Effective QCD - problem set 1
10.10.2017. Tuesday 14:00

room D-02-2

1. Canonical representation of Dirac matrices (or Bjorken-Drell) takes the following form:

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \gamma^k = \begin{bmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{bmatrix}$$

   where σ^k are 2×2 Pauli matrices. It is useful to use diffent representations. Weyl (or chiral) representation differs from the canonical one only by γ^0:

$$\gamma^0_{\rm ch} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \gamma^k_{\rm ch} = \gamma^k.$$

   Another useful representation, known as Majorana representation, takes the following form:

$$\gamma^0_{\rm M} = \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix}, \ \gamma^1_{\rm M} = i \begin{bmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{bmatrix}, \ \gamma^2_{\rm M} = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \gamma^3_{\rm M} = -i \begin{bmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{bmatrix}.$$

   Note that $(\gamma^\mu_M)^* = -\gamma^\mu_M$. Check by explicit calculation that all these reps. satisfy anticommutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$

   Calculate $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in all these representations. Find transformation matrices X and Y defined as:

$$\gamma^\mu_{\rm ch} = X\gamma^\mu X^{-1}, \ \gamma^\mu_{\rm ch} = Y\gamma^\mu_{\rm M} Y^{-1}.$$

2. Remember that originally Dirac equation has been written in a form

$$i\hbar\frac{\partial\psi}{\partial t} = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi$$

   where $\vec{p} = -i\hbar\vec{\nabla}$ is the momentum operator. Relation between γ matrices and β, $\vec{\alpha}$ matrices reads as follows:

$$\gamma^0 = \beta, \ \gamma^k = \beta\alpha^k.$$

   Calculate α^k in all the above representations of Dirac matrices.

3. Spin operator is related to the J generators of the Lorentz transformations

$$J^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

   and reads

$$S_i = \frac{1}{2}\varepsilon_{ijk}J^{jk}.$$

   Calculate $\vec{S}$ in chiral representation. Calculate $\vec{S}^2$.

4. Helicity operator is defined as

$$h = 2\frac{\vec{S}\cdot\vec{k}}{\left|\vec{k}\right|}.$$

Prove that

$$\gamma^5 h = \frac{\gamma^0\vec{\gamma}\cdot\vec{k}}{\left|\vec{k}\right|}.$$

5. Let's denote Lorentz transformations (including boosts, rotations, space and time reflections) in a usual way

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}.$$

In order to calculate Lorentz transformation of the spinors consider matrix X related to the space-time point x^{μ}:

$$X(x) = x_{\mu}\tilde{\sigma}^{\mu} = x^0\sigma^0 + x^1\sigma^1 + x^2\sigma^2 + x^3\sigma^3.$$

To this end it is useful to introduce the following notation

$$\sigma^{\mu} = (1, \vec{\sigma}), \quad \tilde{\sigma}^{\mu} = (1, -\vec{\sigma})$$

remembering that:

$$\partial_{\mu} = (\partial_t, \vec{\nabla}), \quad \partial^{\mu} = (\partial_t, -\vec{\nabla}).$$

Show that

$$\det X = x^{\mu}x_{\mu}.$$

Therefore Lorentz transformation Λ generates $\mathrm{SL}(2, C)$ transformation of matrix X:

$$M^{\dagger}X'M = X$$

where $X' = X(x')$. These transformations preserve determinant. Show that

$$\Lambda^{\mu}{}_{\nu} = \frac{1}{2}\,\mathrm{Tr}\left(\tilde{\sigma}^{\nu}M^{\dagger}\tilde{\sigma}^{\mu}M\right) = \frac{1}{2}\,\mathrm{Tr}\left(\sigma_{\mu}M^{\dagger}\tilde{\sigma}^{\mu}M\right).$$

Note that M are defined up to a phase. The same reasoning can be repeated for matrix $Y(x)$:

$$Y(x) = x_{\mu}\sigma^{\mu}$$

where the pertinent transformation matrix N is defined as

$$N^{\dagger}Y'N = Y.$$

Express Lorentz transformation Λ in terms of N. Prove that $NM^{\dagger} = 1$.

Calculate Λ for

$$M = N = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}$$

and

$$M = \begin{bmatrix} e^{\theta/2} & 0 \\ 0 & e^{-\theta/2} \end{bmatrix}, \quad N = \begin{bmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{bmatrix}$$

where $v/c = \tanh\theta$.