion of soft photons in all ways after the last hard interaction may be handled in the same way, and we again obtain (A-11).

We can summarize our result as follows. If we insert one end of an additional virtual photon line into an open electron line in all possible ways, we will find two types of contributions. The first will be the original factor in the matrix element multiplied by

$$R_s(p, p', k) = \left( \frac{2p_a - k_x}{k^2} + \frac{2p_a' + k_x}{k^2 + 2k \cdot p'} \right) \tag{A-12}$$

which has been chosen to be gauge invariant

$$k_x R_s = 0 \tag{A-13}$$

and to provide a natural cutoff at large $k$. The second type is the residue which is well behaved as $k \to 0$.

The corresponding result for real photon emission is obtained by the same arguments. In this case the infrared contribution to the matrix element is obtained by multiplying the original matrix element by the factor

$$\tilde{R}_s(p, p', k) = \left( \frac{p_a'}{k \cdot p'} - \frac{p_a}{k \cdot p} \right) \tag{A-14}$$

Since $k^2 = 0$, there is now no need to keep the $k$ in the numerator for gauge invariance; it may be incorporated in the terms of form $K(k)$. We simultaneously neglect $k^2 = \lambda^2$ in the denominator since it also gives a change of the form $K(k)$.

The results given in (A-12, 14) apply to any charged particle. For example, with spin zero particles, there are two types of electromagnetic vertices, corresponding to single or double photon emission or absorption. Thus the additional photon may terminate either on one of the boson propagators of the original
diagram or at one of the single photon vertices of the original line, converting it into a double one. Since the latter type of termination does not increase the number of denominators which can be small, it is of the form $K(k)$. The discussion for the case where the photon terminates in a single vertex closely parallels the discussion for the electron line, only it is somewhat simpler. For example, there is no contribution corresponding to the second term of (A-3) or the first and third main terms of the bracket of (A-10).

If we are considering real photon production, each open charged line will contribute a factor $\hat{R}_\mu$ to the matrix element. This factor depends only on the external lines and not the specific form of $G$. We therefore have the result that the matrix element for the emission (or absorption) of one additional photon while the other external lines are held fixed is given by

$$e \frac{1}{2(2\pi)^2 h_0^{1/2}} \sum_i \hat{R}_\mu(p_i, p_i', k) M + K(k),$$

where $M$ is the matrix element for the original process, including all virtual photons and potential interactions and $\hat{R}_\mu(p_i, p_i', k)$ is the emission operator for the $i$th particle. If two or more of the particles in the initial or final states are identical, the total photon emission operator of (A-15) will be symmetric in the variables of those particles and the overall symmetry of the new matrix element will be the same as that of the original one. That is, the individual $\hat{R}_\mu$ are not invariant to the interchanges that take place when contributions from exchange diagrams are added together to form $M$, but their sum is invariant.

If the additional photon is virtual and it terminates on two different electron lines, our analysis can be carried through for each line separately and the result will be a product of two factors of the form of (A-12), with the sign of $k$ reversed in one of the two factors (representing the fact that momentum is taken from one line and added to the other). Now, because of the possibility of exchange, we must consider the effect on each diagram $G$ separately. The correction obtained by adding one virtual photon of momentum $k$ which connects two different charged lines is to multiply the original matrix element corresponding to $G$ by the factor

$$-\frac{1}{2} \left[ \frac{ie^2}{(2\pi)^3(k^2 - \lambda^2)} \right] \sum_{i \not= j} R_\mu(p_i, p_i', k) R_\nu(p_j, p_j', -k).$$

After we have studied the case (b), we will see that the final complete expression is symmetric.

Case (b)

If both ends of a photon line terminate on a single electron line, there will be ultraviolet as well as infrared divergencies. The ultraviolet divergencies are associated with the mass renormalization and the spurious charge renormalization.
The spurious charge renormalization also has an infrared divergence in the usual gauge. The mass renormalization will be handled by the usual procedure of direct cancellation by a counter term. The contributions from various diagrams will be grouped so that the spurious charge renormalization cancels out directly. For simplicity, assume that the electron line initially has no self-energy parts on external lines; it is obvious that our result will be correct for the more general case where there are initially wave function renormalizations.

We first study the class of diagrams where the additional photon does not yield a wave function renormalization. The procedure will be as follows: hold one end of the photon line fixed at a point after the first interaction and sum over all contributions corresponding to insertions of the other end at points before the fixed point. This class of diagrams is illustrated generally in Fig. 2(d), (f), (g); if the fixed point is on the final external line, the class of graphs is illustrated in Fig. 2(a), (e). As is well known, all ultraviolet divergencies for this set will cancel as a consequence of Ward's identity (32) (except for the case where the fixed point is on the final external line; this cancellation requires wave function renormalization). Finally sum over all contributions corresponding to different positions of the fixed point.

The contribution from Fig. 2(d), with the fixed point an interior point, will contain a factor of the form

\[ \bar{u}_{p'} \Gamma_2(p + Q - m)^{-1} \gamma_\mu (p + Q - k - m)^{-1} \]

\[ \cdot \Gamma_1(p - k, q_i)(p - k - m)^{-1} \gamma^\mu u_p = \bar{u}_{p'} \Gamma_2(p + Q - m)^{-1} \]

\[ \cdot \frac{2p - k}{k^2 - 2k \cdot p} (p + Q - k - m)^{-1} \Gamma_1(p - k, q_i)u_p + K(k), \]

where \( \Gamma_1 \) and \( \Gamma_2 \) are matrices representing the factors arising from the parts of the graph before and after the fixed point, respectively. The quantity \( \Gamma \) defined in (A-1) is just the product \( \Gamma_2(p + Q - m)^{-1} \Gamma_1 \), with \( k \) set equal to zero. The matrix \( \Gamma_1(p, q_i) \) satisfies an identity similar to (A-6), and we may accordingly follow the same procedure as in Case (a). The contribution arising from the difference between \( \Gamma_1(p - k, q_i) \) and \( \Gamma_1(p, q_i) \) is combined with the contribution from those graphs in Fig. 2(f) where the variable end of the photon terminates between the first and last interactions in \( \Gamma_1 \). As before, this just corresponds to the substitution (A-8) and by the same arguments as before it is easily seen that this net contribution does contain an ultraviolet divergence (upon integration with respect to \( k \)) which is associated with a vertex part for the last interaction in \( \Gamma_1 \).
The remainder to be studied is
\[
\tilde{a}_{\rho'} \Gamma_2(p' + Q - m)^{-1} \left\{ \frac{2p - k}{k^2 - 2k \cdot p} \left[ 1 + \frac{1}{p + Q - k - m} \right] \right\} + \gamma^\mu \frac{1}{p + Q - k - m} \frac{2(m^2 + k \cdot p)/m}{k^2 - 2k \cdot p} (p + Q - m)^{-1} \Gamma_1 u_p.
\]
(A-17)

The first main term in the curly bracket corresponds to (A-16), with \( k \) set equal to zero in \( \Gamma_1 \); the second term corresponds to the one graph of Fig. 2(f) in which the photon is emitted and reabsorbed on the same internal electron line; and the third term is a factor in the integrand which gives the mass renormalization. We will now argue that this whole expression is of form \( K(k) \). We see that the second part of the first main term together with the second term has the structure of (A-9), if that expression is multiplied by \( \gamma^\mu \) from the left. This means that we have a contribution of the form of (A-10) multiplied by \( \gamma^\mu \) from the left. Examination shows that the contribution corresponding to the first two terms of (A-10) is of the form \( K(k) \) and that the last term contributes to the mass renormalization. Thus the curly bracket of (A-17) reduces to
\[
\frac{2(p - m)}{k^2 - 2k \cdot p} - \frac{(mk + 2k \cdot p)/m}{k^2 - 2k \cdot p} + \frac{3k}{k^2 - 2k \cdot (p + Q) + (Q^2 + 2Q \cdot p)} + K(k).
\]

The new first term is obviously of the form \( K(k) \). Due to cancellations, the second and third terms will also give a net result of the form \( K(k) \); in showing this it is convenient to use symmetry to make the substitution \( k \rightarrow (p + Q)k \cdot (p + Q)/m^2 \) in the second term and \( k \rightarrow (p + Q)k \cdot (p + Q)/(p + Q)^2 \) in the third term. Thus the infrared divergence due to (A-17) disappears; it of course contains an ultraviolet divergence which is necessary (from general principles) to cancel the ultraviolet divergence mentioned at the end of the preceding paragraph.

Finally, we consider the case where the fixed point is on the final external line. The first graph of this case is shown in Fig. 2(a); its contribution is
\[
\tilde{a}_{\rho'} \frac{2p - k}{k^2 - 2k \cdot p} \frac{1}{p' - k - m} \Gamma(p - k, q_i) u_p + K(k).
\]
By the usual argument, this may be replaced by
\[
\frac{(2p - k) \cdot (2p' - k)}{(k^2 - 2k \cdot p)(k^2 - 2k \cdot p')} \tilde{a}_{\rho'} \Gamma(p, q_i) u_p + K(k).
\]
We can now identify the infrared part of the spurious charge renormalization most easily by setting \( p' \) equal to \( p \) in (A-18). After this spurious charge renormalization is removed, the result for Case (b) is found to be
\[
-\frac{1}{2} \left[ \frac{(2p - k)_\mu}{k^2 - 2k \cdot p} - \frac{(2p' - k)_\mu}{k^2 - 2k \cdot p'} \right]^2 \tilde{a}_{\rho'} \Gamma(p, q_i) u_p.
\]
(A-19)
It is again apparent that the separation of the infrared factor is gauge invariant. The square of the individual terms in the bracket are represented diagrammatically in Fig. 2b, c, the usual wave function renormalization diagrams. It is clear that we have to add to (A-15) the following contribution from graphs in which the additional photon is attached to a single electron line.

$$\frac{1}{2} \left[ \frac{i e^2}{(2\pi)^3 (k^2 - \Lambda^2)} \right] \sum_i \left[ \frac{(2p_i - k)_\mu}{k^2 - 2k \cdot p_i} - \frac{(2p_i' - k)_\mu}{(k^2 - 2k \cdot p')} \right]. \quad (A-20)$$

The complete result for both real and virtual photons is summarized in Section 4b. It is interesting to note that the wave function renormalization contained in (A-19) contains all the ultraviolet as well as the infrared divergence, although it does miss some finite renormalization. Details will not be given here, but may be seen most easily by considering the radiative corrections to the lowest order Born approximation electron scattering. In this case the only other possible ultraviolet divergence in the vertex function comes from the contribution of the "magnetic terms" on both initial and final lines; from the particular way that these terms were defined, it turns out that the part that would be infinite actually gives no contribution. By Ward's identity (32), there is then no additional infinity in the wave function renormalization.

**Closed Loops**

Except for the usual charge renormalization associated with real and virtual photon lines, the presence of closed charge loops does not alter our result. To see this, consider a closed loop on which four or more external photon lines terminate (by external we mean here external to the closed loop). If the four-momentum of one of the photons vanishes, the factor associated with this loop will vanish. This follows from the fact that we can always obtain the expression for a zero-momentum photon by differentiating a lower order expression with respect to $P_\mu$, the momentum carried around the loop by the charged particle. We then have an integral of a perfect derivative, and the result is zero. Thus if a soft photon connects to a closed charge loop it will automatically give a contribution of order $K(k)$ (except for vacuum polarization loops where it gives the proper charge renormalization).

**Possible Complications in the Argument**

The construction of a general, yet simple and rigorous, proof of the infrared factorization has not yet been accomplished. The preceding discussion show how some of the more obvious of the confluences (or overlaps) of infrared divergences may be eliminated. In the interests of simplicity in presenting the main line of the argument, several possible sources of further difficulty were glossed over. Some of these will now be mentioned, and arguments will be given why they do not in fact upset the main result.
(i) As we have seen, the infrared divergence phenomenon is associated with those photons which terminate on electron lines whose four-momenta is on, or very close to, the mass shell. We have only considered the possibility that the electron line is external, or nearly external; but what about the possibility that the electron line is internal and lies between two large momentum transfers? This point may be seen by considering (A-10) under the conditions that \( \vec{Q} \) is large, but \((\vec{Q}^2 + 2p \cdot \vec{Q})\) is small. The arguments for neglecting (A-10) no longer hold. This difficulty is dealt with by the observation that \( \vec{Q} \) is, in effect, a variable of integration; with regard to the \( \vec{Q}\)-integration, the first factor in (A-10) is a pole, and there is no obvious difficulty when \( k \to 0 \).

(ii) If the total momentum transfer is small, the approximation \( \tilde{q}_i \to q_i \), used shortly after (A-11), will become invalid if \( k \) is too large. Rather than indicating a breakdown in the proof, however, this may be interpreted as a restriction on how small \( k \) must be if the infrared term in \( k \) is to give a good approximation by itself. Thus if \( k \gtrsim q \) (where \( q \) is the total momentum transfer), we may call the photon “soft” and claim that its infrared contribution, as defined by our particular separation, dominates other contributions. If \( k \gtrsim q \), the photon is “hard” and noninfrared contributions may dominate. These remarks are supported by the intuitive semiclassical arguments given in the Introduction. Since the scattering region cannot be localized to a region of dimensions smaller than \( q^{-1} \), the spectrum is expected to depart from the infrared form for \( k \gtrsim q \). It is important to point out that these remarks do not invalidate the discussion of spurious charge renormalization given in Section 5(b). In that case the virtual photon under discussion has both ends terminating on a single electron line (Case (b) of the present Appendix); there is then no difference between \( \tilde{q}_i \) and \( q_i \) and this particular approximation need not be made.

(iii) The discussion of closed loops could have a difficulty if the differentiation with respect to \( p_n \) makes the integrand singular; then integration by parts could not be carried out to give zero. However, from the structure of these closed loops there is no known reason why such singularities should occur.

The listing we have just made is surely not exhaustive. It is given mainly to show that the detailed presentation of a rigorous proof might be prohibitively complicated, but that the approximations made in our argument are intuitively very plausible.

APPENDIX B. DISCUSSION OF COMBINATORIAL FACTORS WHEN THE BASIC PROCESS CONTAINS REAL OR VIRTUAL PHOTONS

The discussion of Section 2 was somewhat simplified by the fact that the basic process was due to potential scattering and did not contain any virtual photons. Thus it was not necessary somehow to distinguish the virtual photons producing the basic interaction from those which give the radiative corrections to the basic process. The presence of real photons in the basic process causes no difficulty.
If they are not observed (manifesting themselves only by an energy loss), they are already included in the treatment of Sections 2 and 3(a). If they are observed, they are given by the treatment in Section 3(e). The main problem is to establish that the set of Eqs. (2.2) is correct in more general situations.

By definition, $M_0$ is the lowest order contribution to the matrix element for the given process. Let $C(M_0)$ be the set of graphs $G_0$ which contribute to $M_0$. Each of these graphs will have the minimum number of virtual photons, $n_0$, required for the process to take place. The lowest order contribution can thus be written:

$$M_0 = \frac{1}{n_0!} \int \cdots \int \prod_{j=1}^{n_0} \frac{d^4 k_j}{k_j^2 - \lambda^2} \rho_0(k_1 \cdots k_{n_0}). \tag{B-1}$$

The integrand $\rho_0$ contains a sum over contributions from the various graphs in $C(M_0)$. As in Section 2, all photons are inserted into each graph independently. Also, each term in $\rho_0$ contains $n_0 \delta$-functions which fix the $k$'s in terms of the external four-momenta. All higher graphs can be obtained by adding photon lines into the graphs in the set $C(M_0)$, possibly in several different ways. If there are $n$ additional photons inserted in all possible ways, the matrix element may be written:

$$M_n = \frac{1}{(n + n_0)!} \int \cdots \int \prod_{j=1}^{n+n_0} \frac{d^4 k_j}{k_j^2 - \lambda^2} \rho_n(k_1 \cdots k_{n+n_0}). \tag{B-2}$$

Each term in $\rho_n$ contains $n_0 \delta$-functions relating the $k$'s to the external four-momenta, but leaving $n$ of them to be integrated over. The rest of the analysis goes through just as before, and we find

$$\rho_n(k_1 \cdots k_{n+n_0}) = S(k_{n+n_0}) \rho_{n-1}(k_1 \cdots k_{n+n_0-1}) + K(k_{n+n_0}). \tag{B-3}$$

The analysis leading to the infrared exponential follows the same pattern as in Section 2, Eqs. (2.5) through (2.12) particularly. In place of (2.9) we find

$$\rho_n(k_1 \cdots k_{n+n_0}) = \sum_{\text{perm}} \sum_{r=0}^n r!(n_0 - n - r)! \prod_{i=1}^r S(k_i; (\beta_{n-r}(k_{r+1} \cdots k_{n+n_0})), \tag{B 1}$$

while (2.12) becomes

$$m_n = \frac{1}{(n + n_0)!} \int \cdots \int \prod_{i=1}^{n+n_0} \frac{d^4 k_i}{k_i^2 - \lambda^2} \beta_n(k_1 \cdots k_{n+n_0}). \tag{B-5}$$

The integrand $\beta_n$ is a sum of terms, each of which has $n_0 \delta$-functions relating the $k$'s to the external four-momenta.
Details of the calculation of $\tilde{B}_{ij}$ and $B_{ij}$ will be given in this Appendix. The basic integral required for the evaluation of $\tilde{B}_{ij}$ is

$$I_{ij} = \int_0^{k_A} \frac{d^3k}{(k^2 + \lambda^2)^{1/2}} \cdot \frac{1}{k^2 \cdot p_i \cdot k^2 \cdot p_j}$$

$$= 2\pi \int_1 \frac{dx}{x} \int_0^{k_A} \frac{k^2 \cdot dk}{(k^2 + \lambda^2)^{1/2}} \cdot \frac{1}{\omega^2 E_{ij}^2 - k^2 p_z^2}, \quad (C-1)$$

where $2p_z = (1 + x)p_i + (1 - x)p_j$. With the change of variables $z = k/(k^2 + \lambda^2)^{1/2}$ the $k$-integration is then easily evaluated in the limit $\lambda \to 0$:

$$I_{ij} = 2\pi \int_1 \frac{dx}{x} \int_0^{k_A/(k^2 + \lambda^2)^{1/2}} \frac{z^2 \cdot dz}{(1 - z^2)(E_{ij}^2 - z^2 p_z^2)}$$

$$= 2\pi \int_1 \frac{dx}{x} \frac{1}{p_z^2} \left[ \frac{1}{2} \ln \frac{k^2 p_z^2}{\lambda^2 E_{ij}^2} - \tilde{G}_{ij}(x) \right], \quad (C-2)$$

where

$$\tilde{G}_{ij}(x) = \frac{E_i - \left| p_i \right|}{2 \cdot \left| p_z \right|} \ln \left( \frac{E_i + \left| p_i \right|}{E_j - \left| p_j \right|} \right) + \ln \left( \frac{E_j + \left| p_j \right|}{2 E_z} \right). \quad (C-3)$$

In the extreme relativistic limit (with $p_i \cdot p_j \gg m_i m_j$), the main contribution to this integral will come from that region of $x$ where $p_z^2$ is small; in this region $\tilde{G}_{ij}$ tends to be small, cancelling out the small denominator. Thus the $\tilde{G}_{ij}$ contribution to $B_{ij}$ should be of order unity, not log $(E/m)$. The complete result for $B_{ij}$ may then be written

$$\tilde{B}_{ij} = \frac{Z_i \theta_i Z_j \theta_j}{2\pi} \left\{ \ln \frac{k^2 m_i m_j}{\lambda^2 E_i E_j} - \tilde{G}_{ij}(1) - \tilde{G}_{ij}(-1) \right. \right.$$  

$$- \frac{p_i \cdot p_j}{2} \int_1 \frac{dx}{x^2} \left[ \ln \frac{k^2 p_z^2}{\lambda^2 E_{ij}^2} - 2\tilde{G}_{ij}(x) \right]. \quad (C-4)$$

It is not desirable to carry the calculation further at this stage because there will be some cancellations with terms from $B_{ij}$.

For purposes of calculation, it is convenient to rewrite $B_{ij}$ in the form

$$B_{ij} = \frac{-iZ_i \theta_i Z_j \theta_j}{8\pi^3} \int \frac{d^4k}{k^2 - \lambda^2} \left( \frac{2p_i \theta_i}{k^2 - 2k \cdot p_i \theta_i} + \frac{2p_j \theta_j}{k^2 + 2k \cdot p_j \theta_j} \right)^2$$

$$- k^2 \left( \frac{1}{k^2 - 2k \cdot p_i \theta_i} - \frac{1}{k^2 + 2k \cdot p_j \theta_j} \right)^2 \right\}, \quad (C-5)$$
The basic integrals are

\[ I_{ij} = \int \frac{d^3k}{k^3 - \lambda^3} \frac{1}{(k^2 - 2k \cdot p_i \theta_i)(k^2 + 2k \cdot p_j \theta_j)} \]  

(C-6)

and

\[ J_{ij} = \int d^3k \frac{1}{(k^2 - 2k \cdot p_i \theta_i)(k^2 + 2k \cdot p_j \theta_j)} \]  

(C-7)

where \( 2p^ \prime_z = (1 + x)p_i \theta_i - (1 - x)p_j \theta_j \). The constant in (C-7) is infinite, but it cancels out when the various terms are added to the form (C-5); the cancellation is unambiguous since (C-5) converges in the ultraviolet. Using these results, we find that \( B_{ij} \) is given by

\[ B_{ij} = \frac{-Z_i \theta_i Z_j \theta_j}{2\pi} \left\{ \ln \frac{m_i m_j}{\lambda^2} + \frac{p_i \cdot p_j \theta_i \theta_j}{2} \int_{-1}^{1} \ln \frac{p_z r^2}{\lambda^2} \frac{dx}{p_z^2} \right\} + \frac{1}{4} \int_{-1}^{1} \ln \frac{p_z r^2}{m_i m_j} \frac{dx}{p_z^2} \]  

(C-8)

There are now two cases to consider \( (\theta_i = \pm \theta_j) \). If \( \theta_i = - \theta_j \), then \( p_z^2 \) is equal to \( p_z^2 \) and \( B_{ij} \) is real. Thus we have

\[ B_{ij} + \text{Re} B_{ij} = \frac{-Z_i \theta_i Z_j \theta_j}{2\pi} \left\{ \ln \frac{k_i^2}{E_i E_j} + \frac{1}{2} \frac{p_i \cdot p_j}{k_i^2} \int_{-1}^{1} \ln \frac{k_i^2}{E_i E_j} \frac{dx}{p_z^2} \right\} + \frac{1}{4} \int_{-1}^{1} \ln \frac{p_z^2}{m_i m_j} E_i E_j \frac{dx}{p_z^2} + G_{ij}(1) - G_{ij}(-1) - p_i \cdot p_j \int_{-1}^{1} G_{ij}(x) \frac{dx}{p_z^2} \]  

(C-9)

Under the conditions \( p_i \cdot p_j \gg m_i m_j \) and \( (p_i - p_j)^2 < 0 \), the leading (logarithmic) terms of (C-9) are found to be

\[ B_{ij} + \text{Re} B_{ij} \approx \frac{-Z_i \theta_i Z_j \theta_j}{2\pi} \left\{ \ln \frac{2p_i \cdot p_j}{m_i m_j} - 1 \right\} \ln \frac{k_i^2}{E_i E_j} + \frac{1}{2} \ln \frac{(p_i - p_j)^2}{m_i m_j} \ln \frac{m_j}{m_i} \ln \frac{E_j}{E_i} - \frac{1}{2} \ln^2 \frac{E_j}{E_i} \]  

(C-10)

The evaluation of some of these integrals, and also those for the condition \( (p_i - p_j)^2 > 0 \), will be given at the end of this Appendix.

In case \( \theta_i = \theta_j \), \( p_z^2 \) will have two zeroes within the range of integration. These zeroes are to be interpreted by adding a small negative imaginary part to \( p_z^2 \);
this will result in an imaginary contribution to \( B_{ij} \), the physical interpretation of which has been given in Section 4(b). Let the roots of \( p_x^2 = 0 \) be \( x_1 \) and \( x_2 \), with \( x_2 > x_1 \); then the poles of \( (C-8) \) occur at \( x_2 + i\epsilon \) and \( x_1 - i\epsilon \), where \( \epsilon \) is small and positive. We also choose the branch cuts for \( \ln p_x^2 \) so that they do not cross the \( x \) axis; the branch must be chosen so that \( \text{Im} (\ln p_x^2) = 0 \) for \( x = 1 \) or \(-1\). Now it is a property of the special form of the denominator combining integral we have used that if \( y = 1/x \),

\[
p_y^{r^2} = \frac{p_x^2}{x^2} \quad (y = 1/x). \tag{C-11}
\]

This permits us to transform the first integral in \( (C-8) \) as follows:

\[
\int_{-1}^{1} \ln \frac{p_x^2}{\lambda^2} \frac{dx}{p_x^2} = -\int_{-1}^{1} \ln \frac{p_x^2}{\lambda^2} \frac{dx}{p_x^2} + \int_{-\infty}^{\infty} \ln \frac{p_x^2}{\lambda^2} \frac{dx}{p_x^2} + \int_{-1}^{1} \ln x^2 \frac{dx}{p_x^2} \tag{C-12}
\]

The first term on the right side of \( (C-12) \) is real and gives the same contribution to \( B_{ij} \) as in the case of the corresponding term with \( \theta_i = -\theta_j \).

The third term on the right has no logarithmic dependence on \( m_i \) or \( m_j \). To see this, we set \( m_i = m_j = 0 \) in \( p_x^2 \). The integral then becomes

\[
\frac{-2}{p_i \cdot p_j} \int_{-1}^{1} \ln x^2 \frac{dx}{1 - x^2} = \frac{\pi}{p_i \cdot p_j}. \tag{C-13}
\]

More exactly, the integral could be expressed in terms of Spence functions. We shall not give the details here.

The middle term of \( (C-12) \) is also easily evaluated. The singularities occur at \( z_2^{-1} = \frac{-1 - i\epsilon}{2\xi} \) and \( z_1^{-1} = \frac{1 + i\epsilon}{2\xi} \). We therefore write

\[
p_x^2 = A(z_2 x - 1)(z_1 x - 1), \tag{C-14}
\]

where

\[
A = \frac{1}{4} \left| m_i^2 + m_j^2 + 2p_i \cdot p_j \right| > 0.
\]

Then

\[
\int_{-\infty}^{\infty} \frac{\ln \frac{p_x^2}{\lambda^2}}{p_x^2} \, dx = \frac{1}{A} \int_{-\infty}^{\infty} dx \left\{ \ln \frac{A}{\lambda^2} + \ln \left( 1 - \frac{z_2 x - 1}{z_1 x} \right) + \ln \left( 1 - \frac{z_1 x - 1}{z_2 x} \right) \right\}
\]

\[
= \frac{2\pi i}{(z_2 - z_1) A} \left\{ \ln \frac{A}{\lambda^2} + \ln \left( 1 - \frac{z_2}{z_1} \right) + \ln \left( 1 - \frac{z_1}{z_2} \right) \right\} \tag{C-15}
\]

\[
= \frac{2\pi i}{(x_2 - x_1) A} \left\{ \ln \left| \frac{A}{\lambda^2} \right| \left( \frac{x_2 - x_1}{x_1 x_2} \right)^2 - i\pi \theta(x_1 x_2) \right\}.
\]
Since the roots of \( p_z^2 = 0 \) are
\[
x_i = \frac{-(m_i^2 - m_j^2)}{4 A} \pm 2 \left[ \frac{(p_i \cdot p_j)^2 - m_i^2 m_j^2}{4 A} \right]^{1/2}.
\] (C-16)

We find easily
\[
\int_{-\infty}^{\infty} \ln \frac{p_z^2}{\lambda^2} \, dx = \frac{2\pi i}{p_z^2} \left\{ \ln \left[ \frac{m_i^2 + m_j^2 - 2p_i \cdot p_j}{\lambda^2} \right] - i\pi \theta(x_1 x_2) \right\}.
\] (C-17)

In the center-of-mass system \( (p_i = -p_j = p) \), this reduces further to
\[
\frac{2\pi i}{p(E_i + E_j)} \left\{ \ln \left[ \frac{4p^2(E_i + E_j)^2}{\lambda^2 \left[ 2(m_i^2 + m_j^2) - (E_i + E_j)^2 \right]} \right] - i\pi \theta(x_1 x_2) \right\}. \quad \text{(C-18)}
\]

In the extreme relativistic limit for both particles, this is
\[
\approx \frac{2\pi i}{p^2} \ln \frac{2p}{\lambda}. \quad \text{(C-19)}
\]

On the other hand, in the limit that \( m_i \to \infty \), it becomes
\[
\approx \frac{4\pi i}{pm_i} \ln \frac{2p}{\lambda}
\]

and for \( m_i = m_j \), it is
\[
\frac{2\pi i}{pE} \ln \left( \frac{2E}{\lambda} \right).
\]

This confirms the discussion about the Coulomb phase shifts given at the end of Section 4(b). In summary, Eq. (C-9) is correct for both cases \( \theta_i = \pm \theta_j \) except for terms of nonlogarithmic order.

This Appendix will be concluded with a discussion of the integrals occurring in (C-9) and the extraction of the contributions of logarithmic order from them. The simplest integral is
\[
\int_{1-}^{1} \frac{dx}{p_x^2} = \frac{4}{(p_i - p_j)^2} \left( \frac{1}{x_1 - x_2} \right) \ln \left( \frac{(1 - x_1)(1 + x_2)}{(1 + x_1)(1 - x_2)} \right), \quad \text{(C-20)}
\]

where \( x_1 \) and \( x_2 \) are the roots of \( p_x^2 = 0 \). Under the assumption that \( p_i \cdot p_j \gg m_i m_j \) (high-energy approximation), these roots are
\[ x_1 = \frac{m_i^2 - m_j^2 - 2p_i \cdot p_j + (m_i^2 m_j^2 / p_i \cdot p_j)}{(p_i - p_j)^2}, \]  
\[ x_2 = \frac{m_j^2 - m_i^2 + 2p_i \cdot p_j - (m_i^2 m_j^2 / p_i \cdot p_j)}{(p_i - p_j)^2}, \]  
and (C-20) reduces to
\[ \int_{-1}^{1} \frac{dx}{p_x^2} = \frac{2}{p_i \cdot p_j} \ln \left( \frac{2p_i \cdot p_j}{m_i m_j} \right), \]  
(C-22)

The next more complicated integral to be considered is
\[ \int_{-1}^{1} \ln p_x^2 dx - 2 \ln |(p_i - p_j)|^2 / 4 + \sum_i \{(1 - x_i) \ln|x_i - 1| - 2 + (1 + x_i) \ln |x_i + 1|| \} \approx 2 \ln |(p_i - p_j)|^2; \]  
(C-23)

where terms of order unity have been neglected in the approximate form.

The most complicated integral appearing in (C-9) is
\[ \int_{-1}^{1} \frac{\ln E_x^2}{p_x^2} dx = - \frac{1}{p_i \cdot p_j} \int_{-1}^{1} \ln E_x \left( \frac{1}{x - x_1} - \frac{1}{x - x_2} \right) dx. \]  
(C-24)

Evaluation in closed form requires the use of Spence functions; here we shall simply pick out the leading logarithmic orders. To do this, it is necessary to consider two cases separately.

**CASE (a)**

Momentum difference space-like, \((p_i - p_j)^2 < 0\). The roots (C-21) are of opposite sign and greater than 1 in magnitude: \(x_1 > 1, x_2 < -1\). As a definite example, take \(E_j \geq E_i\), then the root of \(E_i = 0\) is
\[ x' = \frac{E_j + E_i}{E_j - E_i} > 1. \]

Neglecting terms of nonlogarithmic order, the integral involving the pole at \(x_2\) is easily evaluated
\[ \int_{-1}^{1} \frac{\ln E_x^2}{x - x_2} dx \approx \ln E_i \ln \frac{2p_i \cdot p_j}{m_j^2}. \]  
(C-25)

The integral involving the pole at \(x_1\) requires special care since \(\ln E_x\) may vary rapidly near \(x = 1\) if \(E_j \gg E_i\). Rewriting the integral
\[ \int_{-1}^{1} \frac{\ln E_x^2}{x - x_1} dx = - \ln E_i \ln \frac{2p_i \cdot p_j}{m_i^2} + \int_{-1}^{1} \ln \left[ 1 + \xi (1 - x) \right] dx, \]
where $\xi = \frac{1}{2}(E_j - E_i)/E_i$, and making the variable change

$$y = \frac{1 + \xi (1 - x_i)}{1 + \xi (1 - x)}$$

we easily find

$$\int_{-1}^{1} \frac{\ln E_x}{x - x_1} dx \approx - \ln E_i \ln \frac{2p_i \cdot p_j}{m_i^2} - \frac{1}{2} \ln^2 \frac{E_j}{E_i} \quad (C-26)$$

and

$$\int_{-1}^{1} \frac{\ln E_x}{p^2_x} dx \approx \frac{1}{p_i \cdot p_j} \left\{ \ln E_i E_j \ln \frac{2p_i \cdot p_j}{m_i m_j} - \ln \frac{E_i}{E_j} \ln \frac{m_i}{m_j} + \frac{1}{2} \ln^2 \frac{E_i}{E_j} \right\}. \quad (C-27)$$

This, together with the two previous integrals, leads to the approximation (C-10).

**Case (b)**

Momentum difference time-like, $(p_i - p_j)^2 > 0$. This case can occur only if the masses are unequal, and large logarithms arise only if the masses are very different. Thus, with $E_j > E_i$, the conditions $m_i^2 > 2p_i \cdot p_j - m_i^2 \gg m_i^2$ must hold, and the roots are ordered as follows: $x_2 > x' > x_1 > 1$. The integral (C-26) remains unchanged; but in place of (C-25) we find

$$\int_{-1}^{1} \frac{\ln E_x}{x - x_1} dx = \ln E_i \ln \frac{2p_i \cdot p_j}{m_i m_j} + \int_{-1}^{1} \frac{\ln [1 + \xi (1 - x)]}{x - x_2} dx.$$

The appropriate change of integration variables is

$$\frac{1 - y}{y} = \frac{1 + \xi (1 - x)}{-1 - \xi (1 - x_2)}$$

and the logarithmic contributions are given by

$$\int_{-1}^{1} \frac{\ln E_x}{x - x_1} dx = \ln E_j \ln \frac{2p_i \cdot p_j}{m_i^2} + \frac{1}{2} \ln^2 \left( \frac{2p_i \cdot p_j}{m_i^2} \right) \quad (C-28)$$

Thus

$$\int_{-1}^{1} \frac{\ln E_x}{p^2_x} dx = \frac{1}{p_i \cdot p_j} \left\{ \ln E_i E_j \ln \frac{2p_i \cdot p_j}{m_i m_j} - \ln \frac{E_i}{E_j} \ln \frac{m_i}{m_j} + \frac{1}{2} \ln^2 \frac{E_i}{E_j} \right\} + \frac{1}{2} \ln^2 \left( \frac{2p_i \cdot p_j}{m_i^2} \right) \quad (C-29)$$

for $m_i^2 > 2p_i \cdot p_j$. The result for the situation where $m_i^2 > 2p_i \cdot p_j$ can be obtained from (C-29) by substituting $m_i^2$ for $m_i^2$ in the last term.

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*Note added in proof:* Overlapping divergences are also treated in a recent work by W. Van Haeringen [Physica 26, 289, 306 (1960)]. Since the present paper was prepared, other papers have appeared in which the overlapping divergence problem has been circumvented or
ignored. One of these (Caianiello and Okubo, submitted to *Nuovo Cimento*) uses a special formation of field theory; no mention is made of overlapping divergences, but the claim is made (private communication from E. R. Caianiello) that because of the symmetrical treatment such divergences never appear. Another treatment [T. Murota, *Progr. Theoret. Phys. Japan* **24**, 1109 (1960)] is based on a separation of the Hamiltonian into hard and soft photon parts and involves operator analysis to treat the soft photon part to all orders. A third approach [Okubo, *Nuovo Cimento* **18**, 70 (1960) and K. E. Erikson, *Nuovo Cimento* **19**, 1010 (1961)] uses a graphical analysis but divides the photons into hard and soft ones with an arbitrary separation energy $E$. Infrared divergences are now associated with all graphs in which all soft photons are external to all hard photons. When all such contributions are combined symmetrically the result is easily summed to an exponential form (as in the present paper) and the cancellation between real and virtual photons may be demonstrated. Eriksson then uses the fact that the result can not actually depend on $E$ to rewrite his expressions in a form similar to that of the present paper. Since this third approach is actually closest to the present paper, some further comparisons may be appropriate. It appears that this approach is more transparent than the proof we have given and it may make our Appendix unnecessary if the only aim is to show the cancellation of the infrared divergences. However, our proof does show that large contributions (relative order $E^2/M^2$) arising from soft photon emission between hard photon interactions are cancelled by the contribution (A-5). Also, our approach indicates that the infrared factor will dominate the $k$-dependence to fairly high energies (there is no need for restrictions like $|k| \ll m$). Finally, we have shown how to rearrange the integrands so that they have no infrared singularities [e.g., see Eq. (2.8)]; this is in contrast to the situation where the integrated quantities have no divergence but separate integrands may require an infrared cutoff.

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