Effective QCD - problem set 8
5.12.2017. Tuesday 14:00
room D-02-2

1. Effective QCD lagrangian from problem set 4 can be supplemented by two 4 derivative terms, one of them taking a form of the commutator squared,
$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4}\,\text{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2}\,\text{Tr}\left(\left[U^\dagger \partial_\mu U, U^\dagger \partial_\nu U\right]^2\right),$$
where e is a free parameter. Calculate the lowest order interaction term by expanding U in terms of the Goldstone boson fields in SU(3).

2. For the SU(2) case calculate the 4 derivative term for the "hedgehog" Ansatz
$$U = \exp(i\,\vec{n}\cdot\vec{\tau}\,P(r)).$$
To this end it is convenient to expand
$$U^\dagger \partial_M U = \frac{i}{2}\xi_M^A\,\tau_A$$
where M stands for space index (note that U is time independent). Convince yourself that
$$\xi_M^A = A\,\delta_{MA} + B\,n_M n_A + C\,n_C\,\epsilon_{CMA}$$
with
$$A = 2\frac{\sin P\cos P}{r}, \quad B = 2\left(P' - \frac{\sin P \cos P}{r}\right), \quad C = 2\frac{\sin^2 P}{r}.$$

3. Collecting together the kinetic term and the 4-th order term one obtains an expression for the hedgehog energy. Show that the energy is finite if $P(0) = n\pi$ (with $P(\infty) = 0$). Instead of solving equation of motion to calculate the numerical value of the energy (which can be done only numerically), one can use variational principle, using an Ansatz
$$P(r) = 2\arctan\left[\left(\frac{r_0}{r}\right)^2\right],$$
where r_0 is a variational parameter. Show that, introducing variable $r/r_0 = t$, one has
$$\sin P = \frac{2t^2}{t^4+1}, \quad \cos P = \frac{t^4-1}{t^4+1}, \quad \frac{dP}{dt} = -\frac{4t}{t^4+1}$$
and the energy (in fact the mass of the soliton) can be calculated numerically:
$$M = 3\pi^2\sqrt{2}\left[F_\pi^2 r_0 + \frac{30}{32e^2}\frac{1}{r_0}\right].$$

4. Effective lagrangians discussed so far contained only even numbers of U fields. In order do describe physically possible transitions like $K^+K^- \to \pi^+\pi^-\pi^0$ we need anomalous term involving five fields U. Such term is called Wess-Zumino-Witten term and it cannot be written as a local action. Witten proposed to introduce the fifth time-like dimension characterized by the coordinate $0 \le \alpha \le 1$

$$x^\mu \to y^i = (x^\mu, \alpha)$$

and extending the U field

$$U = \exp\left(i\alpha \frac{\phi(x)}{F_0}\right),$$

where ϕ is the SU(3) meson field we have used before. Then the nonlocal action takes the following form

$$S_{WZW} = -\frac{i}{240\pi^2} \int\limits_0^1 d\alpha \int d^4x \, \varepsilon^{ijklm} \, \mathrm{Tr}\left(L_i L_j L_k L_l L_m\right),$$

where

$$L_i = U^\dagger \frac{\partial U}{\partial y^i}$$

and ε^{ijklm} is totally antisymmetric. Expanding U in powers of ϕ allows to perform the integral over α. Calcualte the lowest order term in this expansion.