Effective QCD - problem set 4
7.11.2017. Tuesday 14:00

room D-02-2

1. Find n for which
$$\partial_\mu \frac{z^\mu}{(z^2)^n} = 0.$$

2. Define
$$D(z-y) = \langle 0 \,|\, T\left(\phi(x)\phi(y)\right) |\, 0\rangle \,.$$
We have shown that
$$D(z) = \int_0^\infty d\mu^2 \rho(\mu^2)\, D^+(z,\mu^2)$$
where D^+ is a positive frequency part of the free scalar propagator with mass μ and $\rho(\mu^2)$ is a positive non-perturbative function, called spectral density. Show that the canonical commutation relations of scalar fields imply
$$\int_0^\infty d\mu^2 \rho(\mu^2) = 1.$$

3. We have shown that (for $x^0 > y^0$)
$$\langle 0 \,|\, [A_a^\mu(x), P_a(y)] \,|\, 0\rangle = C_a \partial^\mu D(x-y,0) \quad \text{no sum over } a$$
where $D(z,0)$ is a full scalar propagator. Show hat this implies
$$\left\langle 0 \,\middle|\, \left[Q_a^A(t), P_a(y)\right] \,\middle|\, 0\right\rangle = C_a.$$

4. Gell-Mann Oaks Renner relation.

   Recall that we have shown, that
$$i \left\langle 0 \,\middle|\, \left[Q_a^A(t), P_a(y)\right] \,\middle|\, 0\right\rangle = \frac{2}{3} \left\langle \bar{q} q\right\rangle \tag{1}$$
where
$$\begin{aligned} P_a(y) &= i\bar{q}(y)\gamma^5 \lambda^a q(y), \\ Q_a^A(t) &= \int d^3x\, \bar{q}(x)\gamma^0\gamma^5 \frac{\lambda^a}{2} q(x). \end{aligned}$$
Generalize relation (1) to the case when two SU(3) indices are different $\rightarrow Q_a^A(t), P_b(y)$.In particular show that
$$\int d^3x \left\langle 0 \,\middle|\, \left[\bar{u}(x)\gamma^0\gamma^5 u(x), \bar{d}(y)\gamma^5 u(y)\right] \,\middle|\, 0\right\rangle = -\left\langle \bar{u}u + \bar{d}d\right\rangle .$$

This implies that

$$\langle 0| \bar{u}(x)\gamma^\mu\gamma^5 u(x)\, \bar{d}(y)\gamma^5 u(y) \,|0\rangle = C\partial^\mu D^+(x-y,0)$$

where $C = -\left\langle \bar{u}u + \bar{d}d \right\rangle$. Rewrite this relation in momentum space. Remember that this relation is saturated by intermediate zero mass states, Goldstone bosons. In this particular case the only intermediate state is $|\pi^-(p)\rangle$ .Assuming from Lorentz invariance

$$\langle 0| \bar{u}(0)\gamma^\mu\gamma^5 u(0) \left|\pi^-(p)\right\rangle = i\sqrt{2}F_\pi p^\mu, \;\; \langle 0| \bar{u}(0)\gamma^5 u(0) \left|\pi^-(p)\right\rangle = \sqrt{2}G_\pi$$

Relate the product $F_\pi G_\pi$ to C. Using axial current non-conservation in the massive quark case

$$\partial_\mu A^\mu_a = i\bar{q}\left\{\frac{\lambda_a}{2}, M\right\}\gamma^5 q$$

prove that

$$m_\pi^2 F_\pi^2 = (m_u + m_d)\left|\langle \bar{u}u\rangle\right| . \tag{2}$$