Effective QCD - problem set 2
17.10.2017. Tuesday 14:00
room D-02-2

1. (From last set) Let's denote Lorentz transformations (including boosts, rotations, space and time reflections) in a usual way

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}.$$

In order to calculate Lorentz transformation of the spinors consider matrix X related to the space-time point x^{μ}:

$$X(x) = x_{\mu}\tilde{\sigma}^{\mu} = x^0\sigma^0 + x^1\sigma^1 + x^2\sigma^2 + x^3\sigma^3.$$

To this end it is useful to introduce the following notation

$$\sigma^{\mu} = (1, \vec{\sigma}), \quad \tilde{\sigma}^{\mu} = (1, -\vec{\sigma})$$

remembering that:

$$\partial_{\mu} = (\partial_t, \vec{\nabla}), \quad \partial^{\mu} = (\partial_t, -\vec{\nabla}).$$

Show that

$$\det X = x^{\mu} x_{\mu}.$$

Therefore Lorentz transformation Λ generates $\mathrm{SL}(2, C)$ transformation of matrix X:

$$M^{\dagger} X' M = X$$

where $X' = X(x')$. These transformations preserve determinant. Show that

$$\Lambda^{\mu}{}_{\nu} = \frac{1}{2}\,\mathrm{Tr}\left(\tilde{\sigma}^{\nu} M^{\dagger} \tilde{\sigma}^{\mu} M\right) = \frac{1}{2}\,\mathrm{Tr}\left(\sigma_{\mu} M^{\dagger} \tilde{\sigma}^{\mu} M\right).$$

Note that M are defined up to a phase. The same reasoning can be repeated for matrix $Y(x)$:

$$Y(x) = x_{\mu}\sigma^{\mu}$$

where the pertinent transformation matrix N is defined as

$$N^{\dagger} Y' N = Y.$$

Express Lorentz transformation Λ in terms of N. Prove that $NM^{\dagger} = 1$.

Calculate Λ for

$$M = N = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}$$

and

$$M = \begin{bmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{bmatrix}, \quad N = \begin{bmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{bmatrix}$$

where $v/c = \tanh\theta$.

2. Real scalar field lagrangian density reads as follows:

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x).$$

Calculate Hamiltonian. Canonical equal-time quantization rules for real scalar field operators read:

$$\left[\hat{\phi}(t,\vec{x}), \vec{\pi}(t,\vec{x}\,')\right] = i\delta^{(3)}(\vec{x}-\vec{x}\,')$$

and all other possible commutators are zero. Using decomposition

$$\hat{\phi}(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^2\sqrt{2\omega_k}}\left[e^{-i\,kx}\hat{a}(\vec{k}) + e^{+i\,kx}\hat{a}^\dagger(\vec{k})\right]$$

show that the canonical quatization rules are satisfied if

$$\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}\,')$$

and the remaining two commutators vanish.

3. For a system of SU(3) scalar fields $\hat{\phi}_i(x)$ with $i = 1, 2, 3$ that satisfy the following commutation rules

$$\left[\hat{\phi}_i(t,\vec{x}), \hat{\pi}_j(t,\vec{x}\,')\right] = i\delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{ij}$$

one defines charge operators

$$\hat{Q}^a(t) = -i\int d^3\vec{x}\,\hat{\pi}_i(t,\vec{x})T^a_{ij}\hat{\phi}_j(t,\vec{x})$$

where matrices T^a satisfy SU(3) commutation relations:

$$\left[T^a, T^b\right] = if^{abc}T^c.$$

Prove that

$$\left[\hat{Q}^a(t), \hat{Q}^b(t)\right] = if^{abc}\hat{Q}^c(t).$$

4. In the case of fermion fields, commutation relations from problem 2 are replaced by anticommutation relations:

$$\{q_{\alpha,k}(t,\vec{x}), q_{\beta,l}(t,\vec{x}\,')\} = \delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{\alpha\beta}\delta_{kl}$$

where α, β stand for Dirac indices and k, l denote SU(3) matrices. Relevant charges are defined as

$$\begin{aligned}
\hat{Q}^a_{L,R}(t) &= \int d^3\vec{x}\,q^\dagger_{L,R}(t,\vec{x})T^a q_{L,R}(t,\vec{x}),\\
\hat{Q}_V(t) &= \int d^3\vec{x}\,\left[q^\dagger_L(t,\vec{x})q_L(t,\vec{x}) + q^\dagger_R(t,\vec{x})q_R(t,\vec{x})\right]
\end{aligned}$$

where $T^a = \lambda^a/a$ are SU(3) generators (Gell-Mann matrices). Making use of the identity (prove it!)

$$\{ab, cd\} = a\{b, c\}d - ac\{b, d\} + \{a, c\}bd - c\{a, d\}b$$

show that

$$\begin{aligned} \left[\hat{Q}^a_L, \hat{Q}^b_L\right] &= if^{abc}\hat{Q}^c_L, \\ \left[\hat{Q}^a_R, \hat{Q}^b_R\right] &= if^{abc}\hat{Q}^c_R, \\ \left[\hat{Q}^a_L, \hat{Q}^b_R\right] &= 0, \\ \left[\hat{Q}^a_{L,R}, \hat{Q}^b_V\right] &= 0. \end{aligned}$$

5. In this problem we shall solve Diraq equations

$$i\tilde{\sigma}^\mu \partial_\mu \psi_L - m\psi_R = 0, \quad i\sigma^\mu \partial_\mu \psi_R - m\psi_L = 0$$

staring from a solution in the reference frame S' where the particle is at rest:

$$i\partial'_t \psi'_L = m\psi'_R, \quad i\partial'_t \psi'_R = m\psi'_L.$$

Solve these equations for positive energy $E = m$. Decompose solutions in the basis of eigen functions of the spin operator S_3 (remember we are working here in the chiral representations of Dirac matrices). Next, transform these solution to the system S, in which system S' (connected with the particle) moves with velocity $\vec{v} = (0, 0, v),\ v > 0$ along the z axis ($\tanh\theta = \frac{v}{c}$). Find solutions in this frame by applying Lorentz transformation (see problem 1)

$$\psi_L(x) = M^{-1}\psi'_L(x'),\ \psi_R(x) = N^{-1}\psi'_R(x').$$

Construct positive and negative helicity solutions in frame S in terms of Dirac bispinors

$$\psi = \begin{bmatrix} \psi_L \\ \psi_R \end{bmatrix}.$$

Solution for a particle moving in any direction (not only along a z axis) can be obtained by applying rotation to the above solution. Show that rotation preserves helicity. Express such solutions in terms of two dimensional (Weyl spinor) helicity eigenstates

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\left|\pm\right\rangle = \pm\left|\pm\right\rangle.$$